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ON HERMITE–HADAMARD TYPE INEQUALITIES AND ITS APPLICATIONS

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Abstract In this article, we present several results related to Hermite–Hadamard type inequalities for functions whose second-order derivatives are s-convex in the first kind. Using various techniques, including Hölder's inequality and the power mean inequality, we derive new inequalities that extend the classical Hermite–Hadamard framework. Additionally, we explore applications to special means and the midpoint formula, providing further insights into the behavior of s-convex functions and their practical implications.

1 Introduction

Inequalities represent a dynamic and impactful area of research. The practical significance of mathematical inequalities has been well-recognized over the years, contributing to the development of various branches of mathematics and other scientific disciplines. Convex functions establish an elegant and profound connection between analysis and geometry, often evoking a sense of challenge for the reader. The general theory of convex functions serves as a powerful tool for addressing problems in analysis. Inequalities involving convex functions are particularly effective in advancing many areas of mathematics, which has garnered substantial attention in the literature.

Convexity plays a significant role in our daily lives through numerous applications in industry, business, medicine, and art. Breckner [7] introduced the concept of s-convex functions, often referred to as s-convex functions of the second kind in the literature. Building on this, Noor et al. and Awan et al. proposed the definition of s-convex functions of the first kind (see [12]). Notably, both forms of s-convexity reduce to standard convexity when $s = 1$.

The classical Hermite–Hadamard inequality provides upper and lower bounds for the integral average of any convex function defined on a closed interval, using the midpoint and endpoints of the domain. Given the extensive applications of the Hermite–Hadamard inequality and convex functions, it is natural to explore further inequalities of this type involving convex functions. For additional studies on this topic, refer to [9, 10, 11, 15] and the references therein.

To get our generalizations, we start with the definitions and results used in our article.

Definition 1.1. [4] A function $f : [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ is called a convex function if, for any $\sigma, \varsigma \in [a_1, a_2]$ and $t_1 \in [0, 1]$, the following inequality holds:

$$f(t_1\sigma + (1 - t_1)\varsigma) \leq t_1 f(\sigma) + (1 - t_1)f(\varsigma).$$

Theorem 1.2. [8] *Let $f : I = [a_1, a_2] \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I and let $a_1, a_2 \in I$ with $a_1 < a_2$. Then*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma \leq \frac{f(a_1) + f(a_2)}{2}. \tag{1.1}$$

The corresponding inequality is well-known in the literature as the Hermite–Hadamard inequality for convex functions.

Definition 1.3. [5] A function $f : [0,\infty) \to \mathbb{R}$ is called an s-convex function in the first kind if it satisfies the inequality

$$f(t_1\sigma + (1-t_1)\varsigma) \leq {t_1}^s f(\sigma) + (1-{t_1}^s) f(\varsigma),$$

where $\sigma, \varsigma \in [0,\infty)$ and $t_1, s \in [0,1]$.

Here we take a convention that $0^0 = 1$.

This class of convexity is represented by K_1^s. It may be observed that if we put $s = 0$ and $s = 1$ in the above definition, then s-convexity changes into refinement of quasi convex and ordinary convex functions, respectively [3].

The convexity of function is the basis for numerous inequalities in mathematics. Note that, in latest problems linked to the convexity, generalized concepts about convex functions are required to obtain applicable results. One of this generalization is the concept of s-convex functions which can generalize numerous inequalities related to convex functions such as Hermite–Hadamard inequality, trapezoid and midpoint inequality, etc..

Lemma 1.4. [13] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I and let $a_1, a_2 \in I$ with $a_1 < a_2$. If $f'' \in L[a_1, a_2]$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right) \\
=\ & \frac{(a_2-a_1)^2}{16}\left[\int_0^1 {t_1}^2 f''\left(t_1\frac{a_1+a_2}{2} + (1-t_1)a_1\right)dt_1\right. \\
&\left. + \int_0^1 (t_1-1)^2 f''\left(t_1 a_2 + (1-t_1)\frac{a_1+a_2}{2}\right)dt_1\right]. \qquad (1.2)
\end{aligned}$$

Lemma 1.5. [1] *Assume that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the first kind with $s \in (0,1)$, and let $a_1, a_2 \in [0,\infty)$ with $a_1 < a_2$. If $f'' \in L[a_1, a_2]$, then the following inequality holds:*

$$f\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma \leq \frac{f(a_1)+sf(a_2)}{s+1}. \qquad (1.3)$$

The above inequalities are sharp.

Inspired by [14], this paper presents applications to special means and midpoint-type inequalities.

The primary objective of this paper is to establish new Hermite–Hadamard type inequalities for functions whose second derivatives, raised to certain powers, are s-convex in the first kind.

