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GENERALIZATION OF AN ELEMENT PRIME TO ANOTHER ELEMENT IN MULTIPLICATIVE LATTICES

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Abstract. In this paper, we have introduced the notion of a reduction function ϕ on L. We have defined an element ϕ-prime to another element in L and obtained its various characterizations. We have made an attempt to prove many of its properties and investigate the relations between these structures.

1 Introduction

R. P. Dilworth in [6] took a major step in giving a concrete abstract formulation of the ideal theory of commutative rings $\mathbb{R}$ with identity. He intended that the treatment should be purely ideal theoretic, and therefore the system that was chosen for the study was multiplicative lattices. Since then, in the literature, it is seen that extensive research work has been carried on in multiplicative lattices.

The notion of an element prime to another element in a multiplicative lattice L is introduced by F. Alarcon et. al. in [1]. Further, the concept of an element weakly prime to another element in a multiplicative lattice L is introduced by C. S. Manjarekar et. al. in [9]. To generalize these concepts, in this paper, we introduce the notion of an element ϕ-prime to another element in a multiplicative lattice L, by defining a reduction function ϕ on L. The main aim of this paper is to obtain various characterizations of the concept "an element ϕ-prime to another element in L" introduced in this paper, which is generalization of the concept "an element being prime to another element" in L and then investigate many of its properties.

Let us recall a few definitions in a multiplicative lattice. A multiplicative lattice is a complete lattice $(L, \leqslant, 0, 1)$ provided with commutative, associative and join distributive multiplication (denoted by $\cdot$) in which the largest element 1 acts as a multiplicative identity, where $\leqslant$ is a partial order on L. The product of any two elements a and b of L will be denoted by ab instead of $a \cdot b$ for convenience. $\mathbb{Z}_+$ denotes the set of all natural numbers. For $A \subseteq L$, $\vee A$ = the join of A denotes the supremum of A and $\wedge A$ = the meet of A denotes the infimum of A. For brevity, we shall write $a \vee b$ for $\vee\{a, b\}$ and $a \wedge b$ for $\wedge\{a, b\}$ where $a, b \in L$. Note that a^n denotes $a \cdot ... \cdot a$ (repeated n times) where $a \in L$ and $n \in \mathbb{Z}_+$. For $a, b \in L$, $(a : b) = \vee\{x \in L \mid xb \leqslant a\}$ where $(a : b)$ is read as the residuation of a by b. An element $e \in L$ is called meet principal if $a \wedge be = ((a : e) \wedge b)e$ for all $a,\ b \in L$. An element $e \in L$ is called join principal if $(ae \vee b) : e = (b : e) \vee a$ for all $a,\ b \in L$. An element $e \in L$ is called principal if e is both meet principal and join principal. A multiplicative lattice L is said to be a principally generated lattice or simply a PG-lattice if every element of L is a join of principal elements of L. An element $a \in L$ is called compact if for $X \subseteq L$, $a \leqslant \vee X$ imply the existence of a finite number of elements $a_1, a_2, \cdots, a_n$ in X such that $a \leqslant a_1 \vee a_2 \vee \cdots \vee a_n$. If each element of L is a join of compact elements of L, then L is called a compactly generated lattice or simply a CG-lattice. L_* denotes the set of all compact elements of L.

An element $a \in L$ is said to be proper if $a < 1$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a$, for some $n \in \mathbb{Z}_+\}$. A proper element $p \in L$ is called a prime element if $ab \leqslant p$ imply either $a \leqslant p$ or $b \leqslant p$ where $a, b \in L$. A proper element $a \in L$ is said to be idempotent if $a^2 = a$. A proper element $a \in L$ is said to be nilpotent if $a^n = 0$ for some $n \in \mathbb{Z}_+$. A lattice L is said to be modular if for all $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge c$, $\forall a \leqslant c$. In a lattice L, a family $\{a_i\} \subseteq L$ is an ascending (or descending) chain, if $a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots$ (or $a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots$). A lattice L is said to satisfy the ascending chain condition or simply ACC, if for any ascending chain $a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots$, there exists $k \in \mathbb{Z}_+$ such that $a_k = a_n$ $\forall n \geqslant k$. A multiplicative lattice is called a Noether lattice if it is modular, PG and satisfies ACC. In a Noether lattice L, an element $a \in L$ is said to satisfy the restricted cancellation law if for all $b,\ c \in L$, $ab = ac \neq 0$ imply $b = c$ (see [11]).

