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# On Graded $\alpha$-Prime Ideals

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**Abstract**. Let $G$ be a group, $A$ be a commutative graded ring with nonzero unity and $\alpha : A \rightarrow A$ be a fixed graded ring homomorphism. The goal of our article is introducing a newly develoed induction of graded prime ideals, that is graded $\alpha$-prime ideal. A proper graded ideal $I$ of $A$ is said to be a graded $\alpha$-prime ideal if for all homogeneous elements $x, y$ of $A$ with $xy \in I$, we have either $x \in I$ or $\alpha(y) \in I$. We examine some properties of graded $\alpha$-prime ideals comparable to graded prime ideals.

## 1 Introduction

Throughout this article, the following abbreviations will be used:

(i) gr: graded

(ii) prime: pr

(iii) graded ring: gr-R

(iv) such that: s.t.

Among this article, $G$ is a group and $A$ is a commutative ring with nonzero unity 1. We call $A$ a $G-gr-R$ if $A = \bigoplus_{g\in G} A_g$ with the possession $A_gA_h \subseteq A_{gh}$ for each $g, h \in G$, where $A_g$ is an additive subgroup of $A$ for all $g \in G$. The elements of $A_g$ are called homogeneous of degree $g$. If $w \in A$, then $w$ is written uniquely as $\sum_{g\in G} w_g$, where $w_g$ is the component of $w$ in $A_g$ and $w_g = 0$ except finitely many. The set of all homogeneous elements of $A$ is $\bigcup_{g\in G} A_g$ and is signified by $h(A)$. The component $A_e$ is a subring of $A$ and $1 \in A_e$.

Let $A$ be a $gr-R$ and $I$ be an ideal of $A$. Then $I$ is said to be a gr-ideal if $I = \bigoplus_{g\in G}(I \bigcap A_g)$, i.e; for $w \in I$, $w_g \in I$ for all $g \in G$. An deal of a $gr-R$ is not necessarily gr-ideal. For $G-gr-R$'s $A$ and $B$, a ring homomorphism $\alpha : A \rightarrow B$ is called to be a gr-homomorphism if $\alpha(A_g) \subseteq B_g$ for all $g \in G$. For more terminology, see [4, 5, 11].

A proper gr-ideal $I$ of $A$ is said to be a gr-pr-ideal if $wt \in I$ implies $w \in I$ or $t \in I$ for all $w, t \in h(A)$ ([13]). The concept of a gr-pr-ideal performs an indispensable position in the theory of commutative $gr-R$, and it has been extensively studied. Latterly, many generalizations of gr-pr-ideals were proposed and investigated, for example, see [1, 2, 9, 10].

For a gr-ideal $I$ of $A$, the gr-radical of $I$ is expressed as $Grad(I)$ and it is established as :

$$Grad(I) = \{w = \sum_{g\in G} w_g \in A : \forall g \in G, \exists n_g \in \mathbb{N} \text{ s.t. } w_g^{n_g} \in I\}.$$

Note that $Grad(I)$ is always a gr-ideal of $A$ (see [13]). A proper gr-ideal $I$ of $A$ is called gr-primary if $wt \in I$ implies $w \in I$ or $t \in Grad(I)$ for all $w, t \in h(A)$ ([12]).

The concept of $\alpha$-pr-ideals has been introduced in [6]. A proper ideal $I$ of $A$ is said to be an $\alpha$-prime ideal if for all $w, t \in A$ with $wt \in I$, we have either $w \in I$ or $\alpha(t) \in I$, where $\alpha : A \to A$ is a fixed ring homomorphism. In this article, we propose the concept of gr-$\alpha$-prime ideals, and study some of its properties. Let $A$ be a $gr - R$, $I$ be a proper gr-ideal of $A$ and $\alpha : A \to A$ be a fixed gr-R homomorphism. Then $I$ is said to be a gr-$\alpha$-prime ideal if for all $w, t \in h(A)$ with $wt \in I$, we have either $w \in I$ or $\alpha(t) \in I$. Certainly, if $I$ is an $\alpha$-pr-ideal of $A$, $I$ is a gr-ideal of $A$ and $\alpha : A \to A$ is a $gr - R$ homomorphism, then $I$ is a gr-$\alpha$-pr-ideal of $A$. However, we show that a gr-$\alpha$-pr-ideal is not necessarily $\alpha$-prime (Example 2.2). So, it will be meritorious to examine and investigate gr-$\alpha$-prime ideals. Indeed, we follow [6] to examine some properties of gr-$\alpha$-prime ideals comparable to gr-pr-ideals. In consideration of the definition of a gr-$\alpha$-prime ideal, we see that in the case when $\alpha$ is the identity map, gr-$\alpha$-prime ideal and gr-pr-ideal coincide. So, gr-$\alpha$-pr-ideals are recognized as a generalization of gr-pr-ideals. Evidently, every gr-pr-ideal is a gr-$\alpha$-pr-ideal, where $\alpha$ is the identity map. However, we show that a gr-$\alpha$-pr-ideal is not necessarily gr-pr (Example 2.3). Among a variety of outcomes, we show that $I$ is a gr-$\alpha$-pr-ideal of $A$ iff any two gr-ideals $P$ and $K$ of $A$ such that $PK \subseteq I$, either $P \subseteq I$ or $\alpha(K) \subseteq I$ (Theorem 2.10). We prove that if $f : A \to B$ is a gr-R epimorphism, $\alpha$ is a $gr - R$ homomorphism from $A$ to $A$ and from $B$ to $B$, $\alpha$ commutes with $f$, and $I$ is a gr-ideal of $A$ such that $Ker(f) \subseteq I$, then $I$ is a gr-$\alpha$-pr-ideal of $A$ iff $f(I)$ is a gr-$\alpha$-pr-ideal of $B$ (Proposition 2.13). We prove that $Ker(\alpha)$ is in the intersection of all gr-$\alpha$-prime ideals of $A$ (Lemma 2.15). We propose the gr-$\alpha$-radical; $w \in Grad_{\alpha}(I)$ if for all $g \in G$, $\exists$ a positive integer $n_g$ s.t $\alpha(w_g^{n_g}) \in I$, we examine some properties of $Grad_{\alpha}(I)$ in Proposition 2.20 and Proposition 2.21, and then we introduce the concept of graded $\alpha$-primary ideals; a proper gr-ideal $I$ of $A$ is called gr-$\alpha$-primary if $wt \in I$ implies either $w \in I$ or $t \in Grad_{\alpha}(I)$, for $w, t \in h(A)$. We introduce gr- $\alpha$-domains; $A$ is called gr-$\alpha$-domain if $wt = 0$ implies either $w = 0$ or $\alpha(t) = 0$, for $w, t \in h(A)$. Indeed, every gr-domain will be a gr-$\alpha$-domain, but the other side is not essentially true (Example 2.25). We demonstrate if $A$ is a gr-$\alpha$-domain, then $Ker(\alpha)$ is a gr-$\alpha$-pr-ideal of $A$ (Proposition 2.27). At the end, we propose gr-$\alpha$-fields; $A$ is called gr-$\alpha$-field if $A/Ker(\alpha)$ is a gr-field. Actually, every gr-field will be a gr-$\alpha$-field (Proposition 2.30), The opposite is not always true. (Example 2.31).

