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EXISTENCE OF SOLUTIONS FOR KIRCHHOFF-TYPE PROBLEMS INVOLVING THE $p(x)$-BIHARMONIC OPERATOR WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. Under suitable assumptions and using the Mountain Pass Theorem of Ambrosetti and Rabinowitz, we establish the existence of weak solutions for a Kirchhoff-type problem involving the $p(x)$-biharmonic operator with nonlocal boundary conditions.

1 Introduction

Recently, the study of differential equations and variational problems involving variable exponent growth conditions has attracted considerable attention, due to both their mathematical richness and their wide-ranging applications. These include areas such as mathematical biology [21], electrorheological fluids [28, 29], image restoration [10], elasticity [34], and the mathematical modeling of barotropic gas filtration through porous media [5].

Among these, fourth-order differential equations occupy a central place in applied mathematics and physics. They naturally arise in thin film theory, surface diffusion on solids, microelectromechanical systems (MEMS), thin plate theory, interface dynamics, flow in Hele-Shaw cells, and phase field models of multiphase systems (see [12, 20, 27] and references therein). A particularly important case is the $p(x)$-biharmonic equation, which results from the interplay between variable exponent models and fourth-order operators, highlighting the theoretical and practical significance of such problems.

Furthermore, our interest in problems involving nonlocal boundary conditions is motivated by their significance in diffusion processes, where the state of the system at a specific point may depend on its behavior over a broader spatial domain. A representative example arises in diffusion experiments in which a light beam generates an electrical signal proportional to the concentration of a diffused chemical inside a straight glass tube; in such cases, a boundary integral condition is used to recover the unknown concentration (see [9]).

Nonlocal boundary value problems have attracted the attention of many researchers. In [1], Abreu et al. investigated an elliptic eigenvalue problem with a nonlocal boundary condition and established the existence of a principal eigenvalue using the Krein–Rutman theorem. In [8], Benouhiba applied Ekeland's variational principle to demonstrate the existence of infinitely many eigenvalues for a problem involving the $p(x)$-Laplacian.

In this paper, we study the existence of weak solutions for a $p(x)$-Kirchhoff problem with a nonlocal boundary condition, defined as follows:

$$(\mathcal{P})\begin{cases} \mathcal{K}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\Delta^2_{p(x)}u = \lambda f(x)|u|^{q(x)-2}u, \quad x\in\Omega,\\ \mathcal{K}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\frac{\partial}{\partial\nu}\left(|\Delta u|^{p(x)-2}\Delta u\right)(\varsigma) = \eta\int_\Omega g(\varsigma,x)|u|^{\gamma(x)-2}u dx, \quad \varsigma\in\partial\Omega,\\ \frac{\partial u}{\partial\nu}(\varsigma)=0, \quad \varsigma\in\partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 2$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta^2_{p(x)}u=\Delta\left(|\Delta u|^{p(x)-2}\Delta u\right)$ is the $p(x)$-biharmonic operator, ν is a unit outward normal to $\partial\Omega$, p,γ,q are continuous functions on $\overline{\Omega}$, $\lambda,\eta>0$ are real numbers, the functions f and g are defined on Ω and $\partial\Omega\times\Omega$ respectively, and $\mathcal{K}:\mathbb{R}^+\mapsto\mathbb{R}^+$ is a continuous function which represents the Kirchhoff coefficient.

The problem $(\mathcal{P})$ is related to the stationary form of the Kirchhoff equation, originally introduced by **Kirchhoff** in 1883 (see [23]). This equation can be written as

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.1}$$

where E denotes the Young's modulus of the material, ρ is the mass density, ρ_0 the initial tension, h the cross-sectional area, and L the length of the string. This model generalizes the classical D'Alembert wave equation by incorporating the effect of changes in string length during vibration. As such, it provides a more comprehensive framework for analyzing wave propagation, taking into account tension, material properties, and the geometry of the vibrating medium.
Following the abstract formulation proposed by Lions in [25], equation (1.1) has attracted substantial research interest. Numerous variants of the Kirchhoff-type equation have since been investigated, including those involving the $p(x)$-Laplacian (see [11, 18, 31, 33]), the fractional $p(x)$-Laplacian (see [3, 6]), and the $p(x)$-biharmonic operator (see [2, 13, 22, 30, 32]), among others.