R. P. Dilworth gave the following classical example of a multiplicative lattice:
Let $\mathbb{R}$ be a commutative ring with unity and I, J be the ideals of $\mathbb{R}$. If $L(\mathbb{R}) = L$ denotes the set of all ideals of $\mathbb{R}$, then $< L; \wedge, \vee, \cdot >$ is a multiplicative lattice where the lattice operations defined are as follows:

① $I \wedge J = I \cap J$

② $I \vee J = I + J = (I \cup J)$

③ $I \cdot J = \{\sum_{finite} a_i \cdot b_i \in \mathbb{R} \mid a_i \in I \text{ and } b_i \in J\}$

$L(\mathbb{R})$ is known as the ideal lattice or the lattice of ideals of $\mathbb{R}$ and has the following features:

(i) Every principal ideal of $\mathbb{R}$ is a compact element of $L(\mathbb{R})$.

(ii) $L(\mathbb{R})$ is a compactly generated multiplicative lattice.

(iii) Every prime ideal of $\mathbb{R}$ is a prime element of $L(\mathbb{R})$.

Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact. For general background and terminology in multiplicative lattices, one can refer [1], [3], [4], [8] and [10].

The content of the paper is as follows: In Section 2, we introduce the notion of a reduction function on a multiplicative lattice L and an element being ϕ-prime to another element in L to obtain its characterizations (see Theorems 2.10, 2.13). In Theorem 2.15, we give characterization of an element being non-prime to another element in L. The notion of an element being ϕ_β-prime to another element in L is introduced and relations among them are obtained (see Theorem 2.23). By counter example, it is shown that if $b \in L$ is ϕ-prime to a proper element $p \in L$, then b need not be prime to p (see Example 2.24). In different ways, we have proved that if an element $b \in L$ is ϕ-prime to a proper element $p \in L$, then b is prime to p under certain conditions (see Theorems 2.25, 2.27, 2.28, 2.30). One of the ways is achieved by defining an element being n-potent prime to another element of L where $n \geqslant 2$ and $n \in \mathbb{Z}_+$. In an attempt to highlight the future scope of this paper, we conclude this paper by giving an absorbing flavour to the concept of "an element being prime to another element" of L in the last section.

2 An Element ϕ-Prime To Another Element in L

We begin with introducing the notion of a reduction function $\phi_L : L \longrightarrow L \cup \{\emptyset\}$ in the following way, where $\emptyset$ is an empty set.

Definition 2.1. A reduction function on L is a function $\phi_L : L \longrightarrow L \cup \{\emptyset\}$ which satisfies the following two conditions:
①. $\phi_L(a) \leqslant a \quad \forall\, a \in L$.
②. $a \leqslant b$ imply $\phi_L(a) \leqslant \phi_L(b) \quad \forall\, a,\ b \in L$.

From now onwards, for convenience, we denote ϕ_L as ϕ and throughout this paper, ϕ always denotes a reduction function on L.

Now, we recall the definitions of an element being prime to another element and an element being weakly prime to another element in L.

Definition 2.2. (F. Alarcon et. al. [1]) An element $b \in L$ is said to be prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ imply $x \leqslant p$.

Definition 2.3. (C. S. Manjarekar et. al. [9]) An element $b \in L$ is said to be weakly prime to a proper element $p \in L$ if for all $x \in L$, $0 \neq xb \leqslant p$ imply $x \leqslant p$.

Now, we give definitions of an element being almost prime/n-almost prime $(n \geqslant 2)$/ω-prime to another element in L.

Definition 2.4. An element $b \in L$ is said to be almost prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ and $xb \nleqslant p^2$ imply $x \leqslant p$.

