

KANNAN TYPE POLYNOMIAL CONTRACTIONS AND SOME FIXED POINT THEOREMS

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Abstract The concept of polynomial type contractions was introduced by Jleli et al. [15] in 2024. Two novel classes of polynomial type single-valued mappings defined on a metric space are introduced here. Two fixed point theorems are established, for the first class, which is referred to as the class of Kannan type polynomial mapping. Specifically, we start by taking into account the scenario in which the mapping is continuous. Next, we make the continuity criterion weaker. In particular, we recover Kannan's fixed point theorem. Motivated by Berinde [4], next we have introduced a class of almost Kannan type polynomial mapping. We prove a fixed point theorem for almost Kannan type polynomial mapping. Suitable examples for established results are also given.

1 Introduction

Jleli et al. [15] recently proposed the idea of polynomial contractions for self-mappings on a complete metric space (\mathfrak{S}, d) which resulted in new fixed point theorems that greatly expand the Banach contraction principle range of applications. A polynomial contraction is a kind of mapping that demonstrates a particular kind of distance-reducing behaviour in the context of mathematical analysis, especially in metric spaces. Instead of a straightforward multiplicative factor, like in conventional contraction mappings, it is distinguished by a polynomial connection between the distances between points before and after the mapping is performed. Polynomial contractions allow for a polynomial decay in distances. They offer a more comprehensive framework for examining specific kinds of iterative procedures, dynamical systems and modelling some situations where the distance reduction is not consistent. Recently, Moumen et al. [18] introduced polynomial ϕ -contractions in metric space and presented its applications to fractional logistic growth equations. Aldwoah et al. [1] extended the notion of polynomial contractions in b-metric spaces with applications to boundary value problems of diffusion phenomena and integral type contractions. Some related studies are [19], [23], [2].

The objective of this section is to introduce the basic concepts and definitions used throughout this research article which will further help to understand the next section. In 1968, Kannan [16] introduced a contraction map that admits a unique fixed point while having discontinuity in its domain in a complete metric space.

Theorem 1.1. ([16]) *Let (\mathfrak{S}, d) be a complete metric space and let $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a mapping satisfying:*

$$d(Sx, Sy) \leq \alpha[d(x, Sx) + d(y, Sy)] \quad (1.1)$$

for all $x, y \in \mathfrak{S}$, where $\alpha \in [0, \frac{1}{2})$. Then S has a unique fixed point in \mathfrak{S} .

The mapping satisfying condition (1.1) is referred as Kannan type mapping. In contrast to the Banach contraction principle, which necessitates continuity, Kannan's fixed point theorem

is important for handling discontinuous mappings. It provides conditions for the existence of a unique fixed point for a class of mappings that may be discontinuous, specifically those satisfying condition (1.1). This is important because many real-world phenomena are modelled by discontinuous functions, and Kannan’s theorem provides a tool to analyze these situations. Kannan’s fixed point theorem is crucial because Subrahmanyam [22] proved that a metric space is complete if and only if every Kannan mapping has a fixed point. Recently, Lu et al. [17] introduced the best area of Kannan system with degree s in b-metric spaces with constant s . Berinde and Pacurar [8] introduced the notion of enriched Kannan mappings. Further, Debnath [12] generalized Kannan’s fixed point theorem and used it to solve a particular type of integral equation. Several authors have proposed various contraction type mappings and fixed point theorems that do not require the continuity of S , see Rus [21], Taskovic [24], Chatterjea [10] and references there in.

Example 1.2. Consider the initial value problem:

$$\frac{dy}{dt} = f(y) \text{ with } y(0) = y_0,$$

where the function $f(y)$ is defined as:

$$f(y) = \begin{cases} 1 & \text{if } y \leq 0, \\ \frac{1}{2} & \text{if } y > 0. \end{cases}$$

This function $f(y)$ is clearly discontinuous at $y = 0$.