## 2 Graded $\alpha$-Prime Ideals, Graded $\alpha$-Primary Ideals, Graded $\alpha$-Domains and Graded $\alpha$-Fields

In this section, we propose and examine the concepts of gr-$\alpha$-prime ideals, gr-$\alpha$-primary ideals, gr-$\alpha$-domains and gr-$\alpha$-fields.

**Definition 2.1.** *Let $A$ be a $gr - R$, $I$ be a proper gr-ideal of $A$ and $\alpha : A \to A$ be a fixed $gr - R-$homomorphism. Then $I$ is said to be a gr-$\alpha$-pr-ideal if for all $w, y \in h(A)$ with $wy \in I$, we have either $w \in I$ or $\alpha(y) \in I$.*

Certainly, if $I$ is an $\alpha$-pr-ideal of $A$, $I$ is a gr-ideal of $A$ ,and $\alpha$ is a gr-homomorphism, then $I$ is a gr-$\alpha$-pr-ideal of $A$. However, the next example shows that a gr-$\alpha$-pr-ideal is not necessarily $\alpha$-pr:

**Example 2.2.** *Consider $A = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then $A$ is $G - gr$ by $A_0 = \mathbb{Z}$ and $A_1 = i\mathbb{Z}$. Examine the gr-ideal $I = pA$ of $A$, where $p$ is a prime number with $p = c^2 + d^2$, for some $c, d \in \mathbb{Z}$. Consider the gr-homomorphism $\alpha : A \to A$ with $\alpha(a + ib) = a - ib$. Let $wy \in I$ for some $w, y \in h(A)$.*

*Case (1): Assume that $w, y \in A_0$. In this scenario, $w, y \in \mathbb{Z}$ s.t $p$ divides $wy$, and then either $p$ divides $w$ or $p$ divides $y$. It indicates $w \in I$ or $y \in I$. So, $w \in I$ or $\alpha(y) = y \in I$.*

*Case (2): Assume that $w, y \in A_1$. In this scenario, $w = ia$ and $y = ib$ for some $a, b \in \mathbb{Z}$ s.t $p$ divides $wy = -ab$, and then $p$ divides $a$ or $p$ divides $b$ in $\mathbb{Z}$, it indicates $p$ divides $w = ia$ or $p$ divides $y = ib$ in $A$. then we get $w \in I$ or $y \in I$. So, $w \in I$ or $\alpha(y) = -y \in I$.*

*Case (3): Assume that $w \in A_0$ and $y \in A_1$. In this scenario, $w \in \mathbb{Z}$ and $y = ib$ for some $b \in \mathbb{Z}$, so $p$ divides $wy = iwb$ in $A$, that is $iwb = p(\alpha + i\beta)$ for some $\alpha, \beta \in \mathbb{Z}$. then we get $wb = p\beta$, that is $p$ divides $wb$ in $\mathbb{Z}$, and again $p$ divides $w$ or $p$ divides $b$, it indicates $p$ divides $w$ or $p$ divides $y = ib$ in $A$. Thus, $w \in I$ or $y \in I$. So, $w \in I$ or $\alpha(y) = -y \in I$.*

*Hence, $I$ is a gr-$\alpha$-prime ideal of $A$. But $I$ is not $\alpha$-pr since $(c-id)(c+id)\in I$, $(c-id)\notin I$ and $\alpha(c+id)=c-id\notin I$.*

Evidently, every gr-prime ideal is a gr-$\alpha$-pr-ideal, where $\alpha$ is the identity map. However, the next example shows that a gr-$\alpha$-pr-ideal is not necessarily gr-prime. So, gr-$\alpha$-pr-ideals are recognized as a generalization of gr-pr-ideals.