Clearly, an element $b \in L$ is almost prime to every idempotent element p of L.

Definition 2.5. An element $b \in L$ is said to be n-almost prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ and $xb \nleqslant p^n$ imply $x \leqslant p$ where $n \geqslant 2$ and $n \in \mathbb{Z}_+$.

Definition 2.6. An element $b \in L$ is said to be ω-prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ and $xb \nleqslant \bigwedge_{i=1}^{\infty} p^i$ imply $x \leqslant p$.

To unify all these notions under one frame, we introduce the notion of an element of L to be ϕ-prime to another element of L, which is the generalization of the concept of an element to be prime to another element in L.

Definition 2.7. Given a reduction function $\phi : L \longrightarrow L \cup \{\emptyset\}$, an element $b \in L$ is said to be ϕ-prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ and $xb \nleqslant \phi(p)$ imply $x \leqslant p$.

Now, if $\phi_\beta : L \longrightarrow L \cup \{\emptyset\}$ is a reduction function on L where $\beta \in \{\emptyset\} \cup \{0, 1, 2, \cdots\} \cup \{\omega\}$, then an element "$\phi_\beta$-prime to" another element in L is defined by the following settings in the Definition 2.7 of an element ϕ-prime to another element in L, to get the above notions 2.2, 2.3, 2.4, 2.5, 2.6 as its particular cases.

- $\phi_\emptyset(p) = \emptyset$. Then b is prime (or $\phi_\emptyset$-prime) to p.
- $\phi_0(p) = 0$. Then b is weakly prime (or ϕ_0-prime) to p.
- $\phi_2(p) = p^2$. Then b is almost prime (or ϕ_2-prime) to p.
- $\phi_n(p) = p^n$ $(n \geqslant 2)$. Then b is n-almost prime (or ϕ_n-prime) to p.
- $\phi_\omega(p) = \bigwedge_{i=1}^{\infty} p^i$. Then b is ω-prime (or ϕ_ω-prime) to p.

For the sake of completeness, we define $\phi_1(p) = p$ for any element $p \in L$.

Note that the notions of "almost prime" and "2-almost prime" are the same.

It is easy to see that all these functions $\phi_\emptyset$, ϕ_0, ϕ_1, ϕ_2, ϕ_n $(n \geqslant 2)$, ϕ_ω are examples of the reduction function on L.

Definition 2.8. Given any two reduction functions $\gamma, \psi : L \longrightarrow L \cup \{\emptyset\}$, we define $\gamma \leqslant \psi$ if $\gamma(a) \leqslant \psi(a)$ for each $a \in L$.

Clearly, we have the following order:

$$\phi_\emptyset \leqslant \phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2 \leqslant \phi_1$$

According to [7], a proper element $p \in L$ is said to be ϕ-prime if for all $a,\ b \in L$, $ab \leqslant p$ and $ab \nleqslant \phi(p)$ imply either $a \leqslant p$ or $b \leqslant p$, treating ϕ as a reduction function on L. The following result shows that an element of L can be ϕ-prime to a ϕ-prime element of L.

Theorem 2.9. *Let p be a proper element of L and $b \in L$ be such that $b \nleqslant p$. If p is a ϕ-prime element, then b is ϕ-prime to p. Hence, deduce that if p is a prime element, then b is prime to p.*

Proof. Let $xb \leqslant p$ and $xb \nleqslant \phi(p)$ for $x \in L$. As p is a ϕ-prime element and $b \nleqslant p$, we have $x \leqslant p$ and hence b is ϕ-prime to p. The deduction part is obvious. □

The next result is characterization of an element being ϕ-prime to another element of L.

Theorem 2.10. *Let p be a proper element of L and $b \in L$. Then b is ϕ-prime to p if and only if $x \geqslant b$ is ϕ-prime to p for every $x \in L$.*

Proof. Assume that b is ϕ-prime to p. Consider an element $x \in L$ such that $x \geqslant b$. Let $yx \leqslant p$ and $yx \nleqslant \phi(p)$ where $y \in L$. Then $yb \leqslant p$ and $yb \nleqslant \phi(p)$. Since b is ϕ-prime to p, it follows that $y \leqslant p$. This shows that $x \geqslant b$ is ϕ-prime to p for every $x \in L$. The converse is obvious. □

The consequences of the above Theorem 2.10 are given below, whose proofs being obvious are omitted.

