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Gutman Index of Annihilating-ideal Graph of Finite Local PIRs

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Abstract. A ring R is said to be a Principal ideal ring (for short PIR), if for any ideal of R is a principal ideal. (R, M) it's means that R local ring with maximal M. it is well known that (R, M) is a PIR if and only if the maximal ideal M is a principal. $A(R)^*$ is the set of all non-zero annihilating ideals. The annihilating-ideal graph of R, denoted $AG(R)$, is a graph with vertices $A(R)^*$ and two distinct ideal vertices I_1 and I_2 connected by an edge $\{I_1, I_2\}$ if $I_1 I_2 = (0)$. For a graph G, the Gutman index of G is denoted as $Gut(G)$ equal to $\sum_{u_1,u_2 \in V(G)} d(u_1)d(u_2)d(u_1, u_2)$. In this paper, we give some important properties of AG(R) and we obtain Gut($AG(R)$)where R is a finite local PIR.

1 Introduction

Let us assume that R is a commutative ring with identity, and (R, M, n) is a finite local PIR with maximal ideal M and satisfied, $M^n = (0)$, for some integer $n \geq 2$. It is established that this ring structure admits exactly $n - 1$ principal ideals, namely $M^1, M^2, M^3, ..., M^{n-1}$ (See [1, 2]). The interplay between graph theory and ring theory was first introduced by I. Beck in [3]. Later, D. F. Anderson and P. S. Livingston have revisited the concept in their paper [4], their definition has become more widely studied for analyzing the algebraic structure of rings through graph theory. Those two papers have led to the development of zero divisor graph theory which has become a valuable tool for understanding and studying the structure of commutative rings (See [5, 6, 7, 8, 9]). M. Behboodi and Z. Rakeei, [10], have presented the definition of " Annihilating-ideal graph of commutative ring R", is referred to $AG(R)$, its vertices are of all non-zero ideals of R, and two different ideals I_1 and I_2 connected by an edge $\{I_1, I_2\}$ if $I_1 I_2 = (0)$. For a simple connected graph G, " $V(G)$ is the set of all vertices of G, $E(G)$ is the set of all edges of G, $d(u)$ represents the degree of the vertex $u \in V(G)$, $\delta(G) = min_{v \in V(G)} d(v)$, $\Delta(G) = max_{v \in V(G)} d(v)$ and $d(v, u)$ is the value of the distance between the vertices $v, u \in V(G)$. The order of a graph G, denoted by $n(G)$ equal the cardinal number of $V(G)$, while the size of G, equal the cardinal number of $E(G)$ and denoted by $m(G)$, the eccentricity of vertex $u \in G$ is the distance between the vertex $u \in V(G)$ and a vertex furthest from u and denoted by $e(u)$, $rad(G) = min_{u \in V(G)} e(u)$, $diam(G) = max_{u \in V(G)} e(u)$, and $Cen(G) = \{u \in V(G) : e(u) = rad(G)\}$. The Gutman index of G, (for short $Gut(G)$) is defined as $Gut(G) = \sum(u_1)d(u_2)d(u_1, u_2)$, where u_1 and u_2 any vertices in G " additional details can be found in [11, 12].

The value of the Gutman index for the graph $AG(R, M, n)$ has been determined under the assumption that the ring PIR is a finite local. This paper consists of three sections: the first one represents the introduction and the second one illustrates some basic properties of annihilating ideal where R local P.I.R. such as degree and size. In addition, we have found a relation between graphs $AG(R_1)$ and $AG(R_2)$, where $R_1 = (R_1, M_1, n)$ and $R_2 = (R_2, M_2, n + 2)$. In section three, we have used the results in section two to find Gutman index of graphical rings.

2 Some properties of annihilating-ideal graph over finite local PIRs

This part of the study focuses on examining the structural properties of the annihilating-ideal graph associated with a finite local principal ideal ring This part of the study focuses on examining the structural properties of annihilating ideal graph associated with a finite local PIR ring (R, M, n), where M denotes the principal maximal ideal of R with nil-potency index $n \in Z$, and give some basic properties of this graph.