**Example 2.3.** *Consider $A=\mathbb{Z}[X]$ and $G=\mathbb{Z}$. Then $A$ is $G-gr$ by $A_j=\mathbb{Z}X^j$, where $j\geq 0$ and $A_j=0$ otherwise. Consider the gr-ideal $I=\langle 3X\rangle$ of $A$. Clearly, $I$ is not a gr-pr-ideal of $A$ since $3,X\in h(A)$ with $3X\in I$, $3\notin I$ and $X\notin I$. Consider the gr-homomorphism $\alpha:A\to A$ with $\alpha(f(X))=f(0)$. Let $f(X),g(X)\in h(A)$ with $f(X)g(X)\in I$. Then $3X$ divides $f(X)g(X)$, and then $X$ divides $f(X)g(X)$ it indicates $X$ divides $f(X)$ or $X$ divides $g(X)$. If $X$ divides $f(X)$, then $f(X)=Xk(X)$ for some $k(X)\in A$, and then $\alpha(f(X))=f(0)=0\in I$. Similarly, if $X$ divides $g(X)$, then $\alpha(g(X))=0\in I$. Hence, $I$ is a gr-$\alpha$-pr-ideal of $A$.*

**Proposition 2.4.** *Let $A$ be a $gr-R$ and $I$ be a gr-$\alpha$-pr-ideal of $A$. Then $\alpha(I)\subseteq I$.*

*Proof.* Let $w\in I$. Then $w_g\in I$ for all $g\in G$ as $I$ is a gr-ideal. So, $1.w_g=w_g\in I$ indicates $\alpha(w_g)\in I$ for all $g\in G$ as $I$ is gr-$\alpha$-pr, and hence $\alpha(w)=\alpha\left(\sum_{g\in G}w_g\right)=\sum_{g\in G}\alpha(w_g)\in I$. Therefore, $\alpha(I)\subseteq I$. □

**Proposition 2.5.** *Let $A$ be a $gr-R$ and $I$ be a gr-$\alpha$-pr-ideal of $A$. Then $Grad(I)$ is a gr-$\alpha$-pr-ideal of $A$.*

*Proof.* Let $w,y\in h(A)$ with $wy\in Grad(I)$. Then $w^ky^k=(wy)^k\in I$ for some positive integer $k$, and then since $I$ is gr-$\alpha$-pr, either $w^k\in I$ or $(\alpha(y))^k=\alpha(y^k)\in I$, it indicates either $w\in Grad(I)$ or $\alpha(y)\in Grad(I)$. Therefore, $Grad(I)$ is a gr- $\alpha$-pr-ideal of $A$. □

**Lemma 2.6.** *Let $A$ be a $gr-R$ and $I$ be a $gr-\alpha-pr-ideal$ of $A$. Then $S_I=\{x\in A:\alpha(x)\in I\}$ is a gr-ideal of $A$ containing $I$.*

*Proof.* Clearly, $S_I$ is an ideal of $A$ containing $I$. Let $x\in S_I$. Then $x\in A$ with $\alpha(x)\in I$. Now, $x=\sum_{g\in G}x_g$ where $x_g\in A_g$ for all $g\in G$, it indicates $\alpha(x_g)\in\alpha(A_g)\subseteq A_g$ for all $g\in G$. So, $\alpha(x_g)\in h(A)$ for all $g\in G$ with $\sum_{g\in G}\alpha(x_g)=\alpha\left(\sum_{g\in G}x_g\right)=\alpha(x)\in I$. Since $I$ is a gr-ideal, $\alpha(x_g)\in I$ for all $g\in G$, and then $x_g\in S_I$ for all $g\in G$. Therefore, $S_I$ is a gr-ideal of $A$. □

**Proposition 2.7.** *Let $A$ be a $gr-R$ and $I$ be a gr-$\alpha$-pr-ideal of $A$. Then $S_I$ is a gr-$\alpha$-pr-ideal of $A$.*

*Proof.* Let $w,t\in h(A)$ s.t $wt\in S_I$. Then $\alpha(w)\alpha(t)=\alpha(wt)\in I$. Since $I$ is gr-$\alpha$-pr and $\alpha(w),\alpha(t)\in h(A)$, either $\alpha(w)\in I$ or $\alpha(\alpha(t))\in I$, and then either $w\in S_I$ or $\alpha(t)\in S_I$. Hence, $S_I$ is a gr-$\alpha$-pr-ideal of $A$. □

**Lemma 2.8.** *Let $A$ be a $gr-R$, $I$ be a gr-ideal of $A$. Let $x\in h(A)$. Then $I_x=\{y+ax:y\in I,a\in A\}$ is a gr-ideal of $A$ containing $x$ and $I$.*

*Proof.* Evidently, $I_x$ is an ideal of $A$. containing $x$ and $I$. Let $r\in I_x$. Then $r=y+ax$ for some $y\in I$ and $a\in A$, and then $r_g=(y+ax)_g=y_g+(ax)_g=y_g+\sum_{h\in G}a_hx_{h^{-1}g}$ for all $g\in G$. Now, since $y\in I$ and $I$ is a gr-ideal, $y_g\in I$ for all $g\in G$. Also, since $x\in h(A)$, $x\in A_z$ for some $z\in G$, and then for any $g\in G$, $x_g=x$ for $g=z$ and $x_g=0$ otherwise. So, $\sum_{h\in G}a_hx_{h^{-1}g}=a_{gz^{-1}}x$, and then $r_g=y_g+a_{gz^{-1}}x\in I_x$ for all $g\in G$. Therefore, $I_x$ is a gr-ideal of $A$. □

**Proposition 2.9.** *Let $A$ be a $gr-R$ and $I$ be a gr-$\alpha$-pr-ideal of $A$. If $I$ is maximal with respect to the property that $w \in I$ implies $\alpha(w) \in I$ among all gr-ideals of $A$, then $I$ is a gr-pr-ideal of $A$.*

*Proof.* Let $w, y \in h(A)$ with $wy \in I$ and $w \notin I$. Consider the gr-ideal $I_y = \{s + ay : s \in I, a \in A\}$ of $A$, and let $r \in I_y$. Then $r = s + ay$ for some $s \in I$ and $a \in A$, and then $wr = ws + way \in I$. Since $I$ is gr-$\alpha$-pr and $w \notin I$, $\alpha(r) \in I \subseteq I_y$. So, by the maximality of $I$, $I_y = I$, and hence $y \in I$. Therefore, $I$ is a gr-pr-ideal of $A$. □

The next result introduces an characterization for gr-$\alpha$-pr-ideal.

