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NUMERICAL SOLUTION FOR SOLVING INITIAL VALUE PROBLEMS BY USING MODIFIED EULER METHOD

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Abstract *Euler's method (EM) is one of the most fundamental numerical techniques for solving initial value problems (IVPs). Over time, various improvements to Euler-based methods have attracted significant interest. This study proposes a simplified and improved approximation technique for computing numerical solutions to IVPs. Specifically, it enhances the Modified Euler Method (MEM) by introducing a new approach that reduces computational effort while maintaining accuracy. The efficiency of the method proposed in this paper is demonstrated through examples and the graphical results were generated using MATLAB.*

1 Introduction

In science and engineering, differential equations are used to model problems that arise in various fields such as physics, biology, and economics. The majority of these challenges necessarily involve the process in finding the solution of initial value problems (IVPs), which are differential equations with given initial conditions (I.C.) [1]. Numerical methods are essential for solving IVPs, especially when analytical methods are unable to obtain the solution [2, 3, 4].

For solving IVPs numerically, Al-Shimmary et al. proposed an improved sixth-order Runge-Kutta method, which enhances accuracy and stability for complex systems [5]. Hussain et al. used the Jacobi–Gauss–Radau collocation method to solve fractional IVPs, demonstrating its effectiveness in handling non-integer order differential equations [6]. Ochoche achieved significantly better results by improving the Modified Euler Method (MEM), particularly for autonomous systems [7]. The basic properties of the new method introduced in [8] were compared to those of older methods by the Ochoche [9], highlighting the advantages of refinement in numerical schemes.

Because the Euler Method (EM) was simple and computationally efficient, Yusop modified it to improve its performance in solving ordinary differential equations [10]. In his paper, Yusop discussed the major aims of presenting a new algorithm for implementing the MEM and comparing it with another modified Euler method and an exact solution obtained through integration. Later, another researcher developed a new third-order Euler method in 2010, which offered improved convergence and reliability for solving IVPs [11]. His new study scheme indicated that the method is reliable, accurate, and convergent.

Yusop et al. formulated an innovative modified Euler scheme and they have successfully enhanced the efficiency of the Polygon scheme and make it more accurate across diverse step sizes [12]. The Harmonic-Polygon scheme represents a novel approach that integrates the Polygon scheme with the principles of the Harmonic mean, offering better performance in terms of error reduction. Ali and Hasen enhanced the computational resolution of initial value time-lag differential equations, where they employied a higher order Euler methodology, which is particularly *useful for delay differential equations [13].*

The numerical solutions of ordinary differential equations (ODEs) with IVPs using EM, MEM, and Runge-Kutta methods (RKM) have been discussed by Kamruzzaman and Nath, providing comparative insights into their accuracy and efficiency [14]. Hossen et al. compared the numerical results of IVPs using MEM and RKM, showing the strengths and limitations of each method in practical applications [15]. Nurujjaman proposed enhancements to Euler's method for the resolution of first-order ODEs, focusing on improving accuracy through algorithmic adjustments [16].

Cube Polygon has been proposed by Nooraida et al. as a MEM variant for improving accuracy and computational complexity, especially in cases involving geometric interpretations [17]. Ahmad and Charan used the Improved Euler Method (IEM), MEM, and RKM to solve IVPs, contributing to comparative studies in numerical analysis [18]. Sampoornam published an article on numerical exact Euler solutions, improved Euler solutions, as well as the RKM, offering a broad overview of classical methods [19]. Revathi mentioned a number of numerical methods for solving ODEs and their applications, emphasizing the importance of choosing appropriate methods based on problem characteristics [20]. There have been several works on numerical solutions of IVPs using IEM, EM, and RKM, according to the literature review.

The following is the structure of this paper: The enhanced variant of the Modified Euler Method is delineated in Section 2; computational illustrations are elaborated upon in Section 3. Ultimately, Section 4 culminates with an analysis and final reflections.

2 Improving Modified Euler Method (IMEM)

In this section, let us consider the IVP given as follows.

