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A Note on Wiener index and Stress of a Graph

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Abstract. This paper introduces three new concepts related to a simple, connected, undirected graph: (i) the Modified distance matrix (β), (ii) the Shortest Path Counting matrix (μ), and (iii) the Modified Wiener index(ϕ). Accordingly the m^{th} order distance matrices β_m of β and m^{th} order shortest path counting matrices μ_m of μ for $1 \leq m \leq d$ are also defined. A new computation method to for determining β_m, μ_m, μ, β and ϕ is presented. Additionally, a novel formula for computing the Stress of a graph using these matrices is proposed.

1 Introduction

Wiener index introduced by Harry Wiener [6], is defined as the sum of the distances between all distinct unordered pairs of vertices in a graph. It is one of the oldest and widely studied topological indices, both from theoretical and applied perspectives. Analytical expressions for finding the Wiener index are known for many families of graphs. Several recent studies have investigated the computation of the Wiener index and related distance-based invariants for specific classes of graphs, including Isaac graphs and prime graphs of rings ([14], [15]). For instance, the Wiener index of the Wheel graph W_n is $(n-2)(n-1)$ and that of the path graph P_n is $n(n^2-1)/6$, among others. However, a general algorithm for computing the Wiener index of an arbitrary simple, connected, undirected graph has not yet been established. Moreover, Wiener index considers only the distance between each pair of vertices and ignores the number of distinct shortest paths between them. In many structural, chemical, and network-theoretic contexts, the number of distinct shortest paths plays an essential role in describing robustness, redundancy, reactivity, and information flow in the graph. To address this limitation, we introduce an extended framework that incorporates shortest-path multiplicity into distance-based graph analysis. Central to this approach is the shortest path counting matrix μ, whose entries record the number of distinct shortest paths between each pairs of vertices in the graph. Therefore it is natural to consider a revised form of the Wiener index that incorporates the count of distinct shortest paths as well. This, in turn, requires a corresponding modification of the distance matrix.

Stress of a vertex in a graph is the number of geodesics passing through that vertex, and the total Stress of the graph is the sum of the stress values of all its vertices. This centrality measure was introduced by A. Shimbel in 1953[11]. Analytical expressions for computing stress are known for certain families of graphs. In [12], K. Bhargava, N. N. Dattatreya and R. Rajendra, computed the stress values for standard graphs such as $K_{m,n}$ and C_n, and provided characterization of certain stress regular graphs. However, a general algorithm for determining the stress of an arbitrary simple, connected, undirected graph has not yet been established. This article addresses this gap by proposing generalized algorithms to compute stress index from the adjacency matrix, reducing reliance on graph-specific formulae and improving accessibility for broader applications.

Let $G = (V, E)$ be a simple, connected, undirected, finite graph of order n and diameter d, with adjacency matrix A_G throughout this paper.

Definition 1.1. [1] The *distance matrix* D of G is a square symmetric matrix defined by,

$$(D)_{ij} = d_{ij} = \begin{cases} d(v_i, v_j), \text{ if } i \neq j, \\ 0, \text{ otherwise.} \end{cases}$$

where $d(v_i, v_j)$ is the distance between v_i and v_j. The greatest distance between two vertices of G is called *diameter* of G and it is denoted by $Diam(G)$.

Definition 1.2. Hadamard Product[4] Consider the set of all $m \times n$ real matrices $\mathbb{R}^{m\times n}$. For $C, F \in \mathbb{R}^{m\times n}$, the *Hadamard product*(denoted by '$\circ$') of C and F is defined by, $(C \circ F)_{i,j} = (C)_{ij}.(F)_{ij}$, for all i, j.

Remark 1.3. Let $\mathbb{B}_{m\times n}$ denote the set of all $m \times n$ binary matrices of $\mathbb{R}^{m\times n}$. Then for $B_1, B_2 \in \mathbb{B}_{m\times n}$, $(B_1 \circ B_2)_{ij} = \begin{cases} 1, \text{if } (B_1)_{ij} = 1 \text{ and } (B_2)_{ij} = 1, \\ 0, \text{ otherwise.} \end{cases}$. Hence if $F \in \mathbb{B}_{n\times n}$, then $F \circ F = F$.

1.1 Splitting of the Distance Matrix

For convenience, we can decompose the distance matrix D of G into d symmetric matrices, where d is the diameter of G.

Definition 1.4. [4] For a simple, connected, undirected graph G with diameter d, the m^{th} *order distance matrix* σ_m $(1 \leq m \leq d)$ is defined by,

$$(\sigma_m)_{ij} = \begin{cases} m, \text{if } d(v_i, v_j) = m, \\ 0, \text{ otherwise.} \end{cases}$$

, where $d(v_i, v_j)$ denotes the distance between the vertices v_i and v_j.
We set $\sigma_0 = 0$. Note that $\sigma_1 = A_G$, the adjacency matrix of G. Note that each σ_m is a symmetric matrix of order n, for $1 \leq m \leq d$.

Property 1.5. The distance matrix $D = \sum_{m=1}^{d} \sigma_m$ and $\sigma_i \circ \sigma_j = \begin{cases} \sigma_i, \text{ if } i = j, \\ 0, \text{ otherwise.} \end{cases}$ for $1 \leq i, j \leq d$, where $\circ$ denotes the Hadamard product.

1.2 Binary Distance Matrices

Definition 1.6. [4] Let the function $\rho : \mathbb{R} \to B = \{0, 1\}$ be defined by,

$$\rho(a) = \begin{cases} 1, \text{if } a \neq 0, \\ 0, \text{ otherwise.} \end{cases} \quad a \in \mathbb{R}.$$

For a matrix $F \in \mathbb{R}^{m\times n}$, the ρ function is defined by, $\rho(F)_{ij} = \rho(f_{ij})$, $\forall$ i,j. In this way $\rho(F)$ represents the equivalent binary form of the matrix F.