From definition, $V(AG(R, M, n)) = A(R, M, n)^* = \{M^i : 1 \leq i \leq n-1\}$, the two vertices $M^i, M^j \in A(R, M, n)^*$ incident to the edge $\{M^i, M^j\}$ if and only if $M^i M^j = (0)$, thus $M^{i+j} = (0)$ if and only if $i + j \geq n$, and this occurs if and only if $1 \leq i \leq n-1$ and $n - i \leq j \leq n-1$; therefore, $E(AG(R, M, n)) = \{\{M^i, M^j\} : i + j \geq n, 1 \leq i, j \leq n-1\} = \{\{M^i, M^j\} : 1 \leq i \leq n-1, n-i \leq j \leq n-1\}$. Clearly, the vertex M^1 is adjacent with only vertex M^{n-1}, the vertex M^2 is adjacent with only two vertices: M^{n-1} and M^{n-2}, similarly the vertex $M^{\lceil \frac{n}{2} \rceil - 1}$ adjacent with the only $(\lceil \frac{n}{2} \rceil - 1)$- vertices: $M^{\lceil \frac{n}{2} \rceil + 1}, \ldots, M^{n-2}, M^{n-1}$, thus $d(M^i) = i$ for any $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$. Also, the vertex M^{n-1} is adjacent with only $(n-2)$-vertices: $M^1, M^2, \cdots, M^{n-2}$, the vertex M^{n-2} is adjacent with only $(n-3)$-vertices: $M^2, M^3, \ldots, M^{n-3}$ and M^{n-1} , similarly $M^{\lceil \frac{n}{2} \rceil}$ is adjacent with the only $(\lceil \frac{n}{2} \rceil - 1)$-vertices: $M^{\lceil \frac{n}{2} \rceil + 1}$, $M^{\lceil \frac{n}{2} \rceil + 2}$, …, M^{n-1}, thus $d(M^i) = i - 1$ for any $\lceil \frac{n}{2} \rceil \leq i \leq n-1$. Therefore, we can give the first proposition in this paper:

Proposition 2.1. *Suppose that $M^i \in A(R, M, n)^*$ for $i \in \{1, 2, ..., n-1\}$. Then*

$$d(M^i) = \begin{cases} i & \text{if } i \in \{1, ..., \lceil \frac{n}{2} \rceil - 1\} \\ i - 1 & \text{if } i \in \{\lceil \frac{n}{2} \rceil, ..., n-1\}. \end{cases}$$

The next result establishes the fundamental properties of the graph $AG(R, M, n)$ of finite local PIRs:

Theorem 2.2. *The following statement is valid, for each integer n greater than 1:*

1) $AG(R, M, n)$ has order $n - 1$ and the size

$$\begin{cases} \frac{1}{4}(n-1)^2 & \text{:if n is odd,} \\ \frac{1}{4}n(n-2) & \text{:if n is enen.} \end{cases}$$

2) $\delta((AG(R, M, n))) = d(M^1) = 1$ and $\Delta(AG(R, M, n)) = d(M^{n-1}) = n - 2$.

3) $rad\ (AG(R, M, n)) = 1$ and $diam\,(AG(R, M, n)) = 2$.

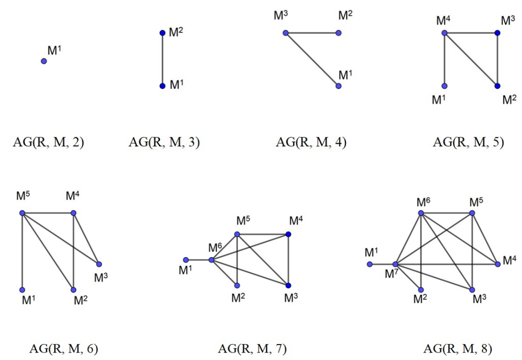
4) $Cen\,(AG(R, M, n)) = \mathrm{A}(R, M, n)^*$ for n=2,3, and $Cen\,(AG(R, M, n)) = \{M^{n-1}\}$ for any $n > 3$.