The well-known Hölder's inequality, in its general integral form, is stated as follows [2]:

Theorem 1.6. *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p$ and $\phi \in L_q$, then $f\phi \in L_1$ and*

$$\int |f(u)\phi(u)|du \leq \|f\|_p \|\phi\|_q, \qquad (1.4)$$

where $f \in L_p$ if $\|f\|_p = \left(\int |f(u)|^p du\right)^{\frac{1}{p}} < \infty$.

Note that if we set $p = q = 2$, the above inequality becomes the Cauchy–Schwarz inequality. Additionally, if we set $q = 1$ and let $p \to \infty$, we obtain

$$\int |f(u)\phi(u)|du \leq \|f\|_\infty \|\phi\|_1,$$

where $\|f\|_\infty$ represents the essential supremum of $|f|$, defined as

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\forall u} |f(u)|.$$

Definition 1.7. Let f and ϕ be real-valued functions defined on $[a_1, a_2]$. If $|f|$ and $|f||\phi|^q$ are integrable on $[a_1, a_2]$, then for $q \geq 1$, the following inequality holds:

$$\int_{a_1}^{a_2} |f(u)||\phi(u)|\,du \leq \left(\int_{a_1}^{a_2} |f(u)|\,du\right)^{1-\frac{1}{q}} \left(\int_{a_1}^{a_2} |f(u)||\phi(u)|^q\,du\right)^{\frac{1}{q}}. \tag{1.5}$$

This inequality is known as the power mean inequality (see [3]).

This article is organized as follows: In the next section, we estimate the bounds of a Hermite–Hadamard inequality by examining the absolute difference between the first and middle terms of (1.1), utilizing second-order differentiable s-convex functions of the first kind. These results include several findings from [13] as special cases. Sections 2 and 3 focus on applications to special means and midpoint formulas, respectively. The final section offers a conclusion, along with remarks and suggestions for future research directions.

2 Main Results

We will now present and prove three generalized results related to Hermite–Hadamard type inequalities for s-convex functions of the first kind, using Definition 1.3, Lemma 1.4, Lemma 1.5, Theorem 1.6, and Definition 1.7.

Theorem 2.1. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a_1, a_2]$, where $a_1, a_2 \in I^\circ$ and $a_1 < a_2$. If $|f''|$ is an s-convex in the first kind on $[a_1, a_2]$ for some fixed $s \in [0, 1]$, then following inequalities hold:*

$$\begin{aligned}
&\left|\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1 + a_2}{2}\right)\right| \leq \frac{(a_2 - a_1)^2}{16}\left[\frac{s\,|f''(a_1)|}{3(s+3)}\right. \\
&\left. + \left\{\frac{1}{3} + \frac{s}{(s+1)(s+2)}\right\}\left|f''\left(\frac{a_1 + a_2}{2}\right)\right| + \frac{2\,|f''(a_2)|}{(s+1)(s+2)(s+3)}\right] \qquad (2.1)\\
\leq\ & \frac{(a_2 - a_1)^2}{48(s+1)^2(s+2)(s+3)}\left[\left(s^4 + 5s^3 + 14s^2 + 22s + 6\right)|f''(a_1)|\right. \\
&\left. + \left(s^4 + 9s^3 + 20s^2 + 12s + 6\right)|f''(a_2)|\right]. \qquad (2.2)
\end{aligned}$$

Proof. Using Lemma 1.4 and the definition of the absolute value, we obtain

$$\begin{aligned}
&\left|\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1 + a_2}{2}\right)\right| \\
\leq\ & \frac{(a_2 - a_1)^2}{16}\left[\int_0^1 {t_1}^2\left|f''\left(t_1\frac{a_1 + a_2}{2} + (1 - t_1)a_1\right)\right|dt_1\right. \\
&\left. + \int_0^1 (1 - t_1)^2\left|f''\left(t_1 a_2 + (1 - t_1)\frac{a_1 + a_2}{2}\right)\right|dt_1\right]. \qquad (2.3)
\end{aligned}$$

According to the assumption, $|f''|$ is an s-convex function in the first kind, so we can take

$$\left|f''\left(t_1\frac{a_1 + a_2}{2} + (1 - t_1)a_1\right)\right| \leq {t_1}^s\left|f''\left(\frac{a_1 + a_2}{2}\right)\right| + (1 - {t_1}^s)\,|f''(a_1)|$$

and

$$\left|f''\left(t_1 a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right| \le {t_1}^s\left|f''(a_2)\right|+(1-{t_1}^s)\left|f''\left(\frac{a_1+a_2}{2}\right)\right|.$$