Remark 1.7. If $F \in R^{n\times n}$, then $\rho(F^m) \in B_{n\times n}$ for all $m = 0, 1, 2, ...$, where $\rho(F^m)$ represents the equivalent binary matrix corresponding to F^m. The ij^{th} entry of A_G^m gives the number of walks of length m joining the vertices v_i and v_j. Accordingly, the binary matrix representation of A_G^m is,

$$\rho(A_G^m) = \begin{cases} 1, \text{ if } \exists \text{ a walk of length } m \text{ joining } v_i \text{ and } v_j, \\ 0, \text{ otherwise.} \end{cases}$$

Here the m^{th} order distance matrices σ_1, σ_2, ..., σ_d are not binary, except for σ_1. Therefore, we can construct their equivalent binary matrices by applying the function ρ.

Definition 1.8. m^{th} **order binary distance matrices:** [4] Let G be a simple, connected, undirected graph with diameter d. Let σ_1, σ_2, ..., σ_d denotes the m^{th} order distance matrices of G. Then the equivalent binary form of the m^{th} order distance matrix σ_m is called the m^{th} *order binary distance matrix* and it is denoted by D_m, $1 \leq m \leq d$.

$$D_m = \rho(\sigma_m) = \begin{cases} 1, \text{if } d(v_i, v_j) = m, \\ 0, \text{ otherwise.} \end{cases}$$

Note that $D_m = \frac{\sigma_m}{m}$, $1 \leq m \leq d$ and $D_0 = I_n$, $D_1 = A_G$.

Property 1.9. Since $D_m = \frac{\sigma_m}{m}$, it follows that $\sigma_m = m.D_m$. Therefore,

$$\sum_{m=1}^{d} m.D_m = \sum_{m=1}^{d} \sigma_m = D.$$

Also, since $D_0 = I_n$, $\sum\limits_{m=0}^{d} m.D_m = D$.

Definition 1.10. *Wiener index* [6] of a graph G is the sum of the distances between all the distinct unordered pairs of vertices of G, and it is denoted by $W(G)$.

$$W(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d(v_i, v_j).$$

2 Main Results

Wiener index does not take into account the number of distinct shortest paths between each pair of vertices. It only considers the distance, without recognizing how many different ways the shortest path can be traversed. To address this limitation, we propose a revised version of the Wiener index that incorporates the count of distinct shortest paths between each pair of vertices. This new approach provides a more comprehensive measure of a graph's connectivity, taking into account not just the distance but also the variety of ways vertices are connected through the shortest paths.

2.1 Shortest Path Counting Matrix

Definition 2.1. The *shortest path counting matrix* μ of G is defined by,

$$\mu_{ij} = \begin{cases} \text{The number of distinct shortest paths between } v_i \text{ and } v_j, \text{ if } i \neq j, \\ 0, \text{ if } i = j. \end{cases}$$

Then μ is an $n \times n$ symmetric matrix. The matrix μ can be decomposed into d $n \times n$ matrices as follows.

Definition 2.2. For $1 \leq m \leq d$, *The* m^{th} *order shortest path counting matrix* μ_m of G is defined by,

$$(\mu_m)_{ij} = \begin{cases} \mu_{ij}, \text{ if } d(v_i, v_j) = m, \\ 0, \text{ otherwise.} \end{cases}$$

Note that $\mu_0 = I_n$, the $n \times n$ identity matrix and $\mu_1 = A_G$, the adjacency matrix.

Property 2.3. The matrices μ_1, μ_2, ..., μ_d are $n \times n$ symmetric matrices and satisfy $\mu = \mu_1 + \mu_2 + ... + \mu_d$. Moreover, $\rho(\mu_m) = D_m$, for $m = 1, 2, .., d$.

Property 2.4. Let f_m denote the number of distinct shortest paths of length m. Then $f_m = \frac{1}{2} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} (\mu_m)_{ij}$, for $m = 1, 2, .., d$.

2.2 Modified Distance Matrix

The classical distance matrix records only the length of a shortest path between two vertices and ignores the multiplicity of such paths. However, in many graphs there may exist several distinct shortest paths of equal length between a given pair of vertices, and this information is lost in the traditional framework. The following modified distance matrix incorporates both the distance $d(v_i, v_j)$ and the number of distinct shortest paths μ_{ij}, thereby capturing the full contribution of all geodesic connections between vertex pairs.

Definition 2.5. Let μ_{ij} denote the number of the distinct shortest paths between the two distinct vertices v_i and v_j of G with distance $d(v_i, v_j)$. Then the *Modified distance matrix* β of G is defined by,

$$\beta_{ij} = \begin{cases} \mu_{ij}.d(v_i, v_j), \text{ if } i \neq j, \\ 0, \text{ otherwise.} \end{cases}$$
$$= \begin{cases} \text{The sum of the lengths of distinct shortest paths} \\ \text{connecting } v_i \text{ and } v_j, \text{ if } i \neq j, \\ 0, \text{ otherwise.} \end{cases}$$

Then β is an $n \times n$ symmetric matrix. Similar to the distance matrix, β can also be decomposed into d matrices of order $n \times n$ as follows.

Definition 2.6. For $1 \leq m \leq d$, the *modified m^{th} order distance matrix β_m* of G is defined by

$$(\beta_m)_{ij} = \begin{cases} \beta_{ij}, \text{ if } d(v_i, v_j) = m, \\ 0, \text{ otherwise.} \end{cases}$$
$$= \begin{cases} \text{The sum of the } \mu_{ij} \text{ number of distinct shortest paths} \\ \text{connecting } v_i \text{ and } v_j \text{ each with length } m, \text{ if } d(v_i, v_j) = m, \\ 0, \text{ otherwise.} \end{cases}$$

Note that $\beta_1 = A_G$, the adjacency matrix. The matrices β_1, β_2, ..., β_d are $n \times n$ symmetric matrices and they satisfy $\beta = \beta_1 + \beta_2 + ... + \beta_d$.