Proof. 1) obviously, $n(AG(R, M, n)) = |A(R, M, n)^*| = n-1$ and since $2m(AG(R, M, n)) = \sum_{i=1}^{n-1} d(M^i) = \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} d(M^i) + \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} d(M^i)$, then by Proposition 2.1 we have $m(AG(R, M, n)) = \frac{1}{2}(\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} i + \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} (i-1)) = \frac{1}{2}(\frac{n(n-3)}{2} + \lceil \frac{n}{2} \rceil) =$

$$\begin{cases} \frac{1}{4}(n-1)^2 & \text{:if n is odd,} \\ \frac{1}{4}n(n-2) & \text{:if n is enen.} \end{cases}$$

(2), (3) and (4)are obvious. □

Now, in the following figures, we find the annihilating-ideal graphs of finite local PIRs whose orders less than or equal 8:



Clearly, $AG(R,M,2) \cong K_1$, $AG(R,M,3) \cong K_2$, $AG(R,M,4) \cong K_{1,2}$, $AG(R,M,5) \cong K_1+(K_2\cup K_1)$, $AG(R,M,6) \cong K_1+(K_{1,2}\cup K_1)$, $AG(R,M,7) \cong K_1+[(K_1+(K_2\cup K_1))\cup K_1]$ and $AG(R,M,8) \cong K_1+[(K_1+(K_{1,2}\cup K_1))\cup K_1]$. We see that there is a relation between these graphs; this relation is showing in the following results:

Proposition 2.3. $< \{M^2, M^3, \dots, M^n\} >_{AG(R,M,n+2)} \cong AG(R,M,n)$.

Proof. Define the mapping $\phi : \{M^2, M^3, \dots, M^n\} \to A(R,M,n)^*$ by $\phi(M^k) = M^{k-1}$ for $k = 2, 3, \dots, n$. Clearly, ϕ is one-to-one and onto mapping. To verify that adjacency is preserved under the mapping ϕ; let $\{M^i, M^j\}$ is an edge in $\{M^2, M^3, \dots, M^n\}$ then $i + j \geq n + 2$ for $2 \leq i, j \leq n+1$, thus $(i-1)+(j-1) \geq n$ for$1 \leq i, j \leq n-1$, therefore $\{M^{i-1}, M^{j-1}\} = \{\phi(M^i), \phi(M^j)\}$ is an edge in $AG(R,M,n)$. Similarly, if $\{M^{i-1}, M^{j-1}\}$ is an edge in $AG(R,M,n)$ then $\{M^i, M^j\}$ is an edge in $<\{M^2, M^3, \dots, M^n\}>_{AG(R,M,n+2)}$ Hence $<\{M^2, M^3, \dots, M^n\}>_{AG(R,M,n+2)} \cong AG(R,M,n)$. □

Corollary 2.4. $AG(R,M,n+2) \cong K_1 + [AG(R,M,n) \cup K_1]$.

Proof. Let $A = \{M^2, M^3, \dots, M^n\}$, then $A(R,M,n+2)^* \cup \{M^1, M^{n+1}\}$. Clearly, the vertex M^{n+1} is adjacent with all vertices of $A \cup \{M^1\}$, and the vertex M^1 is adjacent with the only vertex M^{n+1}, hence $AG(R,M,n+2) \cong K_1 + [<\{M^2, M^3, \dots, M^n\}>_{AG(R,M,n+2)} \cup K_1]$, then by Proposition 2.3 we get the proof. □