The remainder of this paper is structured as follows. Section 2 provides definitions of Lebesgue and Sobolev spaces with variable exponents and presents some important properties. Section 3, the final section, presents and proves the main result (Theorem 3.4) using the Mountain Pass Theorem of Ambrosetti and Rabinowitz [4].

2 Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) := \{\mathcal{V}\in C(\overline{\Omega}) \text{ and } \mathcal{V}(x)>1, \forall x\in\overline{\Omega}\}.$$

Throughout this paper, for any $\mathcal{V}\in C_+(\overline{\Omega})$, we denote

$$1<\mathcal{V}^- := \min_{x\in\overline{\Omega}}\mathcal{V}(x)\leq\mathcal{V}^+ := \max_{x\in\overline{\Omega}}\mathcal{V}(x)<+\infty.$$

Let $p(x)\in C_+(\overline{\Omega})$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) := \left\{u:\Omega\to\mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^{p(x)}dx<\infty\right\}.$$

This space is endowed with the so-called **Luxemburg norm** given by

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} := \inf\left\{\kappa>0 \,:\, \int_\Omega\left|\frac{u(x)}{\kappa}\right|^{p(x)}dx\leq 1\right\}.$$

The Sobolev space with variable exponent $\mathcal{W}^{k,p(x)}(\Omega)$ is defined by:

$$\mathcal{W}^{k,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega)\ :\ D^{\theta}u\in L^{p(x)}(\Omega),\ |\theta|\leq k\right\},$$

with $\theta=(\theta_1,\theta_2,\dots,\theta_N)$ is a multi-index, $|\theta|=\sum_{j=1}^{N}\theta_j$ and $D^{\theta}u=\dfrac{\partial^{|\theta|}u}{\partial^{\theta_1}x_1\dots\partial^{\theta_N}x_N}$.

We define on $\mathcal{W}^{k,p(x)}(\Omega)$ the norm

$$\|u\|_{k,p(x)}=\sum_{|\theta|\leq k}\left|D^{\theta}u\right|_{p(x)}.$$

$L^{p(x)}(\Omega)$ and $\mathcal{W}^{k,p(x)}(\Omega)$ are separable and reflexive Banach spaces.
For further details regarding Sobolev spaces with variable exponents, we refer the reader to references [17, 19, 26].

Proposition 2.1 ([19, 24]).

(i) *$L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, where* $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.

(ii) *For any $\mathcal{R}_j\in L^{p_j(x)}(\Omega)$ $(j=1,2)$ with $\frac{1}{p_1(x)}+\frac{1}{p_2(x)}=1$, we have the* ***Hölder inequality:***

$$\left|\int_{\Omega}\mathcal{R}_1\mathcal{R}_2dx\right|\leq\left(\frac{1}{p_1^-}+\frac{1}{p_2^-}\right)|\mathcal{R}_1|_{p_1(x)}|\mathcal{R}_2|_{p_2(x)}\leq 2|\mathcal{R}_1|_{p_1(x)}|\mathcal{R}_2|_{p_2(x)}.$$

(iii) *For any $\mathcal{R}_j\in L^{p_j(x)}(\Omega)$ $(j=1,2,3)$ such that $\sum_{j=1}^{3}\frac{1}{p_j(x)}=1$, we have:*

$$\begin{aligned}\left|\int_{\Omega}\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3dx\right| &\leq \left(\frac{1}{p_1^-}+\frac{1}{p_2^-}+\frac{1}{p_3^-}\right)|\mathcal{R}_1|_{p_1(x)}|\mathcal{R}_2|_{p_2(x)}|\mathcal{R}_3|_{p_3(x)}\\ &\leq 3\,|\mathcal{R}_1|_{p_1(x)}|\mathcal{R}_2|_{p_2(x)}|\mathcal{R}_3|_{p_3(x)}.\end{aligned}$$

(iv) *If $r_1,r_2\in C_+(\overline{\Omega})$ and $r_1(x)\leq r_2(x)$ for all $x\in\overline{\Omega}$, then $L^{r_2(x)}(\Omega)\hookrightarrow L^{r_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2 ([19, 24]). *Let $\Xi_{r(x)}(\mathcal{H})=\int_{\Omega}|\mathcal{H}|^{r(x)}dx$. For all $\mathcal{H}$, $\mathcal{H}_n\in L^{r(x)}(\Omega)$, we have:*

(i) $|\mathcal{H}|_{r(x)}>1\Longrightarrow|\mathcal{H}|^{r^-}_{r(x)}\leq\Xi_{r(x)}(\mathcal{H})\leq|\mathcal{H}|^{r^+}_{r(x)}$.