Utilizing above two results, (2.3) becomes

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right| \\
\le\ & \frac{(a_2-a_1)^2}{16}\left[\left|f''\left(\frac{a_1+a_2}{2}\right)\right|\int_0^1 ({t_1}^{s+2}+(1-t_1)^2(1-{t_1}^s))dt_1\right. \\
&\left.+\left|f''(a_1)\right|\int_0^1 ({t_1}^2-{t_1}^{s+2})dt_1+\left|f''(a_2)\right|\int_0^1 {t_1}^s(1-t_1)^2 dt_1\right].
\end{aligned}$$

By putting the values of the following facts, result of (2.1) is accomplished.

$$\int_0^1 ({t_1}^2-{t_1}^{s+2})dt_1=\frac{s}{(s+3)},$$

$$\int_0^1 ({t_1}^{s+2}+(1-t_1)^2(1-{t_1}^s))dt_1=\frac{1}{3}+\frac{s}{(s+1)(s+2)}$$

and

$$\int_0^1 {t_1}^s(1-t_1)^2 dt_1=\frac{2}{(s+1)(s+2)(s+3)}.$$

For the second inequality of Theorem 2.1, we know that $|f''|$ is an s-convex in the first kind, it must satisfies

$$\left|f''\left(\frac{a_1+a_2}{2}\right)\right| \le \frac{|f''(a_1)|+s\,|f''(a_2)|}{s+1}.$$

By applying the above inequality to (2.1), we acquired (2.2). □

Remark 2.2. By taking $s=1$ in Theorem 2.1, we acquire the result related to Hermite–Hadamard type inequality for convex function [13]. Also, if we take $s=0$ in Theorem 2.1, we we acquire the result related to Hermite–Hadamard type inequality for refinement of quasi convex function, i.e.,

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right| \le \frac{(a_2-a_1)^2}{48}\left[\left|f''\left(\frac{a_1+a_2}{2}\right)\right|+\left|f''(a_2)\right|\right] \\
\le\ & \frac{(a_2-a_1)^2}{48}\left[\left|f''(a_1)\right|+\left|f''(a_2)\right|\right].
\end{aligned}$$

Theorem 2.3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a_1,a_2]$, where $a_1,a_2 \in I^\circ$ and $a_1 < a_2$. If $|f''|^q$ is an s-convex in the first kind on $[a_1,a_2]$ for some fixed $s \in [0,1]$ and $q \ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following below stated inequalities* *hold:*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right| \le \frac{(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times$$

$$\left[\left(s\left|f''(a_1)\right|^q + \left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(s\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q + \left|f''(a_2)\right|^q\right)^{\frac{1}{q}}\right] \quad (2.4)$$

$$\le \frac{(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}}\left[\left((s^2+s+1)\left|f''(a_1)\right|^q + s\left|f''(a_2)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(s\left|f''(a_1)\right|^q + (s^2+s+1)\left|f''(a_2)\right|^q\right)^{\frac{1}{q}}\right] \quad (2.5)$$

$$\le \frac{(a_2-a_1)^2}{8}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a_1)\right|^q + \left|f''(a_2)\right|^q}{2}\right)^{\frac{1}{q}}. \quad (2.6)$$

Proof. By using Lemma 1.4 and the definition of the absolute value, we have

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\le \frac{(a_2-a_1)^2}{16}\left[\int_0^1 {t_1}^2\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|dt_1\right.$$

$$\left. + \int_0^1 (1-t_1)^2\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|dt_1\right]. \quad (2.7)$$

Applying (1.4) to

$$\int_0^1 {t_1}^2\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|dt_1$$

and

$$\int_0^1 (1-t_1)^2\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|dt_1,$$

we have

$$\int_0^1 {t_1}^2\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|dt_1$$

$$\le \left(\int_0^1 {t_1}^{2p}dt_1\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|^q dt_1\right)^{\frac{1}{q}}$$

and

$$\int_0^1 (1-t_1)^2\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|dt_1$$

$$\le \left(\int_0^1 (1-t_1)^{2p}dt_1\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|^q dt_1\right)^{\frac{1}{q}}.$$

According to the assumption, $|f''|^q$ is an s-convex function in the first kind, so we can take

$$\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|^q \le {t_1}^s\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q+(1-{t_1}^s)\left|f''(a_1)\right|^q$$

and

$$\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|^q \le {t_1}^s\left|f''(a_2)\right|^q+(1-{t_1}^s)\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q.$$