3 Gutman Index of Annihilating-ideal Graph of Finite Local PIRs

If $n = 2, 3, 4$, then we have $AG(R,M,2) \cong K_1$, $AG(R,M,3) \cong K_2$ and $AG(R,M,4) \cong K_{1,2}$. The vertex M^{n-1} is adjacent with for all vertices of $AG(R,M,n)$, hence $diam(AG(R,M,n)) = 2$ for any $n \geq 4$. Since $AG(R,M,n)$ is a connected graph, therefore we can write

$$d(M^i, M^j) = \begin{cases} 1 & \text{:if } i+j \geq n, \\ 2 & \text{:if } i+j < n, \end{cases} \qquad \text{for any} \quad n \geq 4, \text{ and}$$

$$Gut(AG(R,M,n)) = \sum_{M^i, M^j \in A(R,M,n)^*} d(M^i)d(M^j)d(M^i,M^j) = Gut(AG(R,M,n)) =$$

$$\sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i,M^j)=1}} d(M^i)d(M^j)d(M^i,M^j) + \sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i,M^j)=2}} d(M^i)d(M^j)d(M^i,M^j)$$

$$= \sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i,M^j)=1}} d(M^i)d(M^j) + 2 \sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i,M^j)=2}} d(M^i)d(M^j)$$

Theorem 3.1.

$$\sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i,M^j)=1}} d(M^i)d(M^j)d(M^i,M^j) = \begin{cases} \frac{1}{48}(5n^4 - 28n^3 + 64n^2 - 68n + 27) & \text{:if } n \text{ is odd} \\ \frac{1}{48}(5n^4 - 28n^3 + 52n^2 - 32n) & \text{:if } n \text{ is even} \end{cases}$$

Proof. $d(M^i, M^j) = 1$ if and only if $1 \leq i \leq n-1$ and $n-i \leq j \leq n-1$, this holds if and only if either $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ and $n-i \leq j \leq n-1$ or $\lceil \frac{n}{n} \rceil \leq i \leq n-2$ and $i+1 \leq j \leq n-1$. Let,

$$X = \sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i, M^j)=1}} d(M^i)d(M^j)d(M^i, M^j) = \sum_{i=1}^{n-1} \sum_{j=n-i}^{n-1} d(M^i)d(M^j)$$

then we have $X = \sum_{i=1}^{\lceil \frac{n}{2} \rceil -1} \sum_{j=n-i}^{n-1} d(M^i)d(M^j) + \sum_{i=\lceil \frac{n}{2} \rceil}^{n-2} \sum_{j=i+1}^{n-1} d(M^i)d(M^j)$. If $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ and $n-i \leq j \leq n-1$ then we see that $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ and $\lceil \frac{n}{2} \rceil \leq j \leq n-1$ in this case. Let $X_1 = \sum_{i=1}^{\lceil \frac{n}{2} \rceil -1} \sum_{j=n-i}^{n-1} d(M^i)d(M^j)$ then by Proposition 2.1 we get:
$X_1 = \sum_{i=1}^{\lceil \frac{n}{2} \rceil -1} \sum_{j=n-i}^{n-1} i(j-1) = \sum_{i=1}^{\lceil \frac{n}{2} \rceil -1} i(\sum_{j=n-i}^{n-1} j - \sum_{j=n-i}^{n-1} 1) = \sum_{i=1}^{\lceil \frac{n}{2} \rceil -1} i(\frac{n(n-1)}{2} - \frac{(n-i)(n-i-1)}{2} - (n-1) + (n-i-1))$
$= \frac{1}{2} \sum_{i=1}^{\lceil \frac{n}{2} \rceil -1} i((2n-3)i - i^2) = \frac{1}{2} \sum_{i=1}^{\lceil \frac{n}{2} \rceil -1} (2n-3)i^2 - \frac{1}{2} \sum_{i=1}^{\lceil \frac{n}{2} \rceil -1} i^3 = \frac{1}{24}[2(2n-3)(\lceil \frac{n}{2} \rceil - 1)(\lceil \frac{n}{2} \rceil)(2(\lceil \frac{n}{2} \rceil) - 1) + 1 - 3(\lceil \frac{n}{2} \rceil)^2(\lceil \frac{n}{2} \rceil - 1)^2]$. If n is odd, then we have:

$X_1 = \frac{1}{24}[2(2n-3)(\frac{n+1}{2} - 1)(\frac{n+1}{2})(2((\frac{n+1}{2} - 1) + 1) - 3[[\frac{((n+1)}{2})]^2(\frac{n+1}{2} - 1)]^2]$
$= \frac{1}{48}(2n-3)(n-1)(n+1)n - \frac{1}{128}(n+1)^2(n-1)^2$

$$\therefore X_1 = \frac{1}{384}(13n^4 - 24n^3 - 10n^2 + 24n - 3). \tag{3.1}$$

If n is even, then we have:
$X_1 = \frac{1}{24}[2(2n-3)(\frac{n}{2} - 1)(\frac{n}{2})(2((\frac{n}{2} - 1) + 1) - 3[[\frac{(n}{2})]^2(\frac{n}{2} - 1)]^2]$
$= \frac{1}{48}(2n-3)(n-1)(n-2)n - \frac{1}{128}n^2(n-2)^2$

$$\therefore X_1 = \frac{1}{384}(13n^4 - 60n^3 + 92n^2 - 48n). \tag{3.2}$$

Now, when $\lceil \frac{n}{2} \rceil \leq i \leq n-2$ and $i+1 \leq j \leq n-1$, consequently, we obtain $\lceil \frac{n}{2} \rceil \leq i, j \leq n-1$ in this case. Let $X_2 = \sum_{i=\lceil \frac{n}{2} \rceil}^{n-2} \sum_{j=i+1}^{n-1} d(M^i)d(M^j)$. Therefore, we conclude that $\lceil \frac{n}{2} \rceil \leq i, j \leq n-1$ in this case, and by Proposition 2.1 we get:
$X_2 = \sum_{i=\lceil \frac{n}{2} \rceil}^{n-2} \sum_{j=i+1}^{n-1} (i-1)(j-1) = \sum_{i=\lceil \frac{n}{2} \rceil}^{n-2} (i-1)(\sum_{j=i+1}^{n-1} j - \sum_{j=i+1}^{n-1} 1)$

$$= \frac{1}{2} \sum_{i=\lceil \frac{n}{2} \rceil}^{n-2} (i-1)(n(n-1) - i(i+1) - 2(n-1-i)) = \frac{1}{2} \sum_{i=\lceil \frac{n}{2} \rceil}^{n-2} (i-1)((n^2 - 3n + 2) - i^2 + i)$$

$= \frac{1}{2} \sum_{i=\lceil \frac{n}{2} \rceil}^{n-2} [(n^2 - 3n + 1)i + 2i^2 - i^3 - (n^2 - 3n + 2)] = \frac{1}{24}[-12(n^2 - 3n + 2)(n - \lceil \frac{n}{2} \rceil - 1) + (n-2)(n-1)(3n^2 - n - 12) - (\lceil \frac{n}{2} \rceil)(\lceil \frac{n}{2} \rceil - 1)(6n^2 - 18n - 3(\lceil \frac{n}{2} \rceil)^2 + 11(\lceil \frac{n}{2} \rceil) + 2)]$. If n is odd, then we have:
$X_2 = \frac{1}{24}[-12(n^2 - 3n + 2)(n - \frac{(n+1)}{2} - 1) + (n-2)(n-1)(3n^2 - n - 12) - (\frac{n+1}{2})(\frac{n+1}{2} - 1)(6n^2 - 18n - 3(\frac{n+1}{2})^2 + 11\frac{n+1}{2} + 2)]$

$$= \frac{-1}{4}(n^2-3n+2)(n-3) + \frac{1}{24}(n-2)(n-1)(3n^2-n-12) - \frac{1}{384}(n-2)((n-1)(21n^2-56n+27)$$

$$\therefore X_2 = \frac{1}{384}(27n^4 - 200n^3 + 522n^2 - 568n + 219) \tag{3.3}$$