Corollary 2.11. *If $\{b_\alpha\}$ is a family of elements of L such that each b_α is ϕ-prime to a proper element $p \in L$, then $(\underset{\alpha}{\vee} b_\alpha)$ is ϕ-prime to p.*

Corollary 2.12. *If $\{b_\alpha\}$ is a family of elements of L such that $(\underset{\alpha}{\wedge} b_\alpha)$ is ϕ-prime to a proper element $p \in L$, then each b_α is ϕ-prime to p.*

Now, we obtain the characterization of an element being ϕ-prime to another element of L.

Theorem 2.13. *Let p be a proper element of L and $b \in L$. Then the following statements are equivalent:*
①. *b is ϕ-prime to p.*
②. *either $(p : b) = p$ or $(p : b) = (\phi(p) : b)$.*
③. *for every $r \in L_*$, $rb \leqslant p$ and $rb \nleqslant \phi(p)$ imply $r \leqslant p$.*

Proof. ①$\Longrightarrow$②. Suppose ① holds. Let $h \in L_*$ be such that $h \leqslant (p : b)$. Then $hb \leqslant p$. If $hb \leqslant \phi(p)$, then $h \leqslant (\phi(p) : b)$. If $hb \nleqslant \phi(p)$, then since b is ϕ-prime to p, $hb \leqslant p$ and $hb \nleqslant \phi(p)$, it follows that $h \leqslant p$. Hence, by Lemma 2.3.13 from [5], either $(p : b) \leqslant (\phi(p) : b)$ or $(p : b) \leqslant p$. Consequently, either $(p : b) = (\phi(p) : b)$ or $(p : b) = p$.

②$\Longrightarrow$③. Suppose ② holds. Let $rb \leqslant p$ and $rb \nleqslant \phi(p)$ for $r \in L_*$. By ② if $(p : b) = (\phi(p) : b)$, then as $r \leqslant (p : b)$, it follows that $r \leqslant (\phi(p) : b)$ which contradicts $rb \nleqslant \phi(p)$ and so we must have $(p : b) = p$. Therefore, $r \leqslant (p : b)$ gives $r \leqslant p$.

③$\Longrightarrow$①. Suppose ③ holds. Let $xb \leqslant p$ and $xb \nleqslant \phi(p)$ for $x \in L$. Then, as L is compactly generated, there exists $y' \in L_*$ such that $y' \leqslant x$ and $y'b \nleqslant \phi(p)$. Let $y \leqslant x$ be any compact element of L. Then $(y \vee y') \in L_*$ such that $(y \vee y')b \leqslant p$ and $(y \vee y')b \nleqslant \phi(p)$. So by ③, it follows that $(y \vee y') \leqslant p$, which implies $x \leqslant p$ and therefore b is ϕ-prime to p. □

The consequence of the above Theorem 2.13 is presented as the following corollary, whose proof being obvious is omitted.

Corollary 2.14. *An element $b \in L$ is non-prime to a proper element $p \in L$ if and only if $(p : b) \nleqslant p$ i.e. $p < (p : b)$.*

Now, we obtain a characterization of an element of L being non-prime to another element of L.

Theorem 2.15. *An element $b \in L$ is non-prime to a proper element $p \in L$ if and only if $x \leqslant b$ is non-prime to p for every element $x \in L$.*

Proof. Assume that b is non-prime to p. Then by Corollary 2.14, we have $(p : b) \nleqslant p$. Consider an element $x \in L$ such that $x \leqslant b$. Then $(p : b) \leqslant (p : x)$. It follows that $(p : x) \nleqslant p$. This implies that $x \leqslant b$ is non-prime to p for every element $x \in L$, by Corollary 2.14. The converse is obvious. □

The consequences of the above Theorem 2.15 are given below, whose proofs being obvious are omitted.