Property 2.7. $\mu_{ij} = \frac{\beta_{ij}}{d(v_i,v_j)}, \forall\, i, j$ and $\mu_m = \frac{\beta_m}{m}$, $m = 1, 2, ..., d$.

Property 2.8. $\beta = 1.\mu_1 + 2.\mu_2 + ... + d.\mu_d$ and $\mu = \frac{\beta_1}{1} + \frac{\beta_2}{2} + ... + \frac{\beta_d}{d}$.

Property 2.9. $\beta_m = \mu_m \circ \sigma_m$, for $1 \leq m \leq d$, where σ_m is the m^{th} order distance matrix and $\circ$ is the Hadamard product.

The classical Wiener index considers only the distance between each pair of vertices and does not account for the number of distinct shortest paths connecting them. To address this limitation, we define a new graph index by revising the Wiener index to include the count of distinct shortest paths between each pair of vertices. This modified index provides a more comprehensive measure of connectivity in a graph, capturing both the distances and the multiplicity of shortest paths.

Definition 2.10. The *Modified Wiener index* (Φ) of a graph G is defined as,

$$\Phi(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}.d(v_i, v_j)$$

, where μ_{ij} is the number of distinct shortest paths between vertices v_i and v_j of G, and $d(v_i, v_j)$ is the distance between them. By construction, the modified Wiener index satisfies $\phi(G) \geq W(G)$, where $W(G)$ is the Wiener index of G.

To compute the Modified Wiener index, we define a new type of matrix inverse as follows:

Definition 2.11. *The Hadamard inverse matrix* of a matrix $C \in \mathbb{R}^{m\times n}$ is defined by,

$$(C_H^{-1})_{ij} = \begin{cases} 1/c_{ij}, \text{ if } c_{ij} \neq 0, \\ 0, \text{ otherwise.} \end{cases}$$

Then $C \circ C_H^{-1} = C_H^{-1} \circ C = \rho(C)$, where $\circ$ is the Hadamard product and $\rho(C)$ is the binary form of the matrix C.

Property 2.12. For $1 \leq m \leq d$, $\mu_m = (\sigma_m)_H^{-1} \circ \beta_m$ and $\sigma_m = (\mu_m)_H^{-1} \circ \beta_m$, where $(\sigma_m)_H^{-1}$ and $(\mu_m)_H^{-1}$ denote the Hadamard inverses of σ_m and μ_m, respectively.

Definition 2.13. The function $\tau : \mathbb{R} \to \mathbb{R}$ defined by

$$\tau(a) = \begin{cases} a, \text{if } a \geq 0, \\ 0, \text{ otherwise.} \end{cases}$$

For a matrix $F \in \mathbb{R}^{n\times n}$, $(\tau(F))_{ij} = \tau(f_{ij}), \forall\, i, j$. In other words, the τ function replaces all the negative entries of the matrix F with zero and keeps all non negative entries unchanged.

2.3 Walk Sum

Let A_G be the adjacency matrix of G. Then the ij^{th} entry of A_G^m (the m^{th} power of A_G), represents the number of walks of length $m > 0$ between the vertices v_i and v_j of G. That is,

$$(A_G^m)_{ij} = \begin{cases} \text{the number of the walks of length } m \text{ joining the vertices } v_i \text{ and } v_j, \text{(if such walks exist)}, \\ 0, \text{ otherwise.} \end{cases}$$

Then ij^{th} entry of $m.A_G^m$ $(m > 0)$ represents the total sum of the lengths contributed by all these walks of length m joining the vertices v_i and v_j.

$$(m.A_G^m)_{ij} = \begin{cases} m\times \text{ the number of the walks with length } m \\ \text{joining } v_i \text{ and } v_j, \text{(if such walks exist)}, \\ 0, \text{ otherwise.} \end{cases}$$
$$= \begin{cases} \text{the sum of the lengths of the walks each with length } m \text{ joining} \\ v_i \text{ and } v_j, \text{(if such walks exist)}, \\ 0, \text{ otherwise.} \end{cases}$$

Let us call the matrix $(m.A_G^m)$ as m^{th} order Walk sum of the graph and let us denote it by W_{Σ_m}.

Definition 2.14. Let A_G be the adjacency matrix of a simple, connected, undirected graph G. Then the $m^{th}(m > 0)$ *order Walk sum* of G is defined by,

$$W_{\Sigma_m} = m.A_G^m$$

, which represent the sum of the lengths of the walks each with length m between the vertices. The zero order Walk sum of G is defined as $W_{\Sigma_0} = I_n$, the $n \times n$ identity matrix. The first order Walk sum of G is $W_{\Sigma_1} = 1.A_G^1 = A_G$, the adjacency matrix.

Theorem 2.15. *Let A_G be the adjacency matrix of a simple, connected, undirected graph G with diameter d. Then,*

$$\beta_m = \tau[W_{\Sigma_m} - \sum_{k=0}^{(m-1)} W_{\Sigma_m} \circ \rho(W_{\Sigma_k})]$$

for $1 \leq m \leq d$ and $\rho(W_{\Sigma_k})$ is the binary form of W_{Σ_k}.

Proof. Let $(1 \leq m \leq d)$. By the definition of k^{th} $(k > 0))$ order Walk sum,

$$(W_{\Sigma_k})_{ij} = \begin{cases} \text{The sum of the lengths of the walks each with length } k \\ \text{ connecting } v_i \text{ and } v_j, \text{ (if such walks exist),} \\ 0 \text{ , otherwise.} \end{cases}$$

$$\rho(W_{\Sigma_k})_{ij}) = \begin{cases} 1, \text{ if there exist a walk of length } k \text{ connecting } v_i \text{ and } v_j, \text{ (if such walks exist),} \\ 0, \text{ otherwise.} \end{cases}$$

$$(W_{\Sigma_m})_{ij} \circ \rho(W_{\Sigma_k})_{ij} = \begin{cases} (W_{\Sigma_m})_{ij}, \text{ if } \exists \text{ a walk of length } m \text{ and a walk of length } k \text{ connecting } v_i \text{ and } v_j, \\ 0, \text{ otherwise.} \end{cases}$$

$$\sum_{k=0}^{m-1} (W_{\Sigma_m})_{ij} \circ \rho(W_{\Sigma_k})_{ij} = \begin{cases} \sum\limits_{(W_{\Sigma_m})_{ij} \neq 0 \text{ , } \rho((W_{\Sigma_k})_{ij}) \neq 0, \, k<m} (W_{\Sigma_m})_{ij}, \text{ if } \exists \text{ a walk of} \\ \text{ length } m \text{ and a walk of length } k < m \text{ connecting } v_i \text{ and } v_j, \\ 0, \text{ otherwise.} \end{cases}$$