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0 \tag{2.1}$$

This section contains the paper's main idea, as the equation below

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0, 1, \dots \tag{2.2}$$

It is known that Equation (2.2) represents (EM) for approximating the solution to Equation (2.1), see [17]. And

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_n + h, y_{n+1}) \tag{2.3}$$

Equation (2.3) represents the (MEM) for solving (2.1), see [18].

It is clear from the previous methods that MEM performs better than EM. In this paper, a new approach is presented that is simpler than MEM, requires less computation, and yields the same results. Under the following conditions:

Case 1. *If $f(x, y) = g(x)$, then we apply:*

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_n)] \tag{2.4}$$

Case2. *If $f(x, y) = k(y)$, then we apply:*

$$y_{n+1} = y_n + \frac{h}{2}[(1 + h)f(x_n, y_n) + f(x_{n+1}, y_n)] \tag{2.5}$$

Case 3. *If $f(x, y) = h(x, y)$, then we apply:*

$$y_{n+1} = y_n + \frac{h}{2}[(1 + h)f(x_n, y_n) + f(x_{n+1}, y_n)] \tag{2.6}$$

3 Numerical Examples

This section is devoted to presenting some numerical examples using the proposed method and comparing the results with the developed modified Euler method.

Example 3.1. Consider $y' = x$; $y(0) = 0$; $h = 0.1$, $0 \leq x \leq 1$. Where the exact solution provided by $y = \dfrac{x^2}{2}$.

Table 1. Solution of Example 1.

x	Solution using modified Euler's (MEM)	Solution using proposed approach	Exact solution	Error
0.0	0.00000	0.00000	0.00000	0.00000
0.1	0.00005	0.00500	0.00005	0.00000
0.2	0.02000	0.02000	0.02000	0.00000
0.3	0.04500	0.04500	0.04500	0.00000
0.4	0.08000	0.08000	0.08000	0.00000
0.5	0.12500	0.12500	0.12500	0.00000
0.6	0.18000	0.18000	0.18000	0.00000
0.7	0.24500	0.24500	0.24500	0.00000
0.8	0.32000	0.32000	0.32000	0.00000
0.9	0.40500	0.40500	0.40500	0.00000
1.0	0.50000	0.50000	0.50000	0.00000

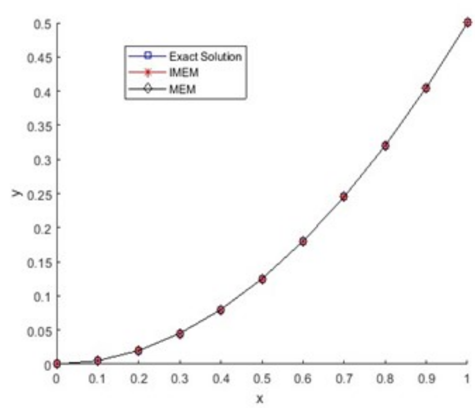

Figure 1. Solutions of Example 1 using Exact, IMEM and MEM methods

The results in Table 1 and Figure 1 show that both the MEM and the IMEM produced identical numerical values for the problem $y' = x$, with initial condition $y(0) = 0$. The exact solution $y = \dfrac{x^2}{2}$ is matched perfectly across all step sizes, and the absolute error is consistently zero. This confirms that IMEM performs equivalently to MEM in simple linear cases, while offering a computationally simpler formulation.

Example 3.2. In this example, consider $y' = y$; $y(0) = 1$; $h = 0.1$, $0 \leq x \leq 1$. Where the exact solution is $y = e^x$.