Corollary 2.16. *If $\{b_\alpha\}$ is a family of elements of L such that each b_α is non-prime to a proper element $p \in L$, then $(\underset{\alpha}{\wedge} b_\alpha)$ is non-prime to p.*

Corollary 2.17. *If $\{b_\alpha\}$ is a family of elements of L such that $(\underset{\alpha}{\vee} b_\alpha)$ is non-prime to a proper element $p \in L$, then each b_α is non-prime to p.*

According to [2], the notation q_p denotes the element $\vee\{x \in L | x$ is non-prime to $p\}$ where $p \in L$. Now, we give lemmas required to obtain some properties of an element q_p of L.

Lemma 2.18. *Let $x, y \in L$. If an element xy is non-prime to a proper element $p \in L$, then either x is non-prime to p, or y is non-prime to p.*

Proof. Assume that x is prime to p and y is prime to p. Let $bxy \leqslant p$ where $b \in L$. As x is prime to p, we have $by \leqslant p$. Further, as y is prime to p, we have $b \leqslant p$. This implies that xy is prime to p, a contradiction. So, the initial assumption is wrong, and we must have either x is non-prime to p, or y is non-prime to p. □

Clearly, $a : (b \vee c) = (a : b) \wedge (a : c)\ \forall a, b, c \in L$. Indeed, we have the following Lemma-11 of [2].

Lemma 2.19. *Let L be a multiplicative lattice and $p \in L$. If $\{b_\alpha\}$ is a family of elements of L, then $\Big(p : (\underset{\alpha}{\vee} b_\alpha)\Big) = \underset{\alpha}{\wedge}(p : b_\alpha)$.*

Proof. The proof is obvious. □

According to [1], a proper element $p \in L$ is called completely meet prime if $(\underset{\alpha}{\wedge} a_\alpha) \leqslant p$ imply $a_\alpha \leqslant p$ for some α.

Lemma 2.20. *Let a proper element $p \in L$ be completely meet prime. If $\{b_\alpha\}$ is a family of elements of L non-prime to p, then $(\underset{\alpha}{\vee} b_\alpha)$ is non-prime to p.*

Proof. By Lemma 2.19, we have $\Big(p : (\underset{\alpha}{\vee} b_\alpha)\Big) = \underset{\alpha}{\wedge}(p : b_\alpha)$. As each b_α is non-prime to p, from Corollary 2.14, it follows that $(p : b_\alpha) \nleqslant p$ for each α. As p is completely meet prime, we have, $\underset{\alpha}{\wedge}(p : b_\alpha) \nleqslant p$ and hence $\Big(p : (\underset{\alpha}{\vee} b_\alpha)\Big) \nleqslant p$. This implies that $(\underset{\alpha}{\vee} b_\alpha)$ is non-prime to p, by Corollary 2.14. □

Now, we are in a position to obtain interesting properties of the element q_p.

Theorem 2.21. *If a proper element $p \in L$ is completely meet prime, then*
①. *q_p is non-prime to p.*
②. *q_p is a prime element of L.*

Proof. ①. As $q_p = \vee\{x \in L | x$ is non-prime to $p\}$, by Lemma 2.20, we have q_p is non-prime to p.
②. Now, let $ab \leqslant q_p$ for $a,\ b \in L$. Then, by Theorem 2.15, ab is non-prime to p, since q_p is non-prime to p. By Lemma 2.18, it follows that either a is non-prime to p or b is non-prime to p. So either $a \leqslant q_p$ or $b \leqslant q_p$, and hence q_p is a prime element of L. □

To obtain the relation among an element being ϕ_β-prime to another element in L, we prove the following lemma.