Let t_{ij} denote the number of walks of length m connecting v_i and v_j that can also be connected by a walk with length less than m. In other words, t_{ij} denotes the number of walks of length m such that $(W_{\Sigma_m})_{ij} \neq 0$ and $\rho((W_{\Sigma_k})_{ij}) = 1$ for some $k < m$. Then

$$\sum_{k=0}^{m-1} (W_{\Sigma_m})_{ij} \circ \rho(W_{\Sigma_k})_{ij} = \begin{cases} t_{ij} \times (W_{\Sigma_m})_{ij}, \text{ if } t_{ij} \geq 1, \\ 0, \text{ otherwise.} \end{cases}$$

Subtracting $\sum\limits_{k=0}^{(m-1)} (W_{\Sigma_m})_{ij} \circ \rho((W_{\Sigma_k})_{ij})$ from $(W_{\Sigma_m})_{ij}$ removes the sum of lengths of those walks with length m for which there also exists a walk of some shorter length $k < m$ connecting v_i and v_j. Since $W_{\Sigma_m} = m.A_G^m$,

$$\sum_{k=0}^{m-1} (W_{\Sigma_m})_{ij} \circ \rho(W_{\Sigma_k})_{ij} = \begin{cases} t_{ij} \times m \times (A_G^m)_{ij}, \text{ if } t_{ij} \geq 1, \\ \\ 0, \text{ otherwise.} \end{cases}$$

$$\begin{aligned} (W_{\Sigma_m})_{ij} - \sum_{k=0}^{m-1} (W_{\Sigma_m})_{ij} \circ \rho(W_{\Sigma_k})_{ij} &= (m.A_G^m)_{ij} - \begin{cases} t_{ij}.m.(A_G^m)_{ij}, \text{ if } t_{ij} \geq 1, \\ 0, \text{ otherwise.} \end{cases} \\ &= \begin{cases} (1 - t_{ij}) \times m \times (A_G^m)_{ij}, \text{ if } t_{ij} > 1, \\ m.(A_G^m)_{ij}, \text{ if } t_{ij} = 0, \\ 0, \text{ otherwise.} \end{cases} \end{aligned}$$

Case(i): $t_{ij} > 1$.
In this case there exist t_{ij} walks of length m connecting v_i and v_j for which there also exists at least one walk of length $< m$ joining the same pair of vertices. Then there cannot exist a shortest path of length m connecting v_i and v_j, because the presence of a walk of length less than m connecting v_i and v_j guarantees the existence of a shortest path connecting v_i and v_j whose length is also less than m. Therefore the quantity $(1 - t_{ij}) \times m \times (A^m)_{ij} < 0$ represents the negative of the total length associated with those walks of length m connecting v_i and v_j that cannot be connected by a shortest path with length m.
We eliminate this undesired negative quantity by applying the τ function to the matrix. When the τ function is applied, any entry of the form $(1 - t_{ij}).m.(A^m)_{ij} < 0$ is mapped to zero, while all non negative entries remain unchanged.
Case(ii): $t_{ij} = 0$.
In this case there exist no walk of length m connecting v_i and v_j that can also be connected by a walk with length less than m. Then the positive quantity $m \times (A^m)_{ij}$, (when $t_{ij} = 0$) will be correspond to the sum of the lengths of the shortest paths each with length m connecting v_i and

v_j in G (because each such path connecting v_i and v_j cannot be connected by a path with length less than m).

$$\therefore \left(\tau \left[(W_{\sigma_m})_{ij} - \sum_{k=0}^{m-1} (W_{\sigma_m})_{ij} \circ \rho((W_{\sigma_k})_{ij}) \right] \right)_{ij}$$

$$= \begin{cases} \text{The sum of the lengths of the shortest paths each with length } m \\ \text{ connecting } v_i \text{ and } v_j, \\ 0, \text{ otherwise.} \end{cases}$$

$$= (\beta_m)_{ij}, \ \forall\, i, j.$$

$$\therefore \beta_m = \tau \left[(W_{\sigma_m}) - \sum_{k=0}^{m-1} (W_{\sigma_m}) \circ \rho(W_{\sigma_k}) \right], 1 \leq m \leq d.$$

□

The above results can be comprised in the form of an algorithm as given below.

2.4 Algorithm for Computing the Wiener Index and Modified Wiener Index:

Let G be a simple, connected, undirected graph with order n, diameter d and adjacency matrix A_G.

(i) Find the m^{th} order walk sum $W_{\sigma_m} = m.A_G^m$, for $1 \leq m \leq d$. Let $W_{\sigma_0} = I_n$.

(ii) Compute the m^{th} order modified distance matrices β_m for $1 \leq m \leq d$ using the formula,

$$\beta_m = \tau \left[(W_{\sigma_m}) - \sum_{k=0}^{m-1} (W_{\sigma_m}) \circ \rho(W_{\sigma_k}) \right].$$

Then compute the modified distance matrix, $\beta = \beta_1 + \beta_2 + ... + \beta_d$.

(iii) Compute the m^{th} order shortest path counting matrix $\mu_m = \frac{\beta_m}{m}$, for $1 \leq m \leq d$.