(ii) $|\mathcal{H}|_{r(x)}<1\Longrightarrow|\mathcal{H}|^{r^+}_{r(x)}\leq\Xi_{r(x)}(\mathcal{H})\leq|\mathcal{H}|^{r^-}_{r(x)}$.

(iii) $|\mathcal{H}|_{r(x)}>1(resp.=1;<1)\Longleftrightarrow\Xi_{r(x)}(\mathcal{H})>1(resp.=1;<1)$.

(iv) $\lim\limits_{n\to\infty}|\mathcal{H}-\mathcal{H}_n|_{r(x)}=0\Longleftrightarrow\lim\limits_{n\to\infty}\Xi_{r(x)}(\mathcal{H}-\mathcal{H}_n)=0$.

Proposition 2.3 ([14]). *Let r_1 and r_2 be measurable functions such that $r_1(x)\in L^{\infty}(\Omega)$ and $1\leq r_1(x)r_2(x)\leq\infty$, a.e. $x\in\Omega$.*
Let $\mathcal{H}\in L^{r_2(x)}(\Omega)$, $\mathcal{H}\neq 0$. We have:

(i) $|\mathcal{H}|_{r_1(x)r_2(x)}\leq 1\Longrightarrow|\mathcal{H}|^{r_1^+}_{r_1(x)r_2(x)}\leq\left||\mathcal{H}|^{r_1(x)}\right|_{r_2(x)}\leq|\mathcal{H}|^{r_1^-}_{r_1(x)r_2(x)}$.

(ii) $|\mathcal{H}|_{r_1(x)r_2(x)}\geq 1\Longrightarrow|\mathcal{H}|^{r_1^-}_{r_1(x)r_2(x)}\leq\left||\mathcal{H}|^{r_1(x)}\right|_{r_2(x)}\leq|\mathcal{H}|^{r_1^+}_{r_1(x)r_2(x)}$.

In particular, when $r_1(x)=r_1$ is constant, we have

$$\left||\mathcal{H}|^{r_1}\right|_{r_2(x)}=|\mathcal{H}|^{r_1}_{r_1r_2(x)}.$$

In the following, we consider the set:

$$\mathcal{M} = \left\{ u \in \mathcal{W}^{2,p(x)}(\Omega) : \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0 \right\}.$$

The norm $\|u\|_{2,p(x)}$ is equivalent to the norm $\|u\| = |\Delta u|_{p(x)}$ in the space $\mathcal{M}$ (see [15, Proposition 2.4]).
Note that $(\mathcal{M}, \|.\|)$ is a Banach, separable, and reflexive space.

Proposition 2.4 ([15]). *Let* $\varphi(\mathcal{H}) = \int_\Omega |\Delta\mathcal{H}|^{r(x)}dx$. *For any* $\mathcal{H}$, $\mathcal{H}_n \in \mathcal{M}$, *we have:*

(i) $\|\mathcal{H}\| < 1(resp. > 1; = 1) \Longleftrightarrow \varphi(\mathcal{H}) < 1(resp. > 1; = 1)$.

(ii) $\|\mathcal{H}\| \geq 1 \Longrightarrow |\mathcal{H}\|^{r^-} \leq \varphi(\mathcal{H}) \leq \|\mathcal{H}\|^{r^+}$.

(iii) $\|\mathcal{H}\| \leq 1 \Longrightarrow \|\mathcal{H}\|^{r^+} \leq \varphi(\mathcal{H}) \leq \|\mathcal{H}\|^{r^-}$.

(iv) $\|\mathcal{H}_n\| \to 0 \Longleftrightarrow \varphi(\mathcal{H}_n) \to 0$.

(v) $\|\mathcal{H}_n\| \to \infty \Longleftrightarrow \varphi(\mathcal{H}_n) \to \infty$.