Lemma 2.22. *Let $\gamma, \psi : L \longrightarrow L \cup \{\emptyset\}$ be reduction functions on L such that $\gamma \leqslant \psi$ and p be proper element of L. If an element $b \in L$ is γ-prime to p, then $b \in L$ is ψ-prime to p.*

Proof. Let an element $b \in L$ be γ-prime to p. Suppose $xb \leqslant p$ and $xb \nleqslant \psi(p)$ for $x \in L$. Then, as $\gamma \leqslant \psi$, we have $xb \leqslant p$ and $xb \nleqslant \gamma(p)$. Since b is γ-prime to p, it follows that $x \leqslant p$ and hence b is ψ-prime to p. □

Theorem 2.23. *Let an element $b \in L$ and a proper element $p \in L$. Then b is $\phi_\emptyset$-prime to p implies b is ϕ_0-prime to p implies b is ϕ_ω-prime to p implies b is $\phi_{(n+1)}$-prime to p implies b is ϕ_n-prime to p (where $n \geqslant 2$) implies b is ϕ_2-prime to p.*

Proof. Obviously, if b is prime to p, then b is weakly prime to p and hence the first implication is done. The remaining implications follow by using Lemma 2.22 on the fact that $\phi_\emptyset \leqslant \phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2 \leqslant \phi_1$. □

Clearly, if an element $b \in L$ is prime to a proper element $p \in L$, then b is ϕ-prime to p. The following example shows that its converse need not be true (by taking ϕ as ϕ_2 for convenience).

Example 2.24. Consider the lattice L of ideals of the ring $R =< Z_{36}\ ,\ +_{36}\ ,\ \times_{36} >$. Then the only ideals of R are the principal ideals (0),(2),(3),(4),(6),(9),(12),(18),(1). Clearly $L = \{$(0),(2),(3),(4),(6),(9),(12),(18),(1)$\}$ is a compactly generated multiplicative lattice. Its lattice structure is shown in Figure-1. From Multiplication table in Figure-2, it is easy to see that the element $(6) \in L$ is weakly prime to $(0) \in L$ and hence $(6) \in L$ is ϕ_2-prime to $(0) \in L$ but (6) is not prime to (0).

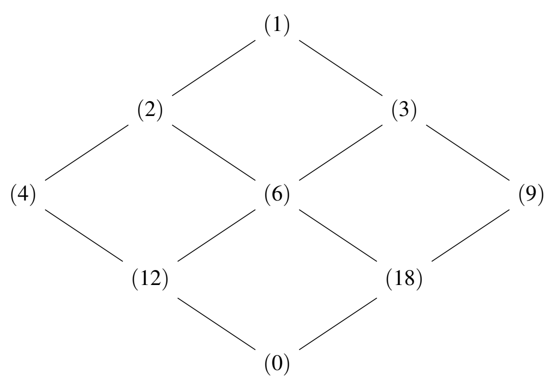

Figure-1 *Lattice of ideals of* Z_{36}

$\times_{36}$	(0)	(2)	(3)	(4)	(6)	(9)	(12)	(18)	(1)
(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
(2)	(0)	(4)	(6)	(4)	(12)	(18)	(12)	(0)	(2)
(3)	(0)	(6)	(9)	(12)	(18)	(9)	(0)	(18)	(3)
(4)	(0)	(4)	(12)	(4)	(12)	(0)	(12)	(0)	(4)
(6)	(0)	(12)	(18)	(12)	(0)	(18)	(0)	(0)	(6)
(9)	(0)	(18)	(9)	(0)	(18)	(9)	(0)	(18)	(9)
(12)	(0)	(12)	(0)	(12)	(0)	(0)	(0)	(0)	(12)
(18)	(0)	(0)	(18)	(0)	(0)	(18)	(0)	(0)	(18)
(1)	(0)	(2)	(3)	(4)	(6)	(9)	(12)	(18)	(1)

Figure-2 *Multiplication Table*

In the following four successive theorems, we show conditions under which, if an element $b \in L$ is ϕ-prime to a proper element p, then b is prime to p.