(iv) Compute the Hadamard inverse $(\mu_m)_H^{-1}$ of the matrix μ_m, for $1 \leq m \leq d$ as follows,

$$((\mu_m)_H^{-1})_{ij} = \begin{cases} 1/(\mu_m)_{ij}, \text{ if } (\mu_m)_{ij} \neq 0 \\ 0, \text{ otherwise.} \end{cases}.$$

(v) Find m^{th} order distance matrix σ_m, for $1 \leq m \leq d$ using the formula,

$$\sigma_m = (\mu_m)_H^{-1} \circ \beta_m.$$

Then compute the distance matrix, $D = \sigma_1 + \sigma_2 + ... + \sigma_d$.

(vi) Finally compute ,

$$\text{the Wiener index, } W(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}.$$

$$\text{the modified Wiener index, } \Phi(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij}.$$

Example 2.16. Consider the following Graph G in Figure-1 with 8 vertices and diameter 4:

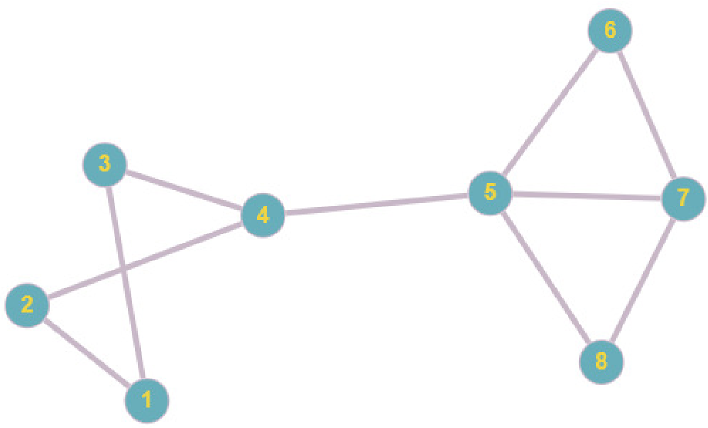

Figure 1. G=(V,E)

$$\text{Let } W_{\sigma_0} = I_8 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, W_{\sigma_1} = A_G = \begin{pmatrix} 0&1&1&0&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&0&0&1&0&1&1&1\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0&1\\ 0&0&0&0&1&0&1&0 \end{pmatrix}.$$

$$W_{\sigma_2} = 2.A^2 = \begin{pmatrix} 4&0&0&4&0&0&0&0\\ 0&4&4&0&2&0&0&0\\ 0&4&4&0&2&0&0&0\\ 4&0&0&6&0&2&2&2\\ 0&2&2&0&8&2&4&2\\ 0&0&0&2&2&4&2&4\\ 0&0&0&2&4&2&6&2\\ 0&0&0&2&2&4&2&4 \end{pmatrix}, W_{\sigma_3} = 3.A^3 = \begin{pmatrix} 0&12&12&0&6&0&0&0\\ 12&0&0&15&0&3&3&3\\ 12&0&0&15&0&3&3&3\\ 0&15&15&0&18&3&6&3\\ 6&0&0&18&12&18&18&18\\ 0&3&3&3&18&6&15&6\\ 0&3&3&6&18&15&12&15\\ 0&3&3&3&18&6&15&6 \end{pmatrix}.$$

$$W_{\sigma_4} = 4.A^4 = \begin{pmatrix} 32&0&0&40&0&8&8&8\\ 0&36&36&0&32&4&8&4\\ 0&36&36&0&32&4&8&4\\ 40&0&0&64&16&32&32&32\\ 0&32&32&16&96&40&64&40\\ 8&4&4&32&40&44&40&44\\ 8&8&8&32&64&40&64&40\\ 8&4&4&32&40&44&40&44 \end{pmatrix}.$$

We have $\beta_1 = A_G$.

$$W_{\sigma_2} - \sum_{k=0}^{k=1} W_{\sigma_2} \circ \rho(W_{\sigma_k}) = W_{\sigma_2} - (W_{\sigma_2} \circ I_8 + W_{\sigma_2} \circ A) = \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 \end{pmatrix}.$$

Since there are no negative entries in $W_{\sigma_2} - \sum_{k=0}^{k=1} W_{\sigma_2} \circ \rho(W_{\sigma_k})$,

$$\beta_2 = \tau(W_{\sigma_2} - \sum_{k=0}^{k=1} W_{\sigma_2} \circ W_{\sigma_k}) = \begin{pmatrix} 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 \end{pmatrix}.$$

$$W_{\sigma_3} - \sum_{k=0}^{k=2} W_{\sigma_3} \circ \rho(W_{\sigma_k}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\beta_3 = \tau(W_{\sigma_3} - \sum_{k=0}^{k=2} W_{\sigma_3} \circ \rho(W_{\sigma_k})) = \begin{pmatrix} 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$W_{\sigma_4} - \sum_{k=0}^{3} W_{\sigma_4} \circ \rho(W_{\sigma_k}) = \begin{pmatrix} -32 & 0 & 0 & 0 & 0 & 8 & 8 & 8 \\ 0 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -36 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -64 & -16 & -32 & -32 & -32 \\ 0 & 0 & 0 & -16 & -192 & -80 & -128 & -80 \\ 8 & 0 & 0 & -32 & -80 & -88 & -80 & -44 \\ 8 & 0 & 0 & -32 & -128 & -80 & -128 & -80 \\ 8 & 0 & 0 & -32 & -80 & -44 & -80 & -88 \end{pmatrix}.$$

$$\beta_4 = \tau(W_{\sigma_4} - \sum_{k=0}^{3} W_{\sigma_4} \circ \rho(W_{\sigma_0})) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\mu_1 = A_G$.

$$\mu_2 = \frac{\beta_2}{2} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

$$\mu_3 = \frac{\beta_3}{3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mu_4 = \frac{\beta_4}{4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Hadamard inverse of μ_2 is,

$$(\mu_2)_H^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, (\mu_3)_H^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$(\mu_4)_H^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ We have $\sigma_1 = A_G$.