Utilizing above four results, (2.7) becomes

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right| \le \frac{(a_2-a_1)^2}{16}\times$$

$$\left[\left(\int_0^1 {t_1}^{2p}dt_1\right)^{\frac{1}{p}}\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q\int_0^1 {t_1}^s dt_1+\left|f''(a_1)\right|^q\int_0^1(1-{t_1}^s)dt_1\right)^{\frac{1}{q}}+\right.$$

$$\left.\left(\int_0^1(1-t_1)^{2p}dt_1\right)^{\frac{1}{p}}\left(\left|f''(a_2)\right|^q\int_0^1 {t_1}^s dt_1+\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q\int_0^1(1-{t_1}^s)dt_1\right)^{\frac{1}{q}}\right].$$

By putting the values of the following facts, result of (2.4) is accomplished.

$$\int_0^1 {t_1}^{2p}dt_1=\frac{1}{2p+1}=\int_0^1(1-t_1)^{2p}dt_1,$$

$$\int_0^1 {t_1}^s dt_1=\frac{1}{s+1}$$

and

$$\int_0^1(1-{t_1}^s)dt_1=\frac{s}{s+1}.$$

For the second inequality of Theorem 2.3, we know that $|f''|^q$ is an s-convex in the first kind, it must satisfies

$$\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q \le \frac{\left|f''(a_1)\right|^q+s\left|f''(a_2)\right|^q}{s+1}.$$

By applying the above inequality to (2.4), we acquired (2.5).

For the last result of Theorem 2.3, we have a function $h:[0,\infty)\to\mathbb{R}$, $h(y)=y^n$, $n\in(0,1]$, which is concave, we can write

$$\frac{{a_1}^n+{a_2}^n}{2}=\frac{h(a_1)+h(a_2)}{2}\le h\left(\frac{a_1+a_2}{2}\right)=\left(\frac{a_1+a_2}{2}\right)^n \tag{2.8}$$

$\forall\, a_1,a_2\ge 0$. For the above inequality, if we choose

$$a_1=(s^2+s+1)\left|f''(a_1)\right|^q+s\left|f''(a_2)\right|^q,\quad a_2=s\left|f''(a_1)\right|^q+(s^2+s+1)\left|f''(a_2)\right|^q$$

and $n=\dfrac{1}{q}$. Applying the inequality (2.8) to the inequality (2.5), we get (2.6). □

Remark 2.4. Theorem 2.3 gives

(i) Result related to Hermite–Hadamard type inequality for convex function by replacing $s = 1$, i.e.,

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right| \leq \frac{(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \times$$

$$\left[\left(\frac{|f''(a_1)|^q + \left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q + |f''(a_2)|^q}{2}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \times$$

$$\left[\left(\frac{3|f''(a_1)|^q + |f''(a_2)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f''(a_1)|^q + 3|f''(a_2)|^q}{4}\right)^{\frac{1}{q}}\right].$$

(ii) Result related to Hermite–Hadamard type inequality for refinement of quasi convex function by replacing $s = 0$, i.e.,

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \frac{(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left|f''\left(\frac{a_1+a_2}{2}\right)\right| + |f''(a_2)|\right]$$

$$\leq \frac{(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[|f''(a_1)| + |f''(a_2)|\right]$$

$$\leq \frac{(a_2-a_1)^2}{8}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a_1)|^q + |f''(a_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 2.5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a_1,a_2]$, where $a_1, a_2 \in I^\circ$ and $a_1 < a_2$. If $|f''|^q$ is an s-convex in the first kind on $[a_1,a_2]$ for some fixed $s \in [0,1]$ and $q \geq 1$, then the following below stated inequalities hold:*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \frac{(a_2-a_1)^2}{48}\left[\left(\frac{s}{s+3}|f''(a_1)|^q + \frac{3}{s+3}\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{s(s^2+6s+11)}{(s+1)(s+2)(s+3)}\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q + \frac{6|f''(a_2)|^q}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\right] \tag{2.9}$$

$$\leq \frac{(a_2-a_1)^2}{48}\left[\left(\frac{s^2+s+3}{(s+1)(s+3)}|f''(a_1)|^q + \frac{3s}{(s+1)(s+3)}|f''(a_2)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{s(s^2+6s+11)|f''(a_1)|^q}{(s+1)^2(s+2)(s+3)} + \frac{s^4+6s^3+11s^2+6s+6}{(s+1)^2(s+2)(s+3)}|f''(a_2)|^q\right)^{\frac{1}{q}}\right] \tag{2.10}$$

$$\leq \frac{(a_2-a_1)^2}{24\{2(s+1)^2(s+2)(s+3)\}^{\frac{1}{q}}}\left[(s^4+5s^3+14s^2+22s+6)|f''(a_1)|^q\right.$$

$$\left.+(s^4+9s^3+20s^2+12s+6)|f''(a_2)|^q\right]^{\frac{1}{q}}. \tag{2.11}$$

Proof. Using Lemma 1.4 and the definition of absolute value, we have

$$
\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right| \\
\le\ & \frac{(a_2-a_1)^2}{16}\left[\int_0^1 t_1{}^2\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|dt_1\right. \\
&\left.+\int_0^1 (1-t_1)^2\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|dt_1\right]. \qquad (2.12)
\end{aligned}
$$