**Theorem 2.10.** *Let $A$ be a $gr-R$ and $I$ be a proper gr-ideal of $A$. Then $I$ is a gr-$\alpha$-prime ideal of $A$ iff for any two gr-ideals $P$ and $K$ of $A$ with $PK \subseteq I$, either $P \subseteq I$ or $\alpha(K) \subseteq I$.*

*Proof.* Suppose that $I$ is a gr-$\alpha$-pr-ideal of $A$. Let $P$ and $K$ be two gr-ideals of $A$ with $PK \subseteq I$ and $P \nsubseteq I$. Then $\exists\, x \in P$ such that $x \notin I$, and then $\exists\, g \in G$ such that $x_g \notin I$. Note that, $x_g \in P$ as $P$ is a gr-ideal. Let $y \in K$. Then $y_h \in K$ for all $h \in G$ as $K$ is a gr-ideal. So, for any $h \in G$, $x_g y_h \in PK \subseteq I$, and since $I$ is gr-$\alpha$-pr with $x_g \notin I$, $\alpha(y_h) \in I$ for all $h \in G$, it indicates $\alpha(y) \in I$. Hence, $\alpha(K) \subseteq I$. Conversely, let $x, y \in h(A)$ such that $xy \in I$. Then $P = \langle x \rangle$ and $K = \langle y \rangle$ are gr-ideals of $A$ with $PK \subseteq I$. By assumption, either $P \subseteq I$ or $\alpha(K) \subseteq I$, it indicates either $x \in I$ or $\alpha(y) \in I$. Therefore, $I$ is a gr-$\alpha$-pr-ideal of $A$. □

**Proposition 2.11.** *Let $A$ be a $gr-R$, $I$ be a gr-$\alpha$-pr-ideal of $A$ and $K$ be a non-empty subset of $h(A)$ with $K \nsubseteq I$ and $\alpha(K) \nsubseteq I$. Then $(I : K) = \{a \in A : aK \subseteq I\}$ is a gr-$\alpha$-pr-ideal of $A$.*

*Proof.* Let $x, y \in h(A)$ with $xy \in (I : K)$. Then $xyK \subseteq I$. Since $\alpha(K) \nsubseteq I$, $\exists\, r \in K$ with $\alpha(r) \notin I$. Now, $xyr \in I$, it indicates $xy \in I$ as $I$ is gr-$\alpha$-pr, and then also either $x \in I \subseteq (I : K)$ or $\alpha(y) \in I \subseteq (I : K)$ as $I$ is gr-$\alpha$-pr. Hence, $(I : K)$ is a gr-$\alpha$-pr-ideal of $A$. □

**Proposition 2.12.** *Let $A$ and $B$ be two $G-gr-R$ and $f : A \to B$ be a $gr-R-$homomorphism. Assume that $\alpha$ and $\beta$ are $gr-R-$homomorphism from $A$ to $A$ and from $B$ to $B$ respectively. If $f \circ \alpha = \beta \circ f$, then for any gr-$\beta$-pr-ideal $P$ of $B$, $f^{-1}(P)$ is a gr-$\alpha$-pr-ideal of $A$.*

*Proof.* Let $P$ be a gr-$\beta$-pr-ideal of $B$. Suppose that $x, y \in h(A)$ with $xy \in f^{-1}(P)$. Then $f(x), f(y) \in h(B)$ such that $f(x)f(y) \in P$, and $P$ being gr-$\beta$-pr it indicates either $f(x) \in P$ or $\beta(f(y)) = f(\alpha(y)) \in P$, that is, either $x \in f^{-1}(P)$ or $\alpha(y) \in f^{-1}(P)$, as desired. □

**Proposition 2.13.** *Let $A$ and $B$ be two $G-gr-R$ and $f : A \to B$ be a $gr-R-$epimorphism. Assume that $\alpha$ and $\beta$ are $gr-R-$homomorphism from $A$ to $A$ and from $B$ to $B$ respectively. If $f \circ \alpha = \beta \circ f$ and $I$ is a gr-ideal of $A$ with $Ker(f) \subseteq I$, then $I$ is a gr-$\alpha$-pr-ideal of $A$ iff $f(I)$ is a gr-$\beta$-pr-ideal of $B$.*

*Proof.* Suppose that $I$ is a gr-$\alpha$-pr-ideal of $A$. Let $w, y \in h(B)$ with $wy \in f(I)$. Since $f$ is gr-epimorphism, $\exists\, a, b \in h(A)$ with $f(a) = w$ and $f(b) = y$, and then $f(ab) = f(a)f(b) = wy \in I$, it indicates $ab \in I$ as $Ker(f) \subseteq I$. Since $I$ is gr-$\alpha$-pr, either $a \in I$ or $\alpha(b) \in I$, and then either $w = f(a) \in f(I)$ or $\beta(y) = \beta(f(b)) = f(\alpha(b)) \in f(I)$. Thus $f(I)$ is a gr-$\alpha$-pr-ideal of $B$. The converse follows from Proposition 2.12. □

Note that, if $\alpha : A \to A$ is a $gr-R-$homomorphism and $I$ is a gr-ideal of $A$, then $\alpha_I : A/I \to A/I$ s.t $\alpha_I(a + I) = \alpha(a) + I$ is a gr-R-homomorphism.