$$\sigma_2 = (\mu_2)_H^{-1} \circ \beta_2 = \begin{pmatrix} 0&0&0&2&0&0&0&0\\ 0&0&2&0&2&0&0&0\\ 0&2&0&0&2&0&0&0\\ 2&0&0&0&0&2&2&2\\ 0&2&2&0&0&0&0&0\\ 0&0&0&2&0&0&0&2\\ 0&0&0&2&0&0&0&0\\ 0&0&0&2&0&2&0&0 \end{pmatrix}, \sigma_3 = (\mu_3)_H^{-1} \circ \beta_3 = \begin{pmatrix} 0&0&0&0&3&0&0&0\\ 0&0&0&0&0&3&3&3\\ 0&0&0&0&0&3&3&3\\ 0&0&0&0&0&0&0&0\\ 3&0&0&0&0&0&0&0\\ 0&3&3&0&0&0&0&0\\ 0&3&3&0&0&0&0&0\\ 0&3&3&0&0&0&0&0 \end{pmatrix}.$$

$$\sigma_4 = (\mu_4)_H^{-1} \circ \beta_4 = \begin{pmatrix} 0&0&0&0&0&4&4&4\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 4&0&0&0&0&0&0&0\\ 4&0&0&0&0&0&0&0\\ 4&0&0&0&0&0&0&0 \end{pmatrix}.$$

The the distance matrix, $D = \sum_{k=1}^{4} \sigma_k = \begin{pmatrix} 0&1&1&2&3&4&4&4\\ 1&0&2&1&2&3&3&3\\ 1&2&0&1&2&3&3&3\\ 2&1&1&0&1&2&2&2\\ 3&2&2&1&0&1&1&1\\ 4&3&3&2&1&0&1&2\\ 4&3&3&2&1&1&0&1\\ 4&3&3&2&1&2&1&0 \end{pmatrix}.$

Hence the Wiener index, $W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d(i,j) = 59.$

The Modified distance matrix, $\beta = \sum_{k=1}^{4} \beta_k = \begin{pmatrix} 0&1&1&4&6&8&8&8\\ 1&0&4&1&2&3&3&3\\ 1&4&0&1&2&3&3&3\\ 4&1&1&0&1&2&2&2\\ 6&2&2&1&0&1&1&1\\ 8&3&3&2&1&0&1&4\\ 8&3&3&2&1&1&0&1\\ 8&3&3&2&1&4&1&0 \end{pmatrix}.$

Thus the modified Wiener index, $\phi(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \beta(i,j) = \frac{1}{2} \times 160 = 80.$

2.5 Stress of a Graph: A new approach:

Stress of a vertex in a graph is the number of geodesics passing through that vertex and the total Stress of the graph is the sum of the stress values of all the vertices in the graph. Let G be a simple, connected, undirected graph with order n and diameter d.

2.6 Stress of a vertex: A new computation method

Let G be a simple, connected, undirected graph with order n and diameter d.
A path $v_i \to ... \to v_r \to ... \to v_j$ joining two vertices v_i and v_j through a vertex v_r $(i \neq j)$ is a shortest path joining v_i and v_j iff $d(v_i, v_r) + d(v_r, v_j) = d(v_i, v_j)$.
Let $d(v_i, v_r) = s$ and $d(v_r, v_j) = t$. Then a path combining two shortest paths $v_i \to ... \to v_r$ and

$v_r \rightarrow ... \rightarrow v_j$ can be counted for the stress of v_r iff $d(v_i, v_j) = s + t$.
Therefore the number of shortest paths joining v_i and v_j passing through v_r =
(The number of shortest paths joining v_i and v_r) $\times$ (The number of shortest paths joining v_r and v_j), counting only those paths for which the combined path $v_i \rightarrow ... \rightarrow v_r \rightarrow ... \rightarrow v_j$ is a shortest path joining v_i and v_j.

By the definition of the s^{th} order shortest path counting matrix(μ_s),

$$(\mu_s)_{ir} = \begin{cases} \text{The number of distinct shortest paths joining} \\ \ v_i \text{ and } v_r \text{ , if } d(v_i, v_r) = s, \\ 0, \text{ otherwise.} \end{cases}$$

Also,

$$(\mu_t)_{rj} = \begin{cases} \text{The number of distinct shortest paths joining} \\ \ v_r \text{ and } v_j \text{ , if } d(v_r, v_j) = t, \\ 0, \text{ otherwise.} \end{cases}$$

By the definition of $(s+t)^{th}$ order binary distance matrix,

$$(D_{s+t})_{ij} = \begin{cases} 1, \text{if } \exists \text{ a shortest path with distance } s+t \text{ joining} \\ v_i \text{ and } v_j, \\ 0, \text{ otherwise.} \end{cases}$$

Then, $(\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij} = \begin{cases} (\mu_s)_{ir} \times (\mu_t)_{rj}, \text{ if } (D_{s+t})_{ij} = 1, \\ 0, \text{ otherwise.} \end{cases}$

$$= \begin{cases} \big(\text{The number of distinct shortest paths joining } v_i \text{ and } v_r \text{ with} \\ \ \text{distance } s \ \big) \times \big(\text{The number of distinct shortest paths with} \\ \ \text{distance } t \text{ joining } v_r \text{ and } v_j\big), \text{ if } (D_{s+t})_{ij} = 1, (\mu_s)_{ir} \neq 0, (\mu_t)_{rj} \neq 0, \\ 0, \text{ otherwise.} \end{cases}$$

But each shortest paths with distance s joining v_i and v_r and each shortest path with distance t joining v_r and v_j can be joined together to form a shortest path with distance $s+t$ joining v_i and v_j, if $(D_{s+t})_{ij} = 1$.

$$\therefore (\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij} = \begin{cases} \text{The number of shortest paths with} \\ \ \text{distance } s+t \text{ joining } v_i \text{ and } v_j \text{ with} \\ \ v_r \text{ as internal vertex, if } (D_{s+t})_{ij} = 1, \\ 0, \text{ otherwise.} \end{cases}$$

Note that :

(i) The product $(\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij} \neq 0$ iff
$(\mu_s)_{ir} \neq 0$ (there exist non zero number of shortest paths with distance s joining v_i and v_r) and $(\mu_t)_{rj} \neq 0$ (there exist non zero number of shortest paths with distance t joining v_r and v_j) and $(D_{s+t})_{ij} \neq 0$ (there exist a shortest path joining v_i and v_j with distance $s+t$).