Now, consider the operator S defined by the integral equation equivalent to this differential equation:

$$S(y(t)) = y_0 + \int_0^t f(y(x))dx$$

To show that S is a Kannan mapping, a constant $\alpha \in [0, \frac{1}{2})$ is required, so that

$$d(Sy_1, Sy_2) \leq \alpha[d(y_1, Sy_1) + d(y_2, Sy_2)]$$

where $d(y_1, y_2) = \sup_{t \in [0, \tau]} |y_1(t) - y_2(t)|$, for some interval $[0, \tau]$. Consider the following cases:

- If $y_1(t) \leq 0$ and $y_2(t) \leq 0$, then $f(y_1(t)) = f(y_2(t)) = 1$, making $d(Sy_1, Sy_2) = 0$.
- If $y_1(t) > 0$ and $y_2(t) > 0$, then $f(y_1(t)) = f(y_2(t)) = \frac{1}{2}$, making $d(Sy_1, Sy_2) = 0$.
- If one value is positive and the other is negative, the situation changes. Assuming a time to where $y_1(t_0) \leq 0$ and $y_2(t_0) > 0$, then $f(y_1(t)) = 1$ and $f(y_2(t)) = \frac{1}{2}$. The difference $d(Sy_1, Sy_2)$ relates to the integral of the difference in $f(y)$. Since $|f(y_1) - f(y_2)| \leq \frac{1}{2}$ (the maximum difference in function value), it's possible to find a suitable α .

This example shows a situation where a differential equation with a discontinuous right-hand side, representing abrupt changes or switching behaviours, has a solution found using the fixed-point iteration of a Kannan mapping. This highlights the usefulness of Kannan’s theorem in cases where the standard Banach contraction principle, which requires continuity, may not directly apply.

Berinde [4] presented an important class of mappings, known as class of weak contractions (or almost contractions) as a generalization of contraction map and established a fixed point theorem that does not require continuity.

Definition 1.3. ([4]) Let (\mathfrak{S}, d) be a metric space. A map $S : \mathfrak{S} \rightarrow \mathfrak{S}$ is called an almost contraction, if there exists $\delta \in [0, 1)$ and $\mathfrak{L} \geq 0$ such that

$$d(Sx, Sy) \leq \delta d(x, y) + \mathfrak{L}d(y, Sx). \tag{1.2}$$

for every $x, y \in \mathfrak{S}$.

- Remark 1.4.** (i) Every strict contraction is an almost contraction.
 (ii) Any Kannan mapping is an almost contraction.
 (iii) Almost contractions can have more than one fixed point(see Example 1[4]).

Berinde[4] established the following fixed point theorem for the above class of mappings.

Theorem 1.5. ([4]) *Let (\mathfrak{S}, d) be a complete metric space and let $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be an almost contraction. Then*

- (i) *S admits at least one fixed point;*
- (ii) *For all $z_0 \in \mathfrak{S}$, the sequence $\{z_n\}$ defined by $z_{n+1} = Sz_n$, converges to a fixed point of S.*

Jleli et al.[15] introduced the class of polynomial contractions and almost polynomial contractions and established following fixed point theorems as a generalization of results obtained by Berinde[4].

Definition 1.6. ([15]) *Let (\mathfrak{S}, d) be a metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a given mapping. We say that S is a polynomial contraction, if there exists $\delta \in [0, 1)$, a natural number $k \geq 1$ and a family of mappings $a_i : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, \infty)$, $i = 0, 1, 2, \dots, k$ such that*

$$\sum_{i=0}^k a_i(Sx, Sy)d^i(Sx, Sy) \leq \delta \sum_{i=0}^k a_i(x, y)d^i(x, y) \tag{1.3}$$

for every $x, y \in \mathfrak{S}$.

Theorem 1.7. ([15]) *Let (\mathfrak{S}, d) be a complete metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a polynomial contraction. Assume that the following conditions hold:*

- (i) *S is continuous;*
- (ii) *There exist $j \in \{1, 2, \dots, k\}$ and $\tau_j > 0$ such that $a_j(x, y) \geq \tau_j$, $x, y \in \mathfrak{S}$.*

Then S admits a unique fixed point $z' \in \mathfrak{S}$. Moreover, for every $z_0 \in \mathfrak{S}$, the Picard sequence $\{z_n\} \subset \mathfrak{S}$ defined by $z_{n+1} = Sz_n$ for all $n \geq 0$ converges to z' .

Definition 1.8. ([15]) *Let (\mathfrak{S}, d) be a metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a given mapping. S is known as almost polynomial contraction, if there exists a finite sequence $\{\mathfrak{L}_i\}_{i=0}^k \subset (0, \infty)$, $\delta \in [0, 1)$, a natural number $k \geq 1$ and a family of mappings $a_i : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, \infty)$, $i = 0, 1, 2, 3, \dots, k$ such that*

$$\sum_{i=0}^k a_i(Sx, Sy)d^i(Sx, Sy) \leq \delta \left[\sum_{i=0}^k a_i(x, y) [d^i(x, y) + \mathfrak{L}_i d^i(y, Sx)] \right] \tag{1.4}$$

for every $x, y \in \mathfrak{S}$.