**Proposition 2.14.** *Let $A$ be a $gr-R$, $I$ and $P$ be two gr-ideals of $A$ s.t $I \subseteq P$. Assume that $\alpha : A \to A$ is a $gr-R-$homomorphism. Then $P/I$ is a gr-$\alpha_I$-pr-ideal of $A/I$ iff $P$ is a gr-$\alpha$-pr-ideal of $A$.*

*Proof.* Suppose that $P/I$ is a gr-$\alpha_I$-pr-ideal of $A/I$. Let $w, y \in h(A)$ such that $wy \in P$. Then $w + I, y + I \in h(A/I)$ with $(w + I)(y + I) = wy + I \in P/I$, and then either $w + I \in P/I$ or $\alpha_I(y + I) \in P/I$, so either $x \in P$ or $\alpha(y) \in P$. Hence, $P$ is a gr-$\alpha$-pr-ideal of $A$. Conversely, let $w + I, y + I \in h(A/I)$ s.t $(w + I)(y + I) \in P/I$. Then $w, y \in h(A)$ with $wy \in P$, and then either $w \in P$ or $\alpha(y) \in P$, so either $w + I \in P/I$ or $\alpha_I(y + I) \in P/I$. Thus $P/I$ is a gr-$\alpha_I$-pr-ideal of $A/I$. □

**Lemma 2.15.** *Let $A$ be a $gr-R$ and $\alpha : A \to A$ be a $gr-R$ homomorphism. Then $Ker(\alpha)$ is in the intersection of all gr-$\alpha$-pr-ideals of $A$.*

*Proof.* Suppose that $a \in Ker(\alpha)$. Then $\alpha(a) = 0$ exists in every gr-$\alpha$-pr-ideal of $A$. So, $a$ exists in the inverse image of every gr-$\alpha$-pr-ideal of $A$ which is repeatedly a gr-$\alpha$-pr-ideal of $A$ by Proposition 2.12. Therefore, $Ker(\alpha)$ is in the intersection of all gr-$\alpha$-pr-ideals of $A$. □

**Lemma 2.16.** *Let $A$ be a gr-domain and $\alpha : A \to A$ be a $gr-R-$ homomorphism. Then $Ker(\alpha)$ is a gr-prime ideal of $A$.*

*Proof.* Suppose that $wb \in Ker(\alpha)$ for $w, b \in h(A)$. Then $\alpha(w), \alpha(b) \in h(A)$ with $\alpha(wb) = \alpha(w)\alpha(b) = 0$ and $A$ being a gr-domain implies either $\alpha(w) = 0$ or $\alpha(b) = 0$, that is, either $w \in Ker(\alpha)$ or $b \in Ker(\alpha)$. Therefore, $Ker(\alpha)$ is a gr-pr-ideal of $A$. □

**Proposition 2.17.** *Let $A$ be a $gr-R$ and $I$ be a gr-$\alpha$-pr-ideal of $A$. Consequently, the following claims are true:*

*(i) If $a \in h(A)$ with $a^k \in I$ for some positive integer $k$, then $\alpha(a) \in I$.*

*(ii) If $x \in h(A)$ with $(\alpha(x))^k \in I$ for some positive integer $k$, then $\alpha(\alpha(x)) \in I$.*

*(iii) $\alpha(\alpha(x)) \in I$ for all homogeneous nilpotent element $x$ of $A$.*

*Proof.* (i) Since $a^k \in I$, $a^{k-1}a \in I$, and then either $a^{k-1} \in I$ or $\alpha(a) \in I$ as $I$ is gr-$\alpha$-pr. If $a^{k-1} \in I$, then $a^{k-2}a \in I$, and then either $a^{k-2} \in I$ or $\alpha(a) \in I$ as $I$ is gr-$\alpha$-pr. Continue on this procedure to obtain surely $\alpha(a) \in I$.

(ii) Since $x \in h(A)$, $x \in A_g$ for a few $g \in G$, and then $\alpha(x) \in \alpha(A_g) \subseteq A_g$. So, $\alpha(x) \in h(A)$, and hence the result holds by (1).

(iii) Let $x \in h(A)$ be a nilpotent. Then $x^k = 0$ for some positive integer $k$, and then $(\alpha(x))^k = \alpha(x^k) = \alpha(0) = 0 \in I$, and hence the result holds by (2).

□

**Theorem 2.18.** *Let $A$ be a gr-domain and $\alpha : A \to A$ be a $gr-R$ homomorphism. Then $N_\alpha(A) = \{x \in A : (\alpha(x))^k = 0,$ for some positive integer $k\}$ is in the intersection of all gr-$\alpha$-pr-ideals of $A$.*

*Proof.* Let $w \in N_\alpha(A)$. Then $\alpha(w^k) = 0$ for some positive integer $k$, it indicates that $w^k \in Ker(\alpha)$. By Lemma 2.16, $w \in Ker(\alpha)$ that is in the intersection of all gr-$\alpha$-pr-ideals of $A$ by Lemma 2.15. So, $w$ belongs to the intersection of all gr-$\alpha$-pr-ideals of $A$. Therefore, $N_\alpha(A)$ is in the intersection of all gr-$\alpha$-pr-ideals of $A$. □