Proposition 2.5 ([16]). *The set* $\mathcal{M}$ *is a closed subspace of* $\mathcal{W}^{2,p(x)}(\Omega)$, *if* $2p(x) \geq N$ *for all* $x \in \overline{\Omega}$.

Proposition 2.6 ([16]). *Let* $p \in C_+(\overline{\Omega})$ *such that* $\forall x \in \overline{\Omega} : 2p(x) > N$. *Then:*

(i) For all $q \in C_+(\Omega)$ *there exists a continuous and compact embedding from* $\mathcal{W}^{2,p(x)}(\Omega)$ *into* $L^{q(x)}(\Omega)$.

(ii) There exists a continuous embedding from $\mathcal{W}^{2,p(x)}(\Omega)$ *into* $C(\overline{\Omega})$.

Proposition 2.7 ([15, 16]). *Let* $\mathcal{Q}(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx$, *then:*

(i) $\mathcal{Q} \in C^1(\mathcal{M}, \mathbb{R})$, *with the derivative given by*

$$\langle \mathcal{Q}'(u), \xi \rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta\xi dx, \text{ for all } u, \xi \in \mathcal{M}.$$

(ii) The mapping $\mathcal{B} := \mathcal{Q}' : \mathcal{M} \to \mathcal{M}^*$ *is of type* $(\mathcal{S}_+)$, *that is, if*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty} \langle \mathcal{Q}'(u_n), u_n - u \rangle \leq 0,$$

then $u_n \longrightarrow u$, *where* $\mathcal{M}^*$ *the dual space of* $\mathcal{M}$.

The following proposition is an expression of the Mountain Pass Theorem without the Palais-Smale condition. This is essentially due to A. Ambrosetti and P. Rabinowitz (see [4]).

Proposition 2.8. *Let* $\mathcal{F}$ *be a Banach space and* $\mathcal{H} \in C^1(\mathcal{F}, \mathbb{R})$. *Assume that:*

- $\mathcal{H}(0) = 0$,
- *there exist* $\rho > 0$ *and* $\alpha > 0$ *such that* $\mathcal{H}(u) \geq \alpha$ *if* $\|u\| = \rho$,
- *there exists a function* $\omega \in \mathcal{F}$ *such that* $\|\omega\| > \rho$ *and* $\mathcal{H}(\omega) \leq 0$.

Let

$$c = \inf_{\varphi\in\Gamma} \max_{t\in[0,1]} \mathcal{H}(\varphi(t)),$$

where

$$\Gamma = \{\varphi \in C([0, 1], \mathcal{F}) \mid \varphi(0) = 0, \varphi(1) = \omega\}.$$

Then, $c \geq \alpha$ *and there exists a sequence* $(u_n) \subset \mathcal{F}$ *satisfying* $\mathcal{H}(u_n) \to c$ *and* $\mathcal{H}'(u_n) \to 0$ *in* $\mathcal{F}^*$ *as* $n \to \infty$.

3 Assumptions and Main result

In the subsequent paragraphs, we will employ the following notation: $z'(x)$ the conjugate exponent of the function $z(x)$ for all $x \in \Omega$, and c_k the positive constants, for $k \in \{1, 2, 3, \cdots\}$. We also set:

$$\tau(x) := \frac{\beta(x)q(x)}{\beta(x) - q(x)}\ , \omega(x) := \frac{s(x)\gamma(x)}{s(x) - \gamma(x)} \quad \text{for all } x \in \overline{\Omega},$$

and

$$|\mathcal{U}|_{\alpha(x)}^{\alpha^i - 1} = \begin{cases} |\mathcal{U}|_{\alpha(x)}^{\alpha^- - 1}, \text{ if } |\mathcal{U}|_{\alpha(x)} \leq 1, \\ \\ |\mathcal{U}|_{\alpha(x)}^{\alpha^+ - 1}, \text{ if } |\mathcal{U}|_{\alpha(x)} \geq 1, \end{cases} \qquad \text{for all } \mathcal{U} \in \mathcal{M} \text{ and } \alpha \in C_+(\overline{\Omega}).$$

Now consider the following assertions:

$(\mathcal{A}_1)$ $f \in L^{\beta(x)}(\Omega)$ where $\beta \in C_+(\overline{\Omega})$ and for all $x \in \Omega$: $f(x) > 0$.