Table 2. Solution of Example 2.

x	Solution using modified Euler's (MEM)	Solution using proposed approach	Exact solution	Error
0.0	1.00000	1.00000	1.00000	0.00000
0.1	1.10500	1.10500	1.10517	0.00017
0.2	1.22103	1.22103	1.22140	0.00038
0.3	1.34923	1.34923	1.34986	0.00063
0.4	1.49092	1.49092	1.49182	0.00090
0.5	1.64745	1.64745	1.64872	0.00127
0.6	1.82043	1.82043	1.82212	0.00169
0.7	2.01157	2.01157	2.01375	0.00218
0.8	2.22279	2.22279	2.22554	0.00275
0.9	2.45618	2.45618	2.45960	0.00342
1.0	2.71408	2.71408	2.71828	0.00420

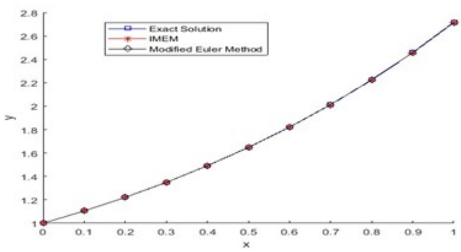

Figure 2. Solutions of Example 2 using Exact IMEM and MEM methods

Table 2 and Figure 2 present the results for the exponential growth problem $y' = y$*, with initial condition* $y(0) = 1$*. Both MEM and IMEM yield identical approximations, and the absolute error compared to the exact solution* $y = e^x$ *remains very small, with a maximum error of approximately* 0.0042 *at* $x = 1.0$*. This demonstrates that IMEM maintains high accuracy even for nonlinear problems, and its performance is on par with MEM throughout the interval.*

Example 3.3. Consider $y' = x + y$; $y(0) = 1$; $h = 0.1$, $0 \leq x \leq 1$ where the exact solution is $y = 2e^x - x - 1$.

Table 3. Solution of Example 3.

x	Solution using modified Euler's (MEM)	Solution using proposed approach	Exact solution	Error
0.0	1.00000	1.00000	1.00000	0.00000
0.1	1.11000	1.11000	1.11034	0.00034
0.2	1.24205	1.24205	1.24281	0.00076
0.3	1.39847	1.39847	1.39972	0.00125
0.4	1.58180	1.58180	1.58365	0.00185
0.5	1.79489	1.79489	1.79744	0.00255
0.6	2.04086	2.04086	2.04424	0.00338
0.7	2.32315	2.32315	2.32751	0.00436
0.8	2.64558	2.64558	2.65108	0.00550
0.9	3.01236	3.01236	3.01921	0.00684
1.0	3.42816	3.42816	3.43656	0.00840

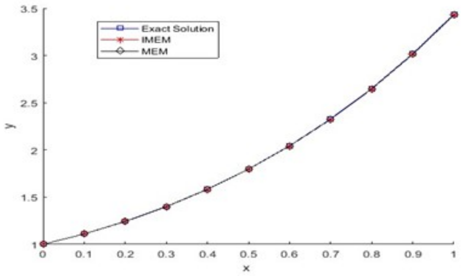

Figure 3. Solutions of Example 3 using Exact, (IMEM) and (MEM) solutions

In Table 3 and Figure 3, we presented the solution for $y' = x + y$ *with initial condition* $y(0) = 1$*. The exact solution* $y = 2e^x - x - 1$ *is closely approximated by both MEM and IMEM. The absolute error increases slightly with each step but remains below* 0.009 *at* $x = 1.0$*, indicating strong accuracy. Once again, IMEM matches MEM in all computed values, reinforcing its reliability and confirming that it can handle more complex functions involving both variables.*

4 Discussion and Conclusions

This research proposes the IMEM as a less complicated and more efficient method than the conventional MEM for numerically solving initial-value problems in ordinary differential equations. IMEM, through three numerical examples, showed the same results as MEM, with absolute errors of a negligible magnitude when compared to exact solutions.

The method is highly flexible in terms of the different forms of $f(x, y)$*, which can be dependent on* x*,* y*, or both. This facilitates a decrease in computational complexity while still retaining the same level of accuracy. Consequently, IMEM is a great choice for use in the teaching process, embedded systems, and real-time simulations.*

Perhaps, subsequent investigations could consider the adaptation of IMEM to differential equation systems, stiff problems, and fractional-order models. Moreover, the incorporation of IMEM into numerical solvers and software packages can significantly increase its usability and influence in the fields of applied mathematics and engineering.

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