**Definition 2.19.** *Let $A$ be a $gr-R$, $I$ be a proper gr-ideal of $A$ and $\alpha : A \to A$ be a $gr-R-$homomorphism. The gr-$\alpha$-radical of $I$ is express by $Grad_\alpha(I)$ and it is in the following terms:*

$$Grad_\alpha(I) = \{x = \sum_{g \in G} x_g \in A : \forall g \in G, \exists n_g \in \mathbb{N} \text{ s.t. } \alpha(x_g^{n_g}) \in I\}.$$

Undoubtedly, $Grad_\alpha(I)$ is a gr-ideal of $A$, $N_\alpha(A) = Grad_\alpha(\{0\})$ and hence $N_\alpha(A)$ is a gr-ideal of $A$. Also, if $I$ is a gr-$\alpha$-pr-ideal of $A$, then $I \subseteq Grad_\alpha(I)$.

**Proposition 2.20.** *Let $A$ be a $gr-R$, $I, J$ be two gr-ideals of $A$ and $\alpha : A \to A$ be a gr-R-homomorphism. Then the following claims are true:*

*(i) If $I \subseteq J$, then $Grad_\alpha(I) \subseteq Grad_\alpha(J)$.*

*(ii) $Grad_\alpha(IJ) = Grad_\alpha(I \bigcap J) = Grad_\alpha(I) \bigcap Grad_\alpha(J)$.*

*(iii) If $\alpha(1) = 1$, then $Grad_\alpha(I) = A$ iff $I = A$.*

*(iv) If $I$ and $J$ are gr-$\alpha$-pr-ideals of $A$, then $Grad_\alpha(I + J) \subseteq Grad_\alpha(Grad_\alpha(I) + Grad_\alpha(J))$.*

*(v) $Grad_\alpha(I^k) = Grad_\alpha(I)$, for all positive integer $k$.*

*Proof.* (i) Let $a \in Grad_\alpha(I)$ ,then $\exists$ a positive integer $k_g$ with $\alpha(a_g^{k_g}) \in I$, since $I \subseteq J$ then $\alpha(a_g^{k_g}) \in J$ then $Grad_\alpha(I) \subseteq Grad_\alpha(J)$.

(ii) Since $IJ \subseteq I \bigcap J$, $Grad_\alpha(IJ) \subseteq Grad_\alpha(I \bigcap J)$. Let $a \in Grad_\alpha(I \bigcap J)$ and $g \in G$. Then $\exists$ a positive integer $k_g$ with $\alpha(a_g^{k_g}) \in I \bigcap J$, and then $\alpha(a_g^{2k_g}) = \alpha(a_g^{k_g})\alpha(a_g^{k_g}) \in IJ$. Hence, $Grad_\alpha(IJ) = Grad_\alpha(I \bigcap J)$. Since $I \bigcap J \subseteq I$ and $I \bigcap J \subseteq J$, $Grad_\alpha(I \bigcap J) \subseteq Grad_\alpha(I) \bigcap Grad_\alpha(J)$. Let $a \in Grad_\alpha(I) \bigcap Grad_\alpha(J)$ and $g \in G$. Then $\exists$ positive integer $n_g, m_g$ s.t $\alpha(a_g^{n_g}) \in I$ and $\alpha(a_g^{m_g}) \in J$, and then $\alpha(a_g^{k_g}) \in I \bigcap J$, where $k_g = max\{n_g, m_g\}$, and hence $a \in Grad_\alpha(I \bigcap J)$. Thus $Grad_\alpha(I \bigcap J) = Grad_\alpha(I) \bigcap Grad_\alpha(J)$.

(iii) Suppose that $Grad_\alpha(I) = A$. Then $1 \in Grad_\alpha(I)$, and then $\exists$ a positive integer $k$ s.t $1 = \alpha(1) = \alpha(1^k) \in I$, and hence $I = A$. Versus, $1 = \alpha(1) \in I$, and then $1 \in Grad_\alpha(I)$, and hence $Grad_\alpha(I) = A$.

(iv) Since $I$ and $J$ are gr-$\alpha$-pr, $I \subseteq Grad_\alpha(I)$ and $J \subseteq Grad_\alpha(J)$, so $I + J \subseteq Grad_\alpha(I) + Grad_\alpha(J)$, and then $Grad_\alpha(I + J) \subseteq Grad_\alpha(Grad_\alpha(I) + Grad_\alpha(J))$.

(v) $Grad_\alpha(I^k) = Grad_\alpha(I) \bigcap ... \bigcap Grad_\alpha(I)$ =Grad$_\alpha(I)$

□

**Proposition 2.21.** *Let $A$ and $B$ be two $G-gr-R$ and $f : A \to B$ be a $gr-R-$homomorphism. Assume that $\alpha$ and $\beta$ are $gr-R-$homomorphism from $A$ to $A$ and from $B$ to $B$ respectively. If $f \circ \alpha = \beta \circ f$. Let $I$ and $J$ be two gr-ideals of $A$ and $B$ respectively. Then the next statements hold:*

*(i)* $f(Grad_\alpha(I)) \subseteq Grad_\beta(f(I))$.

*(ii)* $Grad_\alpha(f^{-1}(J)) \subseteq f^{-1}(Grad_\beta(J))$.

*Proof.* (i) Let $y \in f(Grad_\alpha(I))$. Then $\exists$ $x \in Grad_\alpha(I)$ s.t $f(x) = y$. Let $g \in G$. Then $\exists$ a positive integer $k$ s.t $\alpha(x_g^{k_g}) \in I$, and then $\beta(y_g^{k_g}) = \beta((f(x))_g^{k_g}) = \alpha(f(x_g^{k_g})) = f(\alpha(x_g^{k_g})) \in f(I)$, and hence $y \in Grad_\beta(f(I))$.