Applying (1.7) to

$$\int_0^1 t_1{}^2\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|dt_1$$

and

$$\int_0^1 (1-t_1)^2\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|dt_1,$$

we have

$$
\begin{aligned}
&\int_0^1 t_1{}^2\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|dt_1 \\
\le\ & \left(\int_0^1 t_1{}^2dt_1\right)^{1-\frac{1}{q}}\left(\int_0^1 t_1{}^2\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|^q dt_1\right)^{\frac{1}{q}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 (1-t_1)^2\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|dt_1 \\
\le\ & \left(\int_0^1 (1-t_1)^2dt_1\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t_1)^2\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|^q dt_1\right)^{\frac{1}{q}}.
\end{aligned}
$$

According to the assumption, $|f''|^q$ is an s-convex function in the first kind, we can take

$$\left|f''\left(t_1\frac{a_1+a_2}{2}+(1-t_1)a_1\right)\right|^q \le t_1{}^s\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q+(1-t_1{}^s)\left|f''(a_1)\right|^q$$

and

$$\left|f''\left(t_1a_2+(1-t_1)\frac{a_1+a_2}{2}\right)\right|^q \le t_1{}^s\left|f''(a_2)\right|^q+(1-t_1{}^s)\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q.$$

Utilizing above four results, (2.12) becomes

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right|$$
$$\leq \frac{(a_2-a_1)^2}{16}\left[\left(\int_0^1 {t_1}^2 dt_1\right)^{1-\frac{1}{q}} \times\right.$$
$$\left(\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q \int_0^1 {t_1}^{s+2}dt_1 + |f''(a_1)|^q \int_0^1 ({t_1}^2 - {t_1}^{s+2})dt_1\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t_1)^2 dt_1\right)^{1-\frac{1}{q}} \times$$
$$\left.\left(|f''(a_2)|^q \int_0^1 {t_1}^s(1-t_1)^2 dt_1 + \left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q \int_0^1 (1-{t_1}^s)(1-t_1)^2 dt_1\right)^{\frac{1}{q}}\right].$$

By putting the values of the following facts, result of (2.9) is accomplished.

$$\int_0^1 {t_1}^2 dt_1 = \frac{1}{3} = \int_0^1 (1-t_1)^2 dt_1,$$

$$\int_0^1 {t_1}^{s+2} dt_1 = \frac{1}{s+3},$$

$$\int_0^1 ({t_1}^2 - {t_1}^{s+2})dt_1 = \frac{s}{3(s+3)},$$

$$\int_0^1 {t_1}^s (1-t_1)^2 dt_1 = \frac{2}{(s+1)(s+2)(s+3)}$$

and

$$\int_0^1 (1-{t_1}^s)(1-t_1)^2 dt_1 = \frac{s(s^2+6s+11)}{3(s+1)(s+2)(s+3)}.$$

For the second inequality of Theorem 2.5, we know that $|f''|^q$ is an s-convex in the first kind, it must satisfies

$$\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q \leq \frac{|f''(a_1)|^q + s\,|f''(a_2)|^q}{s+1}.$$

By applying the above inequality to (2.9), we acquired (2.10).

For the last result of Theorem 2.5, take

$$a_1 = \frac{s^2+s+3}{(s+1)(s+3)}\,|f''(a_1)|^q + \frac{3s}{(s+1)(s+3)}\,|f''(a_2)|^q,$$

$$a_2 = \frac{s(s^2+6s+11)}{(s+1)^2(s+2)(s+3)}\,|f''(a_1)|^q + \frac{s^4+6s^3+11s^2+6s+6}{(s+1)^2(s+2)(s+3)}\,|f''(a_2)|^q$$

and $n = \dfrac{1}{q}$ in (2.8) and after applying this inequality to (2.10), we get (2.11). □

Remark 2.6. Theorem 2.3 gives

(i) Result related to Hermite–Hadamard type inequality for convex function by replacing $s = 1$, i.e.,

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right| \leq \frac{(a_2-a_1)^2}{48} \times$$
$$\left[\left(\frac{|f''(a_1)|^q + 3\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q + |f''(a_2)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(a_2-a_1)^2}{48}\left[\left(\frac{5|f''(a_1)|^q + 3|f''(a_2)|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|f''(a_1)|^q + 5|f''(a_2)|^q}{8}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(a_2-a_1)^2}{24}\left(\frac{|f''(a_1)|^q + |f''(a_2)|^q}{2}\right)^{\frac{1}{q}},$$

where first inequality is the same result as we found in [13].