Similarly, if n is even, then we have:

$$X_2 = \frac{1}{384}(27n^4 - 164n^3 + 324n^2 - 208n) \tag{3.4}$$

Now, if n is odd, then by 3.1 and 3.3 we get:
$X = X_1 + X_2 = \frac{1}{384}(13n^4 - 24n^3 - 10n^2 + 24n - 3) + \frac{1}{384}(27n^4 - 200n^3 + 522n^2 - 568n + 219)$

$$= \frac{1}{48}(5n^4 - 28n^3 + 64n^2 - 68n + 27).$$

Also, if n is even, then by 3.2 and 3.4 we get:
$X = X_1 + X_2 = \frac{1}{384}(13n^4 - 60n^3 + 92n^2 - 48n) + \frac{1}{384}(27n^4 - 164n^3 + 324n^2 - 208n)$

$$= \frac{1}{48}(5n^4 - 28n^3 + 52n^2 - 32n).$$

□

Theorem 3.2.

$$\sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i,M^j)=2}} d(M^i)d(M^j)d(M^i,M^j) = \begin{cases} \frac{1}{48}(2n^4 - 8n^3 + 4n^2 + 8n - 6) & \text{:if } n \text{ is odd} \\ \frac{1}{48}(2n^4 - 8n^3 + 4n^2 + 8n) & \text{:if } n \text{ is even} \end{cases}$$

Proof. $d(M^i, M^j) = 2$ if and only if $i+j \leq n$, and this holds if and only if either $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ and $\lceil \frac{n}{2} \rceil \leq j \leq n - i - 1$ or $1 \leq i \leq \lceil \frac{n}{2} \rceil - 2$ and $i + 1 \leq j \leq \lceil \frac{n}{2} \rceil - 1$.
Let $Y = \sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i,M^j)=2}} d(M^i)d(M^j)d(M^i,M^j) = 2\sum_{\substack{M^i, M^j \in A(R,M,n)^* \\ d(M^i,M^j)=2}} d(M^i)d(M^j)$, then we have:

$$Y = 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} \sum_{j=\lceil \frac{n}{2} \rceil}^{n-i-1} d(M^i)d(M^j) + 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} \sum_{j=i+1}^{\lceil \frac{n}{2} \rceil - 1} d(M^i)d(M^j).$$

Let $Y_1 = 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} \sum_{j=\lceil \frac{n}{2} \rceil}^{n-i-1} i(j-1) = 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} i(\sum_{j=\lceil \frac{n}{2} \rceil}^{n-i-1} j - \sum_{j=\lceil \frac{n}{2} \rceil}^{n-i-1} 1)$
$= 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} i[(\frac{(n-i-1)(n-i)}{2} - \frac{\lceil \frac{n}{2} \rceil(\lceil \frac{n}{2} \rceil - 1)}{2}) - (n-i-1) + \lceil \frac{n}{2} \rceil - 1]$
$= \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} i[n^2 - 3n + 2\lceil \frac{n}{2} \rceil - \lceil \frac{n}{2} \rceil(\lceil \frac{n}{2} \rceil - 1) + (3-2n)i + i^2]$
$= (n^2 - 3n + 2\lceil \frac{n}{2} \rceil - \lceil \frac{n}{2} \rceil(\lceil \frac{n}{2} \rceil - 1))\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} i - (3-2n)\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} i^2 + \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} i^3$
$= \frac{1}{2}(n^2 - 3n + 2\lceil \frac{n}{2} \rceil - \lceil \frac{n}{2} \rceil(\lceil \frac{n}{2} \rceil - 1))(\lceil \frac{n}{2} \rceil - (\lceil \frac{n}{2} \rceil(\lceil \frac{n}{2} \rceil - 1)) - \frac{1}{6}(2n-3)\lceil \frac{n}{2} \rceil(\lceil \frac{n}{2} \rceil - 1)(2\lceil \frac{n}{2} \rceil - 1) + \frac{1}{4}\lceil \frac{n}{2} \rceil^2(\lceil \frac{n}{2} \rceil - 1)^2$.
If n is odd, then we have:
$Y_1 = \frac{1}{2}(n^2 - 3n + 2\frac{n+1}{2} - \frac{n+1}{2}(\frac{n+1}{2} - 1))(\frac{n+1}{2}(\frac{n+1}{2} - 1)) - \frac{1}{6}(2n-3)\frac{n+1}{2}(\frac{n+1}{2} - 1)(2\frac{n+1}{2} - 1) + \frac{1}{4}(\frac{n+1}{2})^2(\frac{n+1}{2} - 1)^2$