(ii) Let $x \in Grad_\alpha(f^{-1}(J))$ and $g \in G$. Then $exists$ a positive integer $k_g$ s.t $\alpha(x_g^{k_g}) \in f^{-1}(J)$, and then $\beta((f(x))_g^{k_g}) = \beta(f(x_g^{k_g})) = f(\alpha(x_g^{k_g})) \in J$, and so $f(x) \in Grad_\beta(J)$ that is $x \in f^{-1}(Grad_\alpha(J))$.

□

**Definition 2.22.** *Let $A$ be a $gr-R$, $I$ be a proper gr-ideal of $A$ and $\alpha : A \to A$ be a $gr-R-$homomorphism. Then $I$ is said to be a gr-$\alpha$-primary ideal of $A$ if whenever $x, y \in h(A)$ s.t $xy \in I$, then either $x \in I$ or $\alpha(y^k) \in I$, for some positive integer $k$.*

**Proposition 2.23.** *If $I$ is a gr-$\alpha$-primary ideal of $A$, then $Grad_\alpha(I)$ is a gr-$\alpha$-pr-ideal of $A$.*

*Proof.* Let $w, y \in h(A)$ s.t $wy \in Grad_\alpha(I)$. Then $\alpha(w^k)\alpha(y^k) = \alpha(w^k y^k) = \alpha((wy)^k) \in I$, for some positive integer $k$. Since $I$ is gr-$\alpha$-primary, either $\alpha(w^k) \in I$ or $\alpha((y^k)^n) \in I$, for some positive integer $n$, and then either $\alpha(w^k) \in I$ or $\alpha(y^{kn}) \in I$, it indicates either $w \in Grad_\alpha(I)$ or $\alpha(y) \in Grad_\alpha(I)$. Hence, $Grad_\alpha(I)$ is a gr-$\alpha$-pr-ideal of $A$. □

**Definition 2.24.** *Let $A$ be a $gr-R$ and $\alpha : A \to A$ be a $gr-R$ homomorphism. Then $A$ is said to be gr-$\alpha$-domain if whenever $w, y \in h(A)$ s.t $wy = 0$, then either $w = 0$ or $\alpha(y) = 0$.*

Indeed, every gr-domain is gr-$\alpha$-domain for every $\alpha$. Also, if $\alpha$ is the identity map, then gr-domain and gr-$\alpha$-domain coincide. However, the next example demonstrates that a gr-$\alpha$-domain is not necessarily gr-domain:

**Example 2.25.** *Consider $A = \mathbb{Z}[X]$ and $G = \mathbb{Z}$. Then $A$ is $G-gr$ by $A_j = \mathbb{Z}X^j$, where $j \geq 0$ and $A_j = 0$ otherwise. Consider the gr-ideal $I = \langle X^2 \rangle$ of $A$. So, $A/I$ is a gr-R by $(A/I)_j = (A_j + I)/I$ for all $j \in \mathbb{Z}$. Consider the gr-homomorphism $\alpha : A/I \to A/I$ with $\alpha(f(X) + I) = f(0) + I$. Let $f(X), g(X) \in h(A/I)$ s.t $(f(X) + I)(g(X) + I) = 0 + I$.*

*Then $f(X)g(X) \in I$ that is $X^2$ divides $f(X)g(X)$, and then $X$ divides $f(X)g(X)$, it indicates either $X$ divides $f(X)$ or $X$ divides $g(X)$. If $X$ divides $f(X)$, then $f(X) = Xh(X)$, for some $h(X) \in \mathbb{Z}[X]$, and then $\alpha(f(X) + I) = f(0) + I = 0 + I$. Similarly, if $X$ divides $g(X)$, then $\alpha(g(X) + I) = 0 + I$. Hence, $A/I$ is a gr-$\alpha$-domain. But $A/I$ is not gr-domain since $X + I \in h(A/I)$ with $(X + I)(X + I) = X^2 + I = 0 + I$ and $X + I \neq 0 + I$.*

**Theorem 2.26.** *Let $A$ be a $gr-R$, $I$ be a gr-ideal of $A$ and $\alpha : A \to A$ be a $gr-R-$homomorphism. Then $I$ is a gr-$\alpha$-pr-ideal of $A$ iff $A/I$ is a gr-$\alpha_I$-domain.*

*Proof.* Suppose that $I$ is a gr-$\alpha$-pr-ideal of $A$. Let $w+I, y+I \in h(A/I)$ s.t $(w+I)(y+I) = 0+I$. Then $w, y \in h(A)$ s.t $wy \in I$, and then either $w \in I$ or $\alpha(y) \in I$, so either $w + I = 0 + I$ or $\alpha_I(y + I) = \alpha(y) + I = 0 + I$. Hence, $A/I$ is a gr-$\alpha_I$-domain. Conversely, let $w, y \in h(A)$ s.t $wy \in I$. Then $w+I, y+I \in h(A/I)$ s.t $(w+I)(y+I) = 0+I$, and then either $w+I = 0+I$ or $\alpha(y) + I = \alpha_I(y + I) = 0 + I$, so either $w \in I$ or $\alpha(y) \in I$. Thus $I$ is a gr-$\alpha$-pr-ideal of $A$. □