(ii) If $i = j$, then $(\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij} = 0$. ($\because (D_{s+t})_{ij} = 0$, when $i = j$.)

(iii) If $i = r$ or $j = r$, then $(\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij} = 0$. ($\because (\mu_s)_{rr} = 0$ or $(\mu_t)_{rr} = 0$.)

(iv) The total number of shortest paths joining v_i and v_j with v_r as internal vertex is obtained by taking the sum of the products of the form, $(\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij}$ by varying s, t as $1 \leq s \leq d-1, 1 \leq t \leq d-s$.

(v) The shortest path, $v_i \rightarrow ... \rightarrow v_r \rightarrow ... \rightarrow v_j$ joining v_i and v_j through v_r is again counted as the shortest path, $v_j \rightarrow ... \rightarrow v_r \rightarrow ... \rightarrow v_i$ joining v_j and v_i in the sum
$\sum_{s=1}^{d-1} \sum_{t=1}^{d-s} (\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij}$.

So the stress of the vertex v_r is obtained by taking half of the sum of the products of the form $(\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij}$ where the indices vary over $1 \leq i,j \leq n$, $1 \leq s \leq d-1$, and $1 \leq t \leq d-s$. Thus the new formula for computing the stress of a vertex v_r is given by the following expression:

$$str(v_r) := \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij}. \tag{2.1}$$

Theorem 2.17. *Let G be a simple, connected, undirected graph with order n and diameter d, and let v_r be a vertex of G. Then,*

$$str(v_r) = \frac{1}{2} \times \Big(\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(\mu_t.D_{s+t}).\mu_s\Big)_{rr}. \tag{2.2}$$

Proof. By equation(2.1),

$$\begin{aligned} str(v_r) &= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mu_s)_{ir}.(\mu_t)_{rj}.(D_{s+t})_{ij} \\ &= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}(\mu_s)_{ir}\sum_{j=1}^{n}(\mu_t)_{rj}.(D_{s+t})_{ji} \; (\because D_{s+t} \text{ is symmetric}) \\ &= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}(\mu_s)_{ir}(\mu_t D_{s+t})_{ri} \\ &= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}(\mu_t D_{s+t})_{ri}(\mu_s)_{ir} \\ &= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}((\mu_t D_{s+t}).\mu_s)_{r,r} \end{aligned}$$

Thus, $st(v_r) = \frac{1}{2} \times (r,r)^{th}$ diagonal entry of the matrix $\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(\mu_t.D_{s+t}).\mu_s$.

□

Remark 2.18. By interchanging the summations over i and j in equation (2.1), $str(v_r) = \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{j=1}^{n}\sum_{i=1}^{n}(\mu_s)_{ir} \times (\mu_t)_{rj} \times (D_{s+t})_{ij}$ and compute the same way as in Theorem(2.18), we get,

$$str(v_r) = \frac{1}{2} \times (r,r)^{th} \text{ diagonal entry of the matrix } \sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(\mu_s.D_{s+t}).\mu_t. \tag{2.3}$$

Remark 2.19. Let G be a simple, connected, undirected graph with vertex order n and diameter d. Then the total stress of the graph G is,

$$\begin{aligned} St(G) &= \sum_{r=1}^{n} str(v_r) \\ &= \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}((\mu_t D_{s+t}).\mu_s)_{r,r} \text{ (by equation(2.2)} \\ &= \frac{1}{2} \times \text{Trace of the matrix } \sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(\mu_t D_{s+t}).\mu_s. \end{aligned}$$

$$St(G) = \frac{1}{2} \times Tr\Big(\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(\mu_t D_{s+t}).\mu_s\Big). \tag{2.4}$$

In the same way by equation(2.3),

$$St(G) = \frac{1}{2} \times Tr\Big(\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(\mu_s D_{s+t}).\mu_t\Big). \tag{2.5}$$

.

Theorem 2.20. *Let G be a simple, connected, undirected graph with order n and diameter d. Then the total stress of the graph G is,*

$$St(G) := \frac{1}{2} Tr\Big(\sum_{s=1}^{d-1}\sum_{t=1}^{d-s} D_{s+t}.(\mu_s.\mu_t)\Big). \tag{2.6}$$

Proof.

$$\begin{aligned}
St(G) &= \sum_{r=1}^{n} str(v_r) \\
&= \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mu_s)_{ir}.(\mu_t)_{rj}.(D_{s+t})_{ij} \text{ , (by equation(2.1))} \\
&= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n}(\mu_s)_{ir}.(\mu_t)_{rj}.(D_{s+t})_{ij} \\
&= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}\sum_{j=1}^{n}(D_{s+t})_{ij}.\sum_{r=1}^{n}(\mu_s)_{ir}.(\mu_t)_{rj} \\
&= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{i=1}^{n}\sum_{j=1}^{n}(D_{s+t})_{ij}.(\mu_s.\mu_t)_{ij} \\
&= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{j=1}^{n}\sum_{i=1}^{n}(D_{s+t})_{ji}.(\mu_s.\mu_t)_{ij} \\
&= \frac{1}{2}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}\sum_{j=1}^{n}(D_{s+t}.(\mu_s.\mu_t))_{jj} \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(D_{s+t}.(\mu_s.\mu_t))_{jj} \\
\therefore St(G) &= \frac{1}{2} Tr\Big(\sum_{s=1}^{d-1}\sum_{t=1}^{d-s} D_{s+t}.(\mu_s.\mu_t)\Big).
\end{aligned}$$