(ii) Also if we take $q = 1$ in (2.11), then (2.2) achieved.

(iii) Result related to Hermite–Hadamard type inequality for refinement of quasi convex function by replacing $s = 0$, i.e.,

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\sigma)d\sigma - f\left(\frac{a_1+a_2}{2}\right)\right| \leq \frac{(a_2-a_1)^2}{48}\left[\left|f''\left(\frac{a_1+a_2}{2}\right)\right| + |f''(a_2)|\right]$$
$$\leq \frac{(a_2-a_1)^2}{48}\left[|f''(a_1)| + |f''(a_2)|\right]$$
$$\leq \frac{(a_2-a_1)^2}{24}\left(\frac{|f''(a_1)|^q + |f''(a_2)|^q}{2}\right)^{\frac{1}{q}}.$$

3 Application to Special Means

We begin with the definition of the following special means, as taken from [6].
(1) The Arithmetic Mean:

$$A = A(a_1, a_2) = \frac{a_1+a_2}{2}; \quad a_1, a_2 \in [0, \infty).$$

(2) The p-Logarithmic Mean:

$$L_p = L_p(a_1, a_2) = \left[\frac{{a_2}^{p+1} - {a_1}^{p+1}}{(p+1)(a_2-a_1)}\right]^{\frac{1}{p}}; \quad a_1 \neq a_2,\ a_1, a_2 \in (0, \infty),$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$.

Next, we will establish some relationships between different means using the results obtained in the previous section.

Example 3.1. Let the function $f : [0, 1] \to [0, 1]$ be defined by $f(\sigma) = \sigma^s$. Then

(i) Theorem 2.1 becomes

$$|A^s(a_1, a_2) - L_s^s(a_1, a_2)| \leq \frac{s(s-1)(a_2-a_1)^2}{16} \times$$
$$\left[\frac{s{a_1}^{s-2}}{3(s+3)} + \left(\frac{s^2+6s+2}{3(s+1)(s+2)}\right)A^{s-2}(a_1, a_2) + \frac{2{a_2}^{s-2}}{(s+1)(s+2)(s+3)}\right]$$
$$\leq \frac{s(s-1)(a_2-a_1)^2}{48(s+1)^2(s+2)(s+3)}\left[(s^4+5s^3+14s^2+22s+6){a_1}^{s-2}\right.$$
$$\left.+(s^4+9s^3+20s^2+12s+6){a_2}^{s-2}\right].$$

(ii) Theorem 2.3 becomes

$$\begin{aligned}
&|A^s(a_1,a_2) - L_s^s(a_1,a_2)| \\
\leq\ & \frac{s(s-1)(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times \\
& \left[(sa^{qs} + A^{qs}(a_1,a_2))^{\frac{1}{q}} + (sA^{qs}(a_1,a_2) + {a_2}^{qs})^{\frac{1}{q}}\right] \\
\leq\ & \frac{s(s-1)(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}} \times \\
& \left[\left((s^2+s+1){a_1}^{qs} + sb^{qs}\right)^{\frac{1}{q}} + \left(sa^{qs} + (s^2+s+1){a_2}^{qs}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{s(s-1)(a_2-a_1)^2}{8}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(a_1^{qs}, {a_2}^{qs}\right).
\end{aligned}$$

(iii) Theorem 2.5 becomes

$$\begin{aligned}
&|A^s(a_1,a_2) - L_s^s(a_1,a_2)| \\
\leq\ & \frac{s(s-1)(a_2-a_1)^2}{48}\left[\left(\frac{s}{s+3}{a_1}^{qs} + \frac{3}{s+3}A^{qs}(a_1,a_2)\right)^{\frac{1}{q}}\right. \\
& \left. + \left(\frac{s(s^2+6s+11)}{(s+1)(s+2)(s+3)}A^{qs}(a_1,a_2) + \frac{6}{(s+1)(s+2)(s+3)}{a_2}^{qs}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{s(s-1)(a_2-a_1)^2}{48}\left[\left(\frac{s^2+s+3}{(s+1)(s+3)}{a_1}^{qs} + \frac{3s}{(s+1)(s+3)}{a_2}^{qs}\right)^{\frac{1}{q}}\right. \\
& \left. + \left(\frac{s(s^2+6s+11)}{(s+1)^2(s+2)(s+3)}{a_1}^{qs} + \frac{s^4+6s^3+11s^2+6s+6}{(s+1)^2(s+2)(s+3)}{a_2}^{qs}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{s(s-1)(a_2-a_1)^2}{24\left\{2(s+1)^2(s+2)(s+3)\right\}^{\frac{1}{q}}}\left[(s^4+5s^3+14s^2+22s+6){a_1}^{qs}\right. \\
& \left. +(s^4+9s^3+20s^2+12s+6){a_2}^{qs}\right]^{\frac{1}{q}}.
\end{aligned}$$

Remark 3.2. In a similar way as of Example 3.1, we can find various relationship of our obtained results with well known special means.