**Proposition 2.27.** *Let $A$ be a gr-$\alpha$-domain. Then $Ker(\alpha)$ is a gr-$\alpha$-pr-ideal of $A$.*

*Proof.* Let's say $wb \in Ker(\alpha)$ for $w, b \in h(A)$. Then $\alpha(w), \alpha(b) \in h(A)$ such that $\alpha(wb) = \alpha(w)\alpha(b) = 0$ and $A$ being a gr-$\alpha$-domain implies either $\alpha(w) = 0$ or $\alpha(\alpha(b)) = 0$, that is, either $w \in Ker(\alpha)$ or $\alpha(b) \in Ker(\alpha)$. Therefore, $Ker(\alpha)$ is a gr-$\alpha$-pr-ideal of $A$. □

**Proposition 2.28.** *Let $A$ be a $gr - R$. Then $A$ is a gr-field iff $\{0\}$ and $A$ itself are the only gr-ideals of $A$.*

*Proof.* Let's say $A$ is a gr-field. Let $I$ be a gr-ideal of $A$ and $I \neq \{0\}$. Then $\exists\, 0 \neq x \in I$, and then $\exists\, g \in G$ such that $x_g \neq 0$. Note that, $x_g \in I$ as $I$ is a gr-ideal. Since $A$ is gr-field, $\exists\, y \in A$ such that $x_g y = 1$, and then $1 \in I$, and hence $I = A$. Versus, let $0 \neq x \in h(A)$. Then $I = Ax$ is a gr-ideal of $A$, and then either $I = \{0\}$ or $I = A$. Since $x \neq 0$, $I = A$, and then $1 \in I$, it indicates $xy = 1$ for some $y \in A$, that is, $x$ is unit. □

**Definition 2.29.** *A graded $\alpha$-domain $A$ is said to be a gr-$\alpha$-field if $A/Ker(\alpha)$ is a gr-field.*

**Proposition 2.30.** *Every gr-field is a gr-$\alpha$-field, for all $\alpha$.*

*Proof.* Let $A$ be a gr-field and $\alpha : A \to A$ be a gr-R-homomorphism. Then $A$ is a gr-domain, and then $A$ is a gr-$\alpha$-domain. Since $A$ is a gr-field, $Ker(\alpha) = \{0\}$ or $Ker(\alpha) = A$ by Proposition 2.28, and then $A/Ker(\alpha) = A/\{0\} \approx A$ or $A/Ker(\alpha) = A/A \approx \{0\}$. In both cases, $A/Ker(\alpha)$ is a gr-field. Hence, $A$ is a gr-$\alpha$-field. □

Undoubtedly, the converse of Proposition 2.30 is not true in general, as we see in the next example. Indeed, the converse of Proposition 2.30 is true if $Ker(\alpha) = \{0\}$. So, gr-field and gr-$\alpha$-field coincide when $Ker(\alpha) = \{0\}$.

**Example 2.31.** *Consider $A = K[X]$, where $K$ is a field, and $G = \mathbb{Z}$. Then $A$ is $G$-gr by $A_j = KX^j$, where $j \geq 0$ and $A_j = 0$ otherwise. Consider the gr-homomorphism $\alpha : A \to A$ with $\alpha(f(X)) = f(0)$. Then $Ker(\alpha) = \langle X \rangle$, and then $A/Ker(\alpha) \approx K$ is a gr-field. So, $A$ is a gr-$\alpha$-field. But $A$ is not gr-field since $X \in h(A)$ is non-unit.*

## References


[1] R. Abu-Dawwas, *On graded strongly 1-absorbing primary ideals*, Khayyam Journal of Mathematics, **8** (1), 42–52, (2022).

[2] R. Abu-Dawwas, *On graded radically Principal ideals*, Bulletin of the Korean Mathematical Society, **58** (6), 1401–1407, (2021).

[3] R. Abu-Dawwas and M. Refai, *Graded uniformly pr-ideals*, Bulletin of Korean Mathematical Society, **58** (1), 195–204, (2021).

[4] A. Abu-Qayass and K. Al-Zoubi, *On $A$-gr-$n$-ideals of graded commutative rings*, Palestine Journal of Mathematics, **14** (1), 367–373, (2025).

[5] I. Akray, A. K. Jabbar and S. A. Othman, *Graded $n$-absorbing $I$-ideals*, Palestine Journal of Mathematics, **13** (1), 201–213, (2024).

[6] I. Akray and H. M. Mohammed-Salih, *$\alpha$-prime ideals*, Journal of Mathematical Extension, **1** (16), 1–11, (2022).

[7] A. S. Alshehry, M. Bataineh and R. Abu-Dawwas, *Graded $\phi$-2-absorbing and graded $\phi$-2-absorbing primary submodules*, Mathematics **9** (10), 1083, (2021).

[8] A. S. Alshehry, J.M. Habeb, R. Abu-Dawwas and A. Alrawabdeh, *Graded weakly 2-absorbing ideals over non-commutative graded rings*, Symmetry, **14** (7), 1472, (2022).

[9] K. Al-Zoubi, R. Abu-Dawwas and S. Ceken, *On graded 2-absorbing and graded weakly 2-absorbing ideals*, Hacettepe Journal of Mathematics and Statistics, **48** (3), 724–731, (2019).

[10] M. Bataineh and R. Abu-Dawwas, *On graded 2-prime ideals*, Mathematics, **9**, 493, (2021).

[11] C. Nastasescu and F. Oystaeyen, *Methods of graded rings*, 1836, Berlin, Springer-Verlag, (2004).

[12] M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turkish Journal of Mathematics, **28** (3), 217–229, (2004).

[13] M. Refai, M. Hailat and S. Obiedat, *Graded radicals and graded prime spectra*, Far East Journal of Mathematical Sciences, 59–73, (2000).

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