Theorem 2.25. *Let L be a Noether lattice. Let $0 \neq p \in L$ be a non-nilpotent proper element satisfying the restricted cancellation law. Let $b \in L$ be such that $p < b$. Then b is ϕ-prime to p for some $\phi \leqslant \phi_n$ and for all $n \geqslant 2$ if and only if b is prime to p.*

Proof. Assume that b is prime to p. Then obviously, b is ϕ-prime to p for every ϕ and hence for some $\phi \leqslant \phi_n$, for all $n \geqslant 2$. Conversely, assume that b is ϕ-prime to p for some $\phi \leqslant \phi_n$ and for all $n \geqslant 2$. Then by Lemma 2.22, b is ϕ_n-prime (n-almost prime) to p and for all $n \geqslant 2$. Let $xb \leqslant p$ for $x \in L$. If $xb \nleqslant \phi_n(p)$ for some $n \geqslant 2$, then as b is ϕ_n-prime to p, we have $x \leqslant p$, and we are done. So let $xb \leqslant \phi_n(p)$ for all $n \geqslant 2$. Then $xb \leqslant p^n \leqslant p^2$ as $n \geqslant 2$. This implies $xp \leqslant p^2 \neq 0$ as $p < b$. Hence, $x \leqslant p$ by Lemma 1.11 of [11] and thus b is prime to p. □

Definition 2.26. Let $n \geqslant 2$ and $n \in \mathbb{Z}_+$. An element $b \in L$ is said to be n-potent prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p^n$ imply $x \leqslant p$.

Note that the notions of "potent prime" and "2-potent prime" are the same.

Theorem 2.27. *Let $p \in L$ be a proper element and $b \in L$. Then b is ϕ-prime to p for some $\phi \leqslant \phi_n$ where $n \geqslant 2$ if and only if b is prime to p, provided b is k-potent prime to p for some $k \leqslant n$.*

Proof. Assume that b is prime to p. Then obviously, b is ϕ-prime to p for every ϕ and hence for some $\phi \leqslant \phi_n$ where $n \geqslant 2$. Conversely, assume that b is ϕ-prime to p for some $\phi \leqslant \phi_n$ where $n \geqslant 2$. Then by Lemma 2.22, b is ϕ_n-prime (n-almost prime) to p. Let $xb \leqslant p$ for $x \in L$. If $xb \nleqslant \phi_k(p) = p^k$, then $xb \nleqslant \phi_n(p) = p^n$ as $k \leqslant n$. Since b is ϕ_n-prime to p, we have $x \leqslant p$, and we are done. If $xb \leqslant \phi_k(p) = p^k$, then as b is k-potent prime to p, we have $x \leqslant p$. Hence, b is prime to p. □

Theorem 2.28. *Let an element $b \in L$ be ϕ-prime to a proper element $p \in L$. If $pb \nleqslant \phi(p)$, then b is prime to p.*

Proof. Let $xb \leqslant p$ for $x \in L$. If $xb \nleqslant \phi(p)$, then as b is ϕ-prime to p, we have $x \leqslant p$, and we are done. So assume that $xb \leqslant \phi(p)$. Then, as $pb \nleqslant \phi(p)$, we have $db \nleqslant \phi(p)$ for some $d \leqslant p$ in L. Also $(x \vee d)b = xb \vee db \leqslant p$ and $(x \vee d)b \nleqslant \phi(p)$. As b is ϕ-prime to p, we have $x \leqslant (x \vee d) \leqslant p$ and hence b is prime to p. □

From Theorem 2.28, it follows that, if an element $b \in L$ is ϕ-prime to a proper element $p \in L$ but b is not prime to p, then $pb \leqslant \phi(p)$ and hence $pb \leqslant p$.

Corollary 2.29. *If an element $b \in L$ is weakly prime to a proper element $p \in L$ such that b is not prime to p, then $pb = 0$.*

Proof. The proof is obvious. □

Theorem 2.30. *Let an element $b \in L$ be ϕ-prime to a proper element $p \in L$. If b is prime to $\phi(p)$, then b is prime to p.*

Proof. Let $xb \leqslant p$ for $x \in L$. If $xb \nleqslant \phi(p)$, then as b is ϕ-prime to p, we have $x \leqslant p$, and we are done. Now if $xb \leqslant \phi(p)$, then as b is prime to $\phi(p)$, we have $x \leqslant \phi(p)$. This implies that $x \leqslant p$ because $\phi(p) \leqslant p$, and we are done. □

3 Future Scope

In an attempt to highlight the future scope of the concept of "an element being prime to another element" in L, we add an absorbing flavour to it by defining the following notions.