4 Application to Midpoint Formula

Let d be the division of the interval I such that $d : a_1 = \sigma_0 < \sigma_1 < \cdots < \sigma_{k-1} < \sigma_k = a_2$, f is integrable on $[a_1, a_2]$ and consider the quadrature formula

$$J = \int_{a_1}^{a_2} f(\sigma)d\sigma = M(f,d) + R(f,d),$$

where

$$M(f,d) = \sum_{k=0}^{n-1} f\left(\frac{\sigma_k + \sigma_{k+1}}{2}\right)(\sigma_{k+1} - \sigma_k)$$

is the midpoint formula and $R(f,d)$ denotes the associated approximation error of the interval I.

Now, we are going to drive some estimated for midpoint formula

Theorem 4.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a_1,a_2]$, where $a_1,a_2 \in I^\circ$ and $a_1 < a_2$. If $|f''|$ is an s-convex in the first kind on $[a_1,a_2]$ for some fixed $s \in [0,1]$, then the following inequalities hold:*

$$|R(f,d)| \leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{16} \left[\frac{s|f''(\sigma_k)|}{3(s+3)} + \left\{ \frac{1}{3} + \frac{s}{(s+1)(s+2)} \right\} \left| f''\left(\frac{\sigma_k+\sigma_{k+1}}{2} \right) \right| + \frac{2|f''(\sigma_{k+1})|}{(s+1)(s+2)(s+3)} \right]$$
$$\leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{48(s+1)^2(s+2)(s+3)} \left[(s^4+5s^3+14s^2+22s+6)|f''(\sigma_k)| + (s^4+9s^3+20s^2+12s+6)|f''(\sigma_{k+1})| \right].$$

Proof. We can write the inequality of Theorem 2.1 for $[\sigma_k, \sigma_{k+1}]$ $(k = 0, 1, \dots, k-1)$ of the division d,

$$\left| \int_{\sigma_k}^{\sigma_{k+1}} f(\sigma)d\sigma - (\sigma_{k+1}-\sigma_k) f\left(\frac{\sigma_k+\sigma_{k+1}}{2} \right) \right|$$
$$\leq \frac{(\sigma_{k+1}-\sigma_k)^3}{16} \left[\frac{s|f''(\sigma_k)|}{3(s+3)} + \left\{ \frac{1}{3} + \frac{s}{(s+1)(s+2)} \right\} \left| f''\left(\frac{\sigma_k+\sigma_{k+1}}{2} \right) \right| + \frac{2|f''(\sigma_{k+1})|}{(s+1)(s+2)(s+3)} \right]$$
$$\leq \frac{(\sigma_{k+1}-\sigma_k)^3}{48(s+1)^2(s+2)(s+3)} \left[(s^4+5s^3+14s^2+22s+6)|f''(\sigma_k)| + (s^4+9s^3+20s^2+12s+6)|f''(\sigma_{k+1})| \right].$$

By applying summation from $k = 0$ to $k = n-1$ to the above result, we obtain

$$\left| \int_{a_1}^{a_2} f(\sigma)d\sigma - M(f,d) \right| = \left| \sum_{k=0}^{n-1} \left[\int_{\sigma_k}^{\sigma_{k+1}} f(\sigma)d\sigma - (\sigma_{k+1}-\sigma_k) f\left(\frac{\sigma_k+\sigma_{k+1}}{2} \right) \right] \right|$$
$$\leq \sum_{k=0}^{n-1} \left| \left[\int_{\sigma_k}^{\sigma_{k+1}} f(\sigma)d\sigma - (\sigma_{k+1}-\sigma_k) f\left(\frac{\sigma_k+\sigma_{k+1}}{2} \right) \right] \right|$$
$$\leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{16} \left[\frac{s|f''(\sigma_k)|}{3(s+3)} + \left\{ \frac{1}{3} + \frac{s}{(s+1)(s+2)} \right\} \left| f''\left(\frac{\sigma_k+\sigma_{k+1}}{2} \right) \right| + \frac{2|f''(\sigma_{k+1})|}{(s+1)(s+2)(s+3)} \right]$$
$$\leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{48(s+1)^2(s+2)(s+3)} \left[(s^4+5s^3+14s^2+22s+6)|f''(\sigma_k)| + (s^4+9s^3+20s^2+12s+6)|f''(\sigma_{k+1})| \right].$$