$$\therefore Y_1 = \frac{1}{192}(5n^4 - 24n^3 + 22n^2 + 24n - 27) \tag{3.5}$$

Similarly, if n is even, then we have:

$$Y_1 = \frac{1}{192}(5n^4 - 12n^3 - 20n^2 + 48n) \tag{3.6}$$

Now, if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 2$ and $i + 1 \leq j \leq \lceil \frac{n}{2} \rceil - 1$. Let $Y_2 = 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} \sum_{j=i+1}^{\lceil \frac{n}{2} \rceil - 1} d(M^i)d(M^j)$ we see that $1 \leq i, j \leq \lceil \frac{n}{2} \rceil - 1$ in this case, then by Proposition 2.1 we get:
$Y_2 = 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} \sum_{j=i+1}^{\lceil \frac{n}{2} \rceil - 1} ij = 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} i(\sum_{j=1}^{\lceil \frac{n}{2} \rceil - 1} j - \sum_{j=1}^{i} j) = 2\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} i(\frac{(\lceil \frac{n}{2} \rceil - 1)\lceil \frac{n}{2} \rceil}{2} - \frac{i(i+1)}{2})$
$= \frac{(\lceil \frac{n}{2} \rceil - 1)\lceil \frac{n}{2} \rceil}{2}\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} i - \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} i^3 - \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} i^2$
$= \frac{1}{2}\lceil \frac{n}{2} \rceil(\lceil \frac{n}{2} \rceil - 1)^2(\lceil \frac{n}{2} \rceil - 2) - \frac{1}{4}(\lceil \frac{n}{2} \rceil - 1)^2(\lceil \frac{n}{2} \rceil - 2)^2 - \frac{1}{6}(\lceil \frac{n}{2} \rceil - 1)(\lceil \frac{n}{2} \rceil - 2)(2\lceil \frac{n}{2} \rceil - 3)$.
If n is odd, then we have:
$Y_2 = \frac{1}{2}(\frac{n+1}{2})(\frac{n-1}{2})^2(\frac{n-3}{2}) - \frac{1}{4}(\frac{n-1}{2})^2(\frac{n-3}{2})^2 - \frac{1}{6}(\frac{n-1}{2})(\frac{n-3}{2})(n-2)$

$$\therefore Y_2 = \frac{1}{192}(3n^4 - 8n^3 - 6n^2 + 8n + 3) \tag{3.7}$$

Similarly, if n is even, then we have:

$$Y_2 = \frac{1}{192}(3n^4 - 20n^3 + 36n^2 - 16n) \tag{3.8}$$

Now, if n is odd, then by 3.5 and 3.7 we get:
$Y = Y_1 + Y_2 = \frac{1}{192}(5n^4 - 24n^3 + 22n^2 + 24n - 27) + \frac{1}{192}(3n^4 - 8n^3 - 6n^2 + 8n + 3) = \frac{1}{48}(2n^4 - 8n^3 + 4n^2 + 8n - 6)$.
Also, if n is even, then by 3.6 and 3.8 we get:
$Y = Y_1 + Y_2 = \frac{1}{192}(5n^4 - 12n^3 - 20n^2 + 48n) + \frac{1}{192}(3n^4 - 20n^3 + 36n^2 - 16n) = \frac{1}{48}(2n^4 - 8n^3 + 4n^2 + 8n)$. □

Finally, by Theorems 3.1 and 3.2, we can give the main result of this paper:

Corollary 3.3.

$$Gut(AG(R,M,n)) = \begin{cases} \frac{1}{48}(7n^4 - 36n^3 + 68n^2 - 60n + 21) & \text{: if } n \text{ is odd,} \\ \frac{1}{48}(7n^4 - 36n^3 + 56n^2 - 24n) & \text{: if } n \text{ is even.} \end{cases}$$

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