Theorem 1.9. ([15]) *Let (\mathfrak{S}, d) be a complete metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be an almost polynomial contraction. Assume that the following conditions hold:*

- (i) *S is Picard continuous;*
- (ii) *There exists $j \in \{1, 2, \dots, k\}$ and $\tau_j > 0$ such that $a_j(x, y) \geq \tau_j$, $x, y \in \mathfrak{S}$.*

Then S is a weakly Picard operator.

2 Main Results

In this section we introduce the class of Kannan type polynomial mapping and establish two fixed point theorems for this class. We first consider the case when S is continuous. Next, we make the continuity criterion weaker. In particular, we recover Kannan’s fixed point theorem.

Definition 2.1. Let (\mathfrak{S}, d) be a metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a given mapping. We say that S is a Kannan type polynomial mapping, if there exists $\delta \in [0, \frac{1}{2})$, a natural number $k \geq 1$ and a family of mappings $a_i : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, \infty)$, $i = 0, 1, 2, \dots, k$ such that

$$\sum_{i=0}^k a_i(Sx, Sy)d^i(Sx, Sy) \leq \delta \left[\sum_{i=0}^k a_i(x, Sx)d^i(x, Sx) + \sum_{i=0}^k a_i(y, Sy)d^i(y, Sy) \right] \tag{2.1}$$

for every $x, y \in \mathfrak{S}$.

The study of fixed points for the above class of mappings is our focus. We first consider the case when S is a continuous mapping.

Theorem 2.2. Let (\mathfrak{S}, d) be a complete metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a Kannan type polynomial mapping. Assume that the following conditions hold:

- (i) S is continuous;
- (ii) There exist $j \in \{1, 2, \dots, k\}$ and $\tau_j > 0$ such that $a_j(x, y) \geq \tau_j$, $x, y \in \mathfrak{S}$.

Then S admits a unique fixed point $z' \in \mathfrak{S}$. Moreover, for every $z_0 \in \mathfrak{S}$, the Picard sequence $\{z_n\} \subset \mathfrak{S}$ defined by $z_{n+1} = Sz_n$ for all $n \geq 0$ converges to z' .

Proof. Let $z_0 \in \mathfrak{S}$ be any fixed element and $\{z_n\}$ be the Picard sequence in \mathfrak{S} defined by $z_{n+1} = Sz_n$, $n \geq 0$.

Using (2.1) with $(x, y) = (z_0, z_1)$, we get

$$\sum_{i=0}^k a_i(Sz_0, Sz_1)d^i(Sz_0, Sz_1) \leq \delta \left[\sum_{i=0}^k a_i(z_0, Sz_0)d^i(z_0, Sz_0) + \sum_{i=0}^k a_i(z_1, Sz_1)d^i(z_1, Sz_1) \right],$$

that is,

$$\begin{aligned} \sum_{i=0}^k a_i(z_1, z_2)d^i(z_1, z_2) &\leq \delta \left[\sum_{i=0}^k a_i(z_0, z_1)d^i(z_0, z_1) + \sum_{i=0}^k a_i(z_1, z_2)d^i(z_1, z_2) \right], \\ \Rightarrow (1 - \delta) \sum_{i=0}^k a_i(z_1, z_2)d^i(z_1, z_2) &\leq \delta \sum_{i=0}^k a_i(z_0, z_1)d^i(z_0, z_1), \end{aligned}$$

that is,

$$\sum_{i=0}^k a_i(z_1, z_2)d^i(z_1, z_2) \leq \frac{\delta}{1 - \delta} \sum_{i=0}^k a_i(z_0, z_1)d^i(z_0, z_1). \tag{2.2}$$

Using (2.1) with $(x, y) = (z_1, z_2)$, we get

$$\sum_{i=0}^k a_i(Sz_1, Sz_2)d^i(Sz_1, Sz_2) \leq \delta \left[\sum_{i=0}^k a_i(z_1, Sz_1)d^i(z_1, Sz_1) + \sum_{i=0}^k a_i(z_2, Sz_2)d^i(z_2, Sz_2) \right],$$

that is,

$$\begin{aligned} \sum_{i=0}^k a_i(z_2, z_3)d^i(z_2, z_3) &\leq \delta \left[\sum_{i=0}^k a_i(z_1, z_2)d^i(z_1, z_2) + \sum_{i=0}^k a_i(z_2, z_3)d^i(z_2, z_3) \right], \\ \Rightarrow \sum_{i=0}^k a_i(z_2, z_3)d^i(z_2, z_3) &\leq \frac{\delta}{1 - \delta} \sum_{i=0}^k a_i(z_1, z_2)d^i(z_1, z_2), \end{aligned}$$