Definition 3.1. A proper element $b \in L$ is said to be 1-absorbing to a proper element $p \in L$ if for all proper elements $x_1, x_2 \in L$ such that the product of x_1, x_2 need not collapse to one of the elements among them, $x_1x_2b \leqslant p$ imply either $x_1 \leqslant p$ or $x_2 \leqslant p$.

Generalizing the above Definition 3.1, we get,

Definition 3.2. Let $n \geqslant 1$ in $\mathbb{Z}_+$. A proper element $b \in L$ is said to be n-absorbing to a proper element $p \in L$ if for all proper elements $x_1, \cdots, x_{n+1} \in L$ such that the product of $x_1, \cdots, x_{n+1}$ need not collapse to one of the elements among them, $x_1 \cdots x_{n+1} b \leqslant p$ imply there are n of x_i's whose product is $\leqslant p$.

Particularly, the above Definition 3.2 for $n = 2$ gives the concept of "an element being 2-absorbing to another element" in L.

Definition 3.3. A proper element $b \in L$ is said to be 2-1-absorbing prime to a proper element $p \in L$ if for all proper elements $x_1, x_2, x_3 \in L$ such that the product of x_1, x_2, x_3 need not collapse to one of the elements among them, $x_1 x_2 x_3 b \leqslant p$ imply either $x_1 x_2 \leqslant p$ or $x_3 \leqslant p$.

An element being 2-1-absorbing prime to another element of L is equivalent to saying that "an element being 1-absorbing prime to another element" of L.

Generalizing the above Definition 3.3, we get,

Definition 3.4. Let $n \geqslant 1$ in $\mathbb{Z}_+$. A proper element $b \in L$ is said to be n-1-absorbing prime to a proper element $p \in L$ if for all proper elements $x_1, \cdots, x_{n+1} \in L$ such that the product of $x_1, \cdots, x_{n+1}$ need not collapse to one of the elements among them, $x_1 \cdots x_{n+1} b \leqslant p$ imply either $x_1 \cdots x_n \leqslant p$ or $x_{n+1} \leqslant p$.

Definition 3.5. Given a reduction function $\phi : L \longrightarrow L \cup \{\emptyset\}$, a proper element $b \in L$ is said to be ϕ-1-absorbing to a proper element $p \in L$ if for all proper elements $x_1, x_2 \in L$ such that the product of x_1, x_2 need not collapse to one of the elements among them, $x_1 x_2 b \leqslant p$ and $x_1 x_2 b \nleqslant \phi(p)$ imply either $x_1 \leqslant p$ or $x_2 \leqslant p$.

Similarly, one can define "an element being ϕ-n-absorbing to another element" in L, where $n \geqslant 1$ in $\mathbb{Z}_+$.

Definition 3.6. Given a reduction function $\phi : L \longrightarrow L \cup \{\emptyset\}$, a proper element $b \in L$ is said to be ϕ-2-1-absorbing prime or ϕ-1-absorbing prime to a proper element $p \in L$ if for all proper elements $x_1, x_2, x_3 \in L$ such that the product of x_1, x_2, x_3 need not collapse to one of the elements among them, $x_1 x_2 x_3 b \leqslant p$ and $x_1 x_2 x_3 b \nleqslant \phi(p)$ imply either $x_1 x_2 \leqslant p$ or $x_3 \leqslant p$.

Similarly, one can define "an element being ϕ-n-1-absorbing prime to another element" in L, where $n \geqslant 1$ in $\mathbb{Z}_+$.

4 Conclusion remarks

The main aim of this paper is to generalize the concept of "an element being prime to another element" in a compactly generated multiplicative lattice L and obtain fundamental characterizations and properties related to it. The results investigated, though elementary, are significant, interesting and capable of developing its study in the future.

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