$(\mathcal{A}_2)$ $g(.,.) : \partial\Omega \times \Omega \to \mathbb{R}$ is positive measurable function and $|g(\varsigma, .)|_{s(x)} \in L^1(\partial\Omega)$ for all $\varsigma \in \partial\Omega$, where $s \in C_+(\overline{\Omega})$.

$(\mathcal{A}_3)$ $\gamma(x) < s(x), \quad \dfrac{N}{2} < p(x) < \gamma(x) < q(x) < \beta(x), \ \forall x \in \overline{\Omega},$
with $q, p, s, \gamma, \beta \in C_+(\overline{\Omega})$ and $p^+ < \gamma^- \leq \gamma^+ < q^-$.

$(\mathcal{A}_4)$ For all $T > 0$:

$$M_1\, T^{\rho_1 - 1} \ \leq \ \mathcal{K}(T) \leq \ M_2\, T^{\rho_2 - 1} \quad \text{and} \quad \mathcal{L}(T) \ \geq \ \mathcal{K}(T)T,$$

where $M_2 \geq M_1 > 0$, $\rho_2 \geq \rho_1 > 1$, $\rho_2 < \dfrac{\gamma^-}{p^+}$ and $\mathcal{L}(T) = \int_0^T \mathcal{K}(y)dy$.

Remark 3.1. By Proposition 2.6 and the assumption $(\mathcal{A}_3)$, the embeddings

$$\mathcal{M} \hookrightarrow L^{\beta'(\cdot)q(\cdot)}(\Omega),\ \mathcal{M} \hookrightarrow L^{s'(\cdot)\gamma(\cdot)}(\Omega),\ \mathcal{M} \hookrightarrow L^{\tau(\cdot)}(\Omega) \text{ and } \mathcal{M} \hookrightarrow L^{\omega(\cdot)}(\Omega)$$

are compact and continuous.

Remark 3.2. By assumption $(\mathcal{A}_4)$, we obtain:

$$\frac{M_1}{\rho_1}\, T^{\rho_1} \ \leq \ \mathcal{L}(T) \leq \ \frac{M_2}{\rho_2}\, T^{\rho_2}, \ \ for\ all\ T > 0.$$

Definition 3.3. We say that the function $u \in \mathcal{M}$ is a **weak solution** of problem $(\mathcal{P})$, if:

$$\mathcal{K}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u.\Delta\vartheta dx + \eta\int_{\partial\Omega}\int_\Omega g(\varsigma, x)|u|^{\gamma(x)-2}u\vartheta dx d\varsigma$$

$$= \lambda\int_\Omega f(x)|u|^{q(x)-2}u\vartheta dx,$$

for all $\vartheta \in \mathcal{M}$, where $d\varsigma$ is the surface measure on $\partial\Omega$.

Let $\mathcal{J}_{\lambda,\eta}$ the energy functional corresponding to problem $(\mathcal{P})$ defined by:

$$\begin{aligned} \mathcal{J}_{\lambda,\eta} : \mathcal{M} &\mapsto \mathbb{R} \\ u &\mapsto \mathcal{Q}_1(u) - \lambda\mathcal{Q}_2(u) + \eta\mathcal{Q}_3(u), \end{aligned}$$

where:

$$\mathcal{Q}_1(u) = \mathcal{L}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right),\ \ \mathcal{Q}_2(u) = \int_\Omega \frac{1}{q(x)}f(x)|u|^{q(x)}dx,$$

and

$$\mathcal{Q}_3(u) = \int_{\partial\Omega}\int_{\Omega} \frac{1}{\gamma(x)} g(\varsigma, x)|u|^{\gamma(x)} dx d\varsigma.$$

The main result of this article is expressed as follows:

Theorem 3.4. *If we assume that conditions* $(\mathcal{A}_1) - (\mathcal{A}_4)$ *are satisfied, then the problem* $(\mathcal{P})$ *will have a nontrivial weak solution, for all* $\lambda, \eta > 0$.