□

Remark 2.21. Total stress of a simple, connected, undirected graph G with diameter d can be computed by any of the following three formulae:

$$St(G) = \frac{1}{2} Tr\Big(\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(\mu_t.D_{s+t}).\mu_s\Big).$$

$$St(G) = \frac{1}{2} Tr\Big(\sum_{s=1}^{d-1}\sum_{t=1}^{d-s}(\mu_s.D_{s+t}).\mu_t\Big).$$

$$St(G) = \frac{1}{2} Tr\Big(\sum_{s=1}^{d-1}\sum_{t=1}^{d-s} D_{s+t}.(\mu_s.\mu_t)\Big).$$

Remark 2.22. Let G be a simple, disconnected undirected graph with connected components $K_1, K_2, ..., K_p$. Then $St(G) = St(K_1) + St(K_2) + ... + St(K_p)$.

Remark 2.23. Let Ω_v denote the set of all simple, undirected graphs on the vertex set V. Define a relation $\sim$ on Ω_v by $G \sim H$ if $St(G) = St(H)$, for G, $H \in \Omega$. Then this relation will be an equivalence relation on Ω_v. Let $[G_1]$, $[G_2]$, ..., $[G_F]$ be the associated equivalence classes, where F is the maximum stress value.

2.7 Algorithm for finding the Stress of a graph:

Let G be a simple, connected, undirected graph with order n, diameter d and adjacency matrix A_G.

(i) Find the m^{th} order walk sum $W_{\Sigma_m} = m.(A_G)^m$, for $1 \leq m \leq d$. Let $W_{\Sigma_0} = I_n$.

(ii) Compute the m^{th} order modified distance matrices $\beta_1, \beta_2, ..., \beta_d$ using the formula in Theorem(2.15), $\beta_m = \tau \left[(W_{\Sigma_m}) - \sum_{k=0}^{m-1} (W_{\Sigma_m}) \circ \rho(W_{\Sigma_k}) \right], (1 \leq m \leq d)$.

(iii) Compute the m^{th} order shortest path counting matrix, $\mu_m = \frac{\beta_m}{m}$ and then the m^{th} order binary distance matrix, $D_m = \rho(\mu_m)$, for $1 \leq m \leq d$.

(iv) Compute any one of the following matrix to compute the stress of the vertex v_r, $1 \leq r \leq n$.

$$\sum_{s=1}^{d-1} \sum_{t=1}^{d-s} (\mu_t.D_{s+t}).\mu_s \text{ or } \sum_{s=1}^{d-1} \sum_{t=1}^{d-s} (\mu_s.D_{s+t}).\mu_t.$$

Then $st(v_r) = \frac{1}{2} \times (r,r)^{th}$ diagonal entry of this matrix.

(v) Find the total Stress of the graph G, $St(G) = \sum_{r=1}^{n} st(v_r)$.

Example 2.24. Consider the graph in Figure-1 of Example(2.17). We have already computed the matrices μ_1, μ_2, μ_3 and μ_4 in Example(2.17). Since $D_m = \rho(\mu_m)$, for $1 \leq m \leq d$, we obtain the following:

$$D_1 = \mu_1 = A_G = \begin{pmatrix} 0&1&1&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 0&1&1&0&1&0&0&0 \\ 0&0&0&1&0&1&1&1 \\ 0&0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0&1 \\ 0&0&0&0&1&0&1&0 \end{pmatrix}, D_2 = \rho(\mu_2) = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 1&0&0&0&0&1&1&1 \\ 0&1&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix}.$$

$$D_3 = \rho(\mu_3) = \begin{pmatrix} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \end{pmatrix}, D_4 = \rho(\mu_4) = \begin{pmatrix} 0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \end{pmatrix}.$$

Now, $\sum_{s=1}^{3}\sum_{t=1}^{4-s}(\mu_t.D_{s+t}).\mu_s = \begin{pmatrix} 2 & 0 & 0 & 10 & 0 & 6 & 2 & 6 \\ 0 & 10 & 10 & 0 & 10 & 3 & 6 & 3 \\ 0 & 10 & 10 & 0 & 10 & 3 & 6 & 3 \\ 10 & 0 & 0 & 34 & 0 & 12 & 4 & 12 \\ 0 & 10 & 10 & 0 & 32 & 5 & 12 & 5 \\ 6 & 3 & 3 & 12 & 5 & 0 & 0 & 0 \\ 2 & 6 & 6 & 4 & 12 & 0 & 2 & 0 \\ 6 & 3 & 3 & 12 & 5 & 0 & 0 & 0 \end{pmatrix}.$

$$st(v_r) = \frac{1}{2} \times ((r,r)^{th} \text{ diagonal entry of this matrix.})$$

$$\therefore st(v_1) = \frac{1}{2} \times 2 = 1,\ st(v_2) = \frac{1}{2} \times 10 = 5,\ st(v_3) = \frac{1}{2} \times 10 = 5.$$

$$st(v_4) = \frac{1}{2} \times 34 = 17,\ st(v_5) = \frac{1}{2} \times 32 = 16.$$

$$st(v_6) = \frac{1}{2} \times 0 = 0,\ st(v_7) = \frac{1}{2} \times 2 = 1,\ st(v_8) = \frac{1}{2} \times 0 = 0.$$

Now , $St(G) = \sum_{r=1}^{8} st(v_r) = 45.$

3 Conclusion remarks

This article introduces algorithms for computing modified distance matrix(β), shortest path counting matrix(μ), Wiener index(W), modified Wiener index(ϕ), and Stress index of a simple, connected, undirected graph. Generally it is very difficult to find all these items for a graph with large order. Determining these quantities becomes increasingly difficult as the order of the graph grows. A computer implementation of these algorithms would allow us to compute these matrices and graph indices instantly using only the adjacency matrix as input. Further research is required to extend these algorithms to general simple, connected, undirected weighted graphs.

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