Using (2.2), we get

$$\sum_{i=0}^k a_i(z_2, z_3)d^i(z_2, z_3) \leq \left(\frac{\delta}{1 - \delta}\right)^2 \sum_{i=0}^k a_i(z_0, z_1)d^i(z_0, z_1).$$

Keeping the same approach, we learn by induction that

$$\sum_{i=0}^k a_i(z_n, z_{n+1})d^i(z_n, z_{n+1}) \leq \left(\frac{\delta}{1-\delta}\right)^n \sum_{i=0}^k a_i(z_0, z_1)d^i(z_0, z_1). \tag{2.3}$$

Since

$$a_j(z_n, z_{n+1})d^j(z_n, z_{n+1}) \leq \sum_{i=0}^k a_i(z_n, z_{n+1})d^i(z_n, z_{n+1}),$$

By (ii) we get

$$\tau_j d^j(z_n, z_{n+1}) \leq \sum_{i=0}^k a_i(z_n, z_{n+1})d^i(z_n, z_{n+1}),$$

Thus, we get by (2.3)

$$d^j(z_n, z_{n+1}) \leq \mu^n \sigma_{j,0}, \quad n \geq 0, \tag{2.4}$$

where

$$\mu = \frac{\delta}{1-\delta}, \quad \sigma_{j,0} = (\tau_j)^{-1} \sum_{i=0}^k a_i(z_0, z_1)d^i(z_0, z_1). \tag{2.5}$$

Using triangle inequality and (2.4), we obtain that for all $n \geq 0$ and $m \geq 1$

$$\begin{aligned} d^j(z_n, z_{n+m}) &\leq d^j(z_n, z_{n+1}) + d^j(z_{n+1}, z_{n+2}) + \dots + d^j(z_{n+m-1}, z_{n+m}) \\ &\leq \sigma_{j,0}(\mu^n + \mu^{n+1} + \dots + \mu^{n+m-1}) \\ &= \sigma_{j,0}\mu^n \left[\frac{1-\mu^m}{1-\mu} \right] \\ &\leq \sigma_{j,0} \frac{\mu^n}{1-\mu}, \end{aligned}$$

which gives

$$d(z_n, z_{n+m}) \leq \left(\frac{\sigma_{j,0}}{1-\mu}\right)^{\frac{1}{j}} ((\mu)^{\frac{1}{j}})^n \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

This implies that $\{z_n\}$ is a Cauchy sequence. As (\mathfrak{X}, d) is complete, there exists $z' \in \mathfrak{X}$ such that

$$\lim_{n \rightarrow \infty} d(z_n, z') = 0,$$

Since S is continuous, we get that

$$d(z_{n+1}, Sz') = \lim_{n \rightarrow \infty} d(Sz_n, Sz') = 0.$$

From the uniqueness of the limit, we get that $Sz' = z'$, that is, z' is a fixed point of S .

Uniqueness: Now we show that z' is the unique fixed point of S . Let, if possible $z'' \in \mathfrak{X}$ is another fixed point of S , that is, $Sz'' = z''$ and $d(z', z'') > 0$, then making use of (2.1) with $(x, y) = (z', z'')$, we get

$$\sum_{i=0}^k a_i(Sz', Sz'')d^i(Sz', Sz'') \leq \delta \left[\sum_{i=0}^k a_i(z', Sz')d^i(z', Sz') + \sum_{i=0}^k a_i(z'', Sz'')d^i(z'', Sz'') \right],$$

that is,

$$\sum_{i=0}^k a_i(z', z'')d^i(z', z'') \leq \delta \left[\sum_{i=0}^k a_i(z', z')d^i(z', z') + \sum_{i=0}^k a_i(z'', z'')d^i(z'', z'') \right], \tag{2.6}$$

that is,

$$\sum_{i=0}^k a_i(z', z'')d^i(z', z'') = 0.$$

which is a contradiction since by (ii)

$$\begin{aligned} \sum_{i=0}^k a_i(z', z'')d^i(z', z'') &\geq a_j(z', z'')d^j(z', z''), \\ &\geq \tau_j d^j(z', z''), \\ &> 0. \end{aligned}$$

Consequently, z' is the unique fixed point of S . □

Corollary 2.3. *Let (\mathfrak{S}, d) be a complete metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a continuous mapping. Assume that there exists $\delta \in [0, \frac{1}{2})$, a natural number $k \geq 1$ and a finite sequence $\{a_i\} \subset (0, \infty)$ such that*

$$\sum_{i=1}^k a_i d^i(Sx, Sy) \leq \delta \left[\sum_{i=1}^k a_i (d^i(x, Sx) + d^i(y, Sy)) \right] \tag{2.7}$$

for every $x, y \in \mathfrak{S}$. Then S admits a unique fixed point $z' \in \mathfrak{S}$. Moreover, for every $z_0 \in \mathfrak{S}$, the Picard sequence $\{z_n\} \subset \mathfrak{S}$ defined by $z_{n+1} = Sz_n$ for all $n \geq 0$, converges to z' .