Before proving the last theorem, we will need the following lemmas:

Lemma 3.5. *If* $(\mathcal{A}_1) - (\mathcal{A}_4)$ *are satisfied, then* $\mathcal{J}_{\lambda,\eta}$ *is of class* $C^1(\mathcal{M}, \mathbb{R})$ *and verifies:*

$$\begin{aligned}
\langle \mathcal{J}'_{\lambda,\eta}(u), \vartheta \rangle &= \mathcal{K}\left(\int_{\Omega} \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_{\Omega}|\Delta u|^{p(x)-2}\Delta u.\Delta\vartheta dx \\
&+ \eta \int_{\partial\Omega}\int_{\Omega} g(\varsigma, x)|u|^{\delta(x)-2}u\vartheta dx d\varsigma - \lambda\int_{\Omega} f(x)|u|^{q(x)-2}u\vartheta dx, \quad \forall \vartheta \in \mathcal{M}.
\end{aligned}$$

Proof. Firstly, we will show that $\mathcal{Q}_3 \in C^1(\mathcal{M}, \mathbb{R})$. For this, we need to verify that for all $\upsilon \in \mathcal{M}$,

$$\lim_{t\to 0} \frac{\mathcal{Q}_3(u + t\upsilon) - \mathcal{Q}_3(u)}{t} = \langle \mathcal{Q}'_3(u), \upsilon \rangle,$$

and $\mathcal{Q}_3 : \mathcal{M} \mapsto \mathcal{M}^*$ is continuous.
Indeed, the application : $t \to \frac{1}{\gamma(x)} g(\varsigma, x)|u + t\upsilon|^{\gamma(x)}$ is differentiable for all $(\varsigma, x) \in \partial\Omega \times \Omega$. Then, by Propositions 2.1 and 2.3, for $|t| < 1$, we have:

$$\begin{aligned}
\int_{\partial\Omega}\int_{\Omega}\left|\frac{\partial}{\partial t}\left(\frac{1}{\gamma(x)}g(\varsigma, x)|u + t\upsilon|^{\gamma(x)}\right)\right| dx d\varsigma &= \int_{\partial\Omega}\int_{\Omega}\left|g(\varsigma, x)|u + t\upsilon|^{\gamma(x)-2}(u + t\upsilon)\upsilon\right| dx d\varsigma \\
&\le \int_{\partial\Omega}\int_{\Omega} g(\varsigma, x)\big(|u| + |t\upsilon|\big)^{\gamma(x)-1}|\upsilon| dx d\varsigma \\
&\le \int_{\partial\Omega}\int_{\Omega} g(\varsigma, x)\big(|u| + |t||\upsilon|\big)^{\gamma(x)-1}|\upsilon| dx d\varsigma \\
&\le \int_{\partial\Omega} 3|g(\varsigma, .)|_{s(x)}\left|\big(|u| + |\upsilon|\big)^{\gamma(x)-1}\right|_{\frac{\gamma(x)}{\gamma(x)-1}}|\upsilon|_{\frac{s(x)\gamma(x)}{s(x)-\gamma(x)}} d\varsigma \\
&\le 3|\upsilon|_{\omega(x)}\big||u| + |\upsilon|\big|_{\gamma(x)}^{\gamma^i-1}\int_{\partial\Omega}|g(\varsigma, .)|_{s(x)} d\varsigma \\
&< +\infty,
\end{aligned}$$

due to the fact that: $u, \upsilon \in \mathcal{M} \hookrightarrow L^{\gamma(x)}(\Omega)$, $\upsilon \in \mathcal{M} \hookrightarrow L^{\omega(x)}(\Omega)$, $g(., x) \in L^{r(x)}(\Omega)$ and $|g(\varsigma, .)|_{s(x)} \in L^1(\partial\Omega)$.
So, from the theorem of differentiation under the integral sign, we deduce that:

$$\begin{aligned}
\lim_{t\to 0}\frac{\mathcal{Q}_3(u + t\upsilon) - \mathcal{Q}_3(u)}{t} &= \frac{d}{dt}\mathcal{Q}_3(u + t\upsilon)|_{t=0} \\
&= \frac{d}{dt}\left(\int_{\partial\Omega}\int_{\Omega}\frac{1}{\gamma(x)}g(\varsigma, x)|u + t\upsilon|^{\gamma(x)} dx d\varsigma\right)|_{t=0} \\
&= \int_{\partial\Omega}\int_{\Omega}\frac{\partial}{\partial t}\left(\frac{1}{\gamma(x)}g(\varsigma, x)|u + t\upsilon|^{\gamma(x)}\right)|_{t=0} dx d\varsigma \\
&= \int_{\partial\Omega}\int_{\Omega}\left(g(\varsigma, x)|u + t\upsilon|^{\gamma(x)-2}(u + t\upsilon)\upsilon\right)|_{t=0} dx d\varsigma \\
&= \int_{\partial\Omega}\int_{\Omega} g(\varsigma, x)|u|^{\gamma(x)-2}u\upsilon dx d\varsigma \\
&= \langle \mathcal{Q}'_3(u), \upsilon \rangle.
\end{aligned}$$