□

Theorem 4.2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a_1,a_2]$, where $a_1,a_2 \in I^\circ$ and $a_1 < a_2$. If $|f''|^q$ is an s-convex in the first kind on $[a_1,a_2]$ for some fixed $s \in [0,1]$ and $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following below stated inequalities* *hold:*

$$|R(f,d)| \leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{16} \left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times$$

$$\left[\left(s|f''(\sigma_k)|^q + \left|f''\left(\frac{\sigma_k+\sigma_{k+1}}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(s\left|f''\left(\frac{\sigma_k+\sigma_{k+1}}{2}\right)\right|^q + |f''(\sigma_{k+1})|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{16} \left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \left(\frac{1}{s+1}\right)^{\frac{2}{q}} \left[\left((s^2+s+1)|f''(\sigma_k)|^q + s|f''(\sigma_{k+1})|^q\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(s|f''(\sigma_k)|^q + (s^2+s+1)|f''(\sigma_{k+1})|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{8} \left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \left(\frac{|f''(\sigma_k)|^q + |f''(\sigma_{k+1})|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. The proof follows with the same technique of the previous theorem, applying to the inequality of Theorem 2.3 on $[\sigma_k, \sigma_{k+1}]$. □

Theorem 4.3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a_1,a_2]$, where $a_1, a_2 \in I^\circ$ and $a_1 < a_2$. If $|f''|^q$ is an s-convex in the first kind on $[a_1,a_2]$ for some fixed $s \in [0,1]$ and $q \geq 1$, then the following below stated inequalities hold:*

$$|R(f,d)| \leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{48} \left[\left(\frac{s}{s+3}|f''(\sigma_k)|^q + \frac{3}{s+3}\left|f''\left(\frac{\sigma_k+\sigma_{k+1}}{2}\right)\right|^q\right)^{\frac{1}{q}} + \right.$$

$$\left.\left(\frac{s(s^2+6s+11)}{(s+1)(s+2)(s+3)}\left|f''\left(\frac{\sigma_k+\sigma_{k+1}}{2}\right)\right|^q + \frac{6}{(s+1)(s+2)(s+3)}|f''(\sigma_{k+1})|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{48} \left[\left(\frac{s^2+s+3}{(s+1)(s+3)}|f''(\sigma_k)|^q + \frac{3s}{(s+1)(s+3)}|f''(\sigma_{k+1})|^q\right)^{\frac{1}{q}} + \right.$$

$$\left.\left(\frac{s(s^2+6s+11)}{(s+1)^2(s+2)(s+3)}|f''(\sigma_k)|^q + \frac{s^4+6s^3+11s^2+6s+6}{(s+1)^2(s+2)(s+3)}|f''(\sigma_{k+1})|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \sum_{k=0}^{n-1} \frac{(\sigma_{k+1}-\sigma_k)^3}{24\left\{2(s+1)^2(s+2)(s+3)\right\}^{\frac{1}{q}}} \left[(s^4+5s^3+14s^2+22s+6)|f''(\sigma_k)|^q\right.$$

$$\left.+(s^4+9s^3+20s^2+12s+6)|f''(\sigma_{k+1})|^q\right]^{\frac{1}{q}}.$$

Proof. The proof follows with the same technique of Theorem 4.1, applying to the inequality of Theorem 2.5 on $[\sigma_k, \sigma_{k+1}]$. □

5 Conclusion and Remarks

5.1 Conclusion

The Hermite–Hadamard dual inequality is one of the most well-known inequalities, with numerous generalizations and variants found in the literature. In this paper, we have extended this inequality using the class of s-convex functions of the first kind. Consequently, our results encompass all known findings for convex functions and provide refinements for quasi-convex functions. In Section 2, we presented three different results estimating the bounds of the absolute difference between the left and middle terms of the Hermite–Hadamard dual inequality. We employed various techniques, including the power mean and Hölder's inequality, capturing several results previously discussed in [13]. Sections 3 and 4 focus entirely on applications related to well-known special means and midpoint formulas.

Now, we will provide some remarks and future ideas for readers.

5.2 Remarks and Future Ideas

(i) All the inequalities presented in this article can also be formulated in reverse for concave functions, using the simple relation that a function f is concave if and only if $-f$ is convex.

(ii) There is potential to explore the Fejér inequality by incorporating symmetric weights.

(iii) This work could be extended to the time scale domain.

(iv) It may be interesting to express all results stated in this article in a discrete context.

(v) Additionally, the results discussed here could be adapted for fractional calculus.

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