Proof. Taking $a_0 \equiv 0$, that is, $a_0(x, y) = 0$ for all $x, y \in \mathfrak{S}$ and $i \in \{1, 2, \dots, k\}$ in Theorem 2.2 □

Remark 2.4. Note that Corollary 2.3 reduces to Kannan fixed point theorem on taking $k = 1$ and $a_1 = 1$, (2.7) becomes

$$d(Sx, Sy) \leq \delta [d(x, Sx) + d(y, Sy)]$$

Example 2.5. Let $\mathfrak{S} = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be the mapping defined as follows: $S\mu_1 = \mu_1, S\mu_2 = \mu_3, S\mu_3 = \mu_4, S\mu_4 = \mu_1$. Let d denotes the discrete metric on \mathfrak{S} , that is,

$$d(\mu_i, \mu_j) = \begin{cases} 1, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Define the mapping $a_0 : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, \infty)$ as

$$\begin{aligned} a_0(\mu_i, \mu_j) &= a_0(\mu_j, \mu_i), \\ a_0(\mu_i, \mu_i) &= 0, \\ a_0(\mu_1, \mu_2) &= a_0(\mu_2, \mu_3) = 9, \\ a_0(\mu_1, \mu_3) &= a_0(\mu_3, \mu_4) = 3, \\ a_0(\mu_1, \mu_4) &= \frac{1}{4}, \\ a_0(\mu_2, \mu_4) &= 6. \end{aligned}$$

We show that

$$a_0(Sx, Sy) + d(Sx, Sy) \leq \frac{4}{10} [a_0(x, Sx) + d(x, Sx) + a_0(y, Sy) + d(y, Sy)], \tag{2.8}$$

for every $x, y \in \mathfrak{S}$, that is, S is a Kannan type polynomial contraction with $k = 1, a_1 \equiv 1$ and $\delta = \frac{4}{10}$.

If $x = y$ or $(x, y) = (\mu_1, \mu_4)$, then (2.8) holds trivially. Then, by symmetry, we just have to show that (2.8) holds for all $\mu_i, \mu_j \in \mathfrak{S}$ with $1 \leq i < j \leq 4$ and $(i, j) \neq (1, 4)$. Following are the different values of

$a_0(S\mu_i, S\mu_j) + d(S\mu_i, S\mu_j)$ and $a_0(\mu_i, S\mu_i) + d(\mu_i, S\mu_i) + a_0(\mu_j, S\mu_j) + d(\mu_j, S\mu_j)$ for all $1 \leq i < j \leq 4$.

$$\begin{aligned} a_0(S\mu_1, S\mu_2) + d(S\mu_1, S\mu_2) &= 4, \\ a_0(\mu_1, S\mu_1) + d(\mu_1, S\mu_1) + a_0(\mu_2, S\mu_2) + d(\mu_2, S\mu_2) &= 10. \\ a_0(S\mu_1, S\mu_3) + d(S\mu_1, S\mu_3) &= \frac{1}{4}, \\ a_0(\mu_1, S\mu_1) + d(\mu_1, S\mu_1) + a_0(\mu_3, S\mu_3) + d(\mu_3, S\mu_3) &= 4. \\ a_0(S\mu_2, S\mu_3) + d(S\mu_2, S\mu_3) &= 4, \\ a_0(\mu_2, S\mu_2) + d(\mu_2, S\mu_2) + a_0(\mu_3, S\mu_3) + d(\mu_3, S\mu_3) &= 14. \\ a_0(S\mu_2, S\mu_4) + d(S\mu_2, S\mu_4) &= 4, \\ a_0(\mu_2, S\mu_2) + d(\mu_2, S\mu_2) + a_0(\mu_4, S\mu_4) + d(\mu_4, S\mu_4) &= \frac{45}{4}. \end{aligned}$$

Then all the conditions of Theorem 2.2 are satisfied, (ii) is satisfied with $\alpha_j = 1$. S admits a unique fixed point $z' = \mu_1$, which confirms our result.