As $\mathcal{M}$ is continuously embedded in $L^{\omega(x)}(\Omega)$, then exists $c_1 > 0$ such that:

$$|\upsilon|_{\omega(x)} \leq c_1\|\upsilon\|, \ \ \forall \upsilon \in \mathcal{M}. \tag{3.1}$$

Using the relation (3.1) and Propositions 2.1 and 2.3, we get:

$$\begin{aligned} |\langle \mathcal{Q}_3'(u), \upsilon\rangle| = \left|\int_{\partial\Omega}\int_{\Omega} g(\varsigma, x)|u|^{\gamma(x)-2}u\upsilon dx d\varsigma\right| &\leq 3|u|_{\gamma(x)}^{\gamma^i-1}|\upsilon|_{\omega(x)}\int_{\partial\Omega}|g(\varsigma,.)|_{s(x)}d\varsigma \\ &\leq 3c_1|u|_{\gamma(x)}^{\gamma^i-1}\|\upsilon\|\int_{\partial\Omega}|g(\varsigma,.)|_{s(x)}d\varsigma. \end{aligned}$$

Hence, there exists $c_2 = 3c_1|u|_{\gamma(x)}^{\gamma^i-1}\int_{\partial\Omega}|g(\varsigma,.)|_{s(x)}d\varsigma > 0$, such that:

$$\left|\langle \mathcal{Q}_3'(u), \upsilon\rangle\right| \leq c_2\|\upsilon\|, \quad \forall \upsilon \in \mathcal{M}. \tag{3.2}$$

From the last relation and linearity of $\mathcal{Q}_3'(u)$, we deduce that: $\mathcal{Q}_3'(u) \in \mathcal{M}^*$.
We know that the map: $L^{\gamma(x)}(\Omega) \ni u \mapsto |u|^{\gamma(x)-2}u \in L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)$ is continuous (see [7, Lemma 1]), so we conclude that $\mathcal{Q}_3$ is Frechet differentiable.
Finally,

$$\mathcal{Q}_3 \in C^1(\mathcal{M}, \mathbb{R}) \text{ and } \forall (u, \upsilon) \in \mathcal{M}^2 : \langle \mathcal{Q}_3'(u), \upsilon\rangle = \int_{\partial\Omega}\int_{\Omega} g(\varsigma, x)|u|^{\gamma(x)-2}u\upsilon dx d\varsigma. \tag{3.3}$$

In a comparable way, one can establish that:

$$\mathcal{Q}_2 \in C^1(\mathcal{M}, \mathbb{R}) \text{ and } \forall (u, \upsilon) \in \mathcal{M}^2 : \langle \mathcal{Q}_2'(u), \upsilon\rangle = \int_{\Omega} f(x)|u|^{q(x)-2}u\upsilon dx. \tag{3.4}$$

On the other hand, from Proposition 2.7 we get:

$$\mathcal{Q}_1 \in C^1(\mathcal{M}, \mathbb{R}) \text{ and } \langle \mathcal{Q}_1'(u), \upsilon\rangle = \mathcal{K}\left(\int_{\Omega}\frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_{\Omega}|\Delta u|^{p(x)-2}\Delta u.\Delta\vartheta dx, \tag{3.5}$$

for all $(u, \upsilon) \in \mathcal{M}^2$.
By combining (3.3) – (3.5), we conclude that $\mathcal{J}_{\lambda,\eta} \in C^1(\mathcal{M}, \mathbb{R})$ and

$$\begin{aligned} \langle \mathcal{J}_{\lambda,\eta}'(u), \vartheta\rangle &= \mathcal{K}\left(\int_{\Omega}\frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_{\Omega}|\Delta u|^{p(x)-2}\Delta u.\Delta\vartheta dx \\ &+ \eta\int_{\partial\Omega}\int_{\Omega} g(\varsigma, x)|u|^{\delta(x)-2}u\vartheta dx d\varsigma - \lambda\int_{\Omega} f(x)|u|^{q(x)-2}u\vartheta dx, \end{aligned}$$

for all $(u, \vartheta) \in \mathcal{M}^2$.
The proof of Lemma 3.5 is complete. □

Remark 3.6. It should be noted that the critical points of the energy functional $\mathcal{J}_{\lambda,\eta}$ correspond to the weak solutions of problem$(\mathcal{P})$.