The continuity requirement placed on Theorem 2.2 is now weakened.

Definition 2.6. ([15]) Let (\mathfrak{S}, d) be a metric space. A mapping $S : \mathfrak{S} \rightarrow \mathfrak{S}$ is called Picard continuous, if for all $x, y \in \mathfrak{S}$, we have

$$\lim_{n \rightarrow \infty} d(S^n x, y) = 0 \Rightarrow \lim_{n \rightarrow \infty} d(S(S^n x), Sy) = 0$$

where $S^0 x = x$ and $S^{n+1} x = S(S^n x)$ for all $n \geq 0$.

Note that, if $S : \mathfrak{S} \rightarrow \mathfrak{S}$ is continuous then S is Picard continuous. However, the converse is not true [15].

Theorem 2.7. Let (\mathfrak{S}, d) be a complete metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a Kannan type polynomial mapping. Assume that the following conditions hold:

- (i) S is Picard continuous;
- (ii) There exists $j \in \{1, 2, \dots, k\}$ and $\tau_j > 0$ such that $a_j(x, y) \geq \tau_j$, $x, y \in \mathfrak{S}$.

Then S admits a unique fixed point $z' \in \mathfrak{S}$. Moreover, for every $z_0 \in \mathfrak{S}$, the Picard sequence $\{z_n\} \subset \mathfrak{S}$ defined by $z_{n+1} = Sz_n$ for all $n \geq 0$, converges to z' .

Proof. Let $z_0 \in \mathfrak{S}$ be any fixed element and $\{z_n\}$ be the Picard sequence in \mathfrak{S} defined by $z_{n+1} = Sz_n$, $n \geq 0$ that is, $z_n = S^n z_0$, $n \geq 0$.

We know that $\{z_n\}$ is a Cauchy sequence based on the proof of Theorem 2.2, which implies by the completeness of (\mathfrak{S}, d) that there exists $z' \in \mathfrak{S}$ such that

$$\lim_{n \rightarrow \infty} d(S^n z_0, z') = 0.$$

Using the Picard continuity of S , we get that

$$\lim_{n \rightarrow \infty} d(S^{n+1} z_0, Sz') = \lim_{n \rightarrow \infty} d(S(S^n z_0), Sz') = 0,$$

which by uniqueness of the limit implies that z' is a fixed point of S . The rest of the proof is similar to that of Theorem 2.2. □

3 Almost Kannan polynomial mapping

Inspired by Berinde [4], we present the class of almost Kannan polynomial mapping below.

Definition 3.1. Let (\mathfrak{S}, d) be a metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a given mapping. S is known as almost Kannan polynomial mapping, if there exists a finite sequence $\{\mathfrak{L}_i\}_{i=0}^k \subset (0, \infty)$, $\delta \in [0, \frac{1}{2})$, a natural number $k \geq 1$ and a family of mappings $a_i : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, \infty)$, $i = 0, 1, 2, 3, \dots, k$ such that

$$\begin{aligned} \sum_{i=0}^k a_i(Sx, Sy)d^i(Sx, Sy) &\leq \delta \left[\sum_{i=0}^k a_i(x, Sx)\{d^i(x, Sx) + \mathfrak{L}_i d^i(y, Sx)\} \right. \\ &\quad \left. + \sum_{i=0}^k a_i(y, Sy)\{d^i(y, Sy) + \mathfrak{L}_i d^i(y, Sx)\} \right], \end{aligned} \tag{3.1}$$

for every $x, y \in \mathfrak{S}$.

Let us recall the notion of weakly Picard operator, introduced by Rus [21].

Definition 3.2. ([21]) Let (\mathfrak{S}, d) be a metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a given mapping. S is known as weakly Picard operator, if

- (i) The set of fixed points of S is nonempty;
- (ii) For all $z_0 \in \mathfrak{S}$, the Picard sequence $\{S^n z_0\}$ is convergent and its limit belongs to the set of fixed points of S .

Theorem 3.3. Let (\mathfrak{S}, d) be a complete metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be an almost Kannan polynomial mapping. Assume that the following conditions hold:

- (i) S is Picard continuous;
- (ii) There exists $j \in \{1, 2, \dots, k\}$ and $\tau_j > 0$ such that $a_j(x, y) \geq \tau_j$, $x, y \in \mathfrak{S}$.

Then S is a weakly Picard operator.

Proof. Let $z_0 \in \mathfrak{S}$ be any fixed element and $\{z_n\}$ be the Picard sequence in \mathfrak{S} defined by $z_{n+1} = Sz_n$, $n \geq 0$.

Using (3.1) with $(x, y) = (z_0, z_1)$, we get

$$\begin{aligned} \sum_{i=0}^k a_i(Sz_0, Sz_1)d^i(Sz_0, Sz_1) &\leq \delta \left[\sum_{i=0}^k a_i(z_0, Sz_0)\{d^i(z_0, Sz_0) + \mathfrak{L}_i d^i(z_1, Sz_0)\} \right. \\ &\quad \left. + \sum_{i=0}^k a_i(z_1, Sz_1)\{d^i(z_1, Sz_1) + \mathfrak{L}_i d^i(z_1, Sz_0)\} \right], \end{aligned}$$

that is,

$$\begin{aligned} \sum_{i=0}^k a_i(z_1, z_2)d^i(z_1, z_2) &\leq \delta \left[\sum_{i=0}^k a_i(z_0, z_1)\{d^i(z_0, z_1) + \mathfrak{L}_i d^i(z_1, z_1)\} \right. \\ &\quad \left. + \sum_{i=0}^k a_i(z_1, z_2)\{d^i(z_1, z_2) + \mathfrak{L}_i d^i(z_1, z_1)\} \right], \\ &\leq \delta \left[\sum_{i=0}^k a_i(z_0, z_1)\{d^i(z_0, z_1) + \sum_{i=0}^k a_i(z_1, z_2)\{d^i(z_1, z_2)\} \right], \\ \sum_{i=0}^k a_i(z_1, z_2)d^i(z_1, z_2) &\leq \frac{\delta}{1 - \delta} \sum_{i=0}^k a_i(z_0, z_1)d^i(z_0, z_1). \end{aligned}$$

Again Using (3.1) with $(x, y) = (z_1, z_2)$, we get

$$\begin{aligned} \sum_{i=0}^k a_i(Sz_1, Sz_2)d^i(Sz_1, Sz_2) &\leq \delta \left[\sum_{i=0}^k a_i(z_1, Sz_1)\{d^i(z_1, Sz_1) + \mathfrak{L}_i d^i(z_2, Sz_1)\} \right. \\ &\quad \left. + \sum_{i=0}^k a_i(z_2, Sz_2)\{d^i(z_2, Sz_2) + \mathfrak{L}_i d^i(z_2, Sz_1)\} \right], \end{aligned}$$

that is,

$$\begin{aligned} \sum_{i=0}^k a_i(z_2, z_3) d^i(z_2, z_3) &\leq \delta \left[\sum_{i=0}^k a_i(z_1, z_2) d^i(z_1, z_2) \right. \\ &\quad \left. + \sum_{i=0}^k a_i(z_2, z_3) d^i(z_2, z_3) \right], \\ \sum_{i=0}^k a_i(z_2, z_3) d^i(z_2, z_3) &\leq \left(\frac{\delta}{1-\delta} \right)^2 \sum_{i=0}^k a_i(z_0, z_1) d^i(z_0, z_1). \end{aligned}$$

Keeping the same approach, we learn by induction that

$$\sum_{i=0}^k a_i(z_n, z_{n+1}) d^i(z_n, z_{n+1}) \leq \left(\frac{\delta}{1-\delta} \right)^n \sum_{i=0}^k a_i(z_0, z_1) d^i(z_0, z_1), n \geq 0$$

which implies by (ii) that

$$d^j(z_n, z_{n+1}) \leq \mu^n \sigma_{j,0}, n \geq 0,$$

where

$$\mu = \frac{\delta}{1-\delta}, \sigma_{j,0} = (\tau_j)^{-1} \sum_{i=0}^k a_i(z_0, z_1) d^i(z_0, z_1).$$

Next proceeding as in the proof of Theorem 2.2, we get that $\{z_n\}$ is a Cauchy sequence, which implies by the completeness of (\mathfrak{S}, d) the existence of $z' \in \mathfrak{S}$ such that $\lim_{n \rightarrow \infty} d(z_n, z') = 0$. Since S is Picard continuous, we get $\lim_{n \rightarrow \infty} d(z_{n+1}, Sz') = 0$ which implies by the uniqueness of the limit that $z' = Sz'$. □

Corollary 3.4. *Let (\mathfrak{S}, d) be a complete metric space and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be a given Picard continuous mapping. Assume that there exists $\delta \in [0, \frac{1}{2})$, a natural number $k \geq 1$ and two finite sequence $\{a_i\}_{i=1}^k, \{\mathfrak{L}_i\}_{i=1}^k \subset (0, \infty)$ such that*

$$\sum_{i=1}^k a_i d^i(Sx, Sy) \leq \delta \sum_{i=1}^k a_i \{d^i(x, Sx) + d^i(y, Sy) + 2\mathfrak{L}_i d^i(y, Sx)\} \tag{3.2}$$

for every $x, y \in \mathfrak{S}$. Then S is a weakly Picard operator.

Proof. Note that (3.2) is a special case of (3.1) with $a_0 \equiv 0$ and a_i is constant for all $i \in \{1, 2, 3, \dots, k\}$. □

Remark 3.5. Taking $k = 1, a_1 = 1$ and $\mathfrak{L}_1 = 0$, Corollary 3.4 reduces to Kannan fixed point Theorem.

Example 3.6. Let $\mathfrak{S} = [0, 1]$ and $S : \mathfrak{S} \rightarrow \mathfrak{S}$ be the mapping defined by

$$Sx = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x < 1, \\ 0, & \text{if } x = 1. \end{cases}$$

Define d to be the standard metric on \mathfrak{S} , that is,

$$d(x, y) = |x - y|, x, y \in \mathfrak{S}.$$

Note that S is Picard continuous.

Consider the mapping $a_0 : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, \infty)$ defined by

$$a_0(x, y) = |2x^2 - x| + |2y^2 - y|, x, y \in \mathfrak{S}.$$

We will show that

$$a_0(Sx, Sy) + d(Sx, Sy) \leq \frac{1}{3} [2a_0(x, Sx) + 2a_0(y, Sy) + d(x, Sx) + 2d(y, Sx) + d(y, Sy)] \tag{3.3}$$

for all $x, y \in \mathfrak{S}$, that is, S is an almost Kannan polynomial contraction with $k = 1, a_1 \equiv 1, \mathfrak{L}_0 = \mathfrak{L}_1 = 1$ and $\delta = \frac{1}{3}$. We discuss the possible cases.

Case 1: $0 \leq x, y < 1$. In this case, we have

$$a_0(Sx, Sy) + d(Sx, Sy) = a_0\left(\frac{1}{2}, \frac{1}{2}\right) + d\left(\frac{1}{2}, \frac{1}{2}\right) = 0.$$

Then (3.3) holds.

Case 2: $0 \leq x < 1, y = 1$. In this case, we have

$$\begin{aligned} a_0(Sx, Sy) + d(Sx, Sy) &= a_0\left(\frac{1}{2}, 0\right) + d\left(\frac{1}{2}, 0\right) \\ &= 0 + \frac{1}{2} \\ &= \frac{1}{2} \\ &\leq \frac{1}{3}[2a_0(x, Sx) + 2a_0(y, Sy) + d(x, Sx) + 2d(y, Sx) + d(y, Sy)]. \end{aligned}$$

Then (3.3) holds.

Case 3: $x = 1, 0 \leq y < 1$. In this case, we have

$$\begin{aligned} a_0(Sx, Sy) + d(Sx, Sy) &= a_0\left(0, \frac{1}{2}\right) + d\left(0, \frac{1}{2}\right) \\ &= 0 + \frac{1}{2} \\ &= \frac{1}{2} \\ &\leq \frac{2}{3} \\ &\leq \frac{1}{3}(2a_0(x, Sx)) \\ &\leq \frac{1}{3}[2a_0(x, Sx) + 2a_0(y, Sy) + d(x, Sx) + 2d(y, Sx) + d(y, Sy)]. \end{aligned}$$

Then (3.3) holds.

Case 4: $x = y = 1$. In this case, we have

$$a_0(Sx, Sy) + d(Sx, Sy) = a_0(0, 0) + d(0, 0) = 0.$$

Then (3.3) holds.

Thus, (3.3) is satisfied for all $x, y \in \mathfrak{S}$. Note that condition (ii) of Theorem 3.3 holds with $j = 1$ and $\alpha_1 = 1$. Then Theorem 3.3 applies and $z' = \frac{1}{2}$ is a fixed point of S .

4 Conclusion remarks

This paper aims is to obtain new class of polynomial mapping namely the class of Kannan type polynomial mapping and the class of almost Kannan type polynomial mapping. Also, some fixed point theorems are established, for each class in usual metric space. Any interested researchers can search for the existence and uniqueness of fixed points for maps satisfying different polynomial contraction conditions in metric space or any other generalization of metric space to conduct their thesis work on this topic.

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