Lemma 3.7. *Suppose that the assumptions* $(\mathcal{A}_1) - (\mathcal{A}_4)$ *are satisfied. Then, for all* $\lambda, \eta > 0$, *there exist* $\Theta > 0$ *and* $\delta > 0$, *such that:*

$$\mathcal{J}_{\lambda,\eta}(u) \geq \delta > 0, \ \forall u \in \mathcal{M} \ \ with \ \ \|u\| = \Theta.$$

Proof. Since $\mathcal{M}$ is continuously embedded in $L^{\beta'(x)q(x)}(\Omega)$ and $L^{s'(x)\gamma(x)}(\Omega)$, then $\exists c_3, c_4 > 0$, such that

$$|u|_{\beta'(x)q(x)} \leq c_3\|u\| \quad \text{and} \quad |u|_{s'(x)\gamma(x)} \leq c_4\|u\|, \quad \forall u \in \mathcal{M}. \tag{3.6}$$

Let $\Theta \in (0, 1)$, such that $\Theta < min\left(1, c_3^{-1}, c_4^{-1}\right)$; therefore,

$$|u|_{\beta'(x)q(x)} \leq 1 \quad and \quad |u|_{s'(x)\gamma(x)} \leq 1, \quad \forall u \in \mathcal{M} \text{ with } \|u\| = \Theta. \tag{3.7}$$

Using the relations (3.6) – (3.7), the Hölder inequality and Proposition 2.3, for any $u \in \mathcal{M}$ with $\|u\| = \Theta$, we obtain:

$$\mathcal{Q}_2(u) = \int_\Omega \frac{1}{q(x)} f(x)|u|^{q(x)}dx \leq \frac{2}{q^-}|f(x)|_{\beta(x)} \left||u|^{q(x)}\right|_{\beta'(x)} \leq \frac{2}{q^-}|f(x)|_{\beta(x)}|u|^{q^-}_{\beta'(x)q(x)}$$
$$\leq c_5|f(x)|_{\beta(x)}\|u\|^{q^-}, \quad (3.8)$$

where $c_5 = \dfrac{2c_3^{q^-}}{q^-}$.

In addition, we also obtain:

$$\int_\Omega \frac{1}{\gamma(x)} g(\varsigma, x)|u|^{\gamma(x)}dx \geq -\frac{2}{\gamma^-}|g(\varsigma, .)|_{s(x)} \left||u|^{\gamma(x)}\right|_{s'(x)} \geq -\frac{2}{\gamma^-}|g(\varsigma, .)|_{s(x)}|u|^{\gamma^-}_{s'(x)\gamma(x)}$$
$$\geq -c_6|g(\varsigma, .)|_{s(x)}\ \|u\|^{\gamma^-},$$

where $c_6 = \dfrac{2c_4^{\gamma^-}}{\gamma^-}$.

Then,

$$\mathcal{Q}_3(u) = \int_{\partial\Omega}\int_\Omega \frac{1}{\gamma(x)} g(\varsigma, x)|u|^{\gamma(x)}dxd\varsigma \geq -c_6\ \|u\|^{\gamma^-}\int_{\partial\Omega}|g(\varsigma, .)|_{s(x)}d\varsigma. \quad (3.9)$$

By Remark 3.2 and Proposition 2.4, for all $u \in \mathcal{M}$ with $\|u\| < 1$, we have

$$\mathcal{Q}_1(u) = \mathcal{L}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right) \geq \frac{M_1}{\rho_1}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)^{\rho_1}$$
$$\geq \frac{M_1}{\rho_1(p^+)^{\rho_1}}\|u\|^{\rho_1 p^+}. \quad (3.10)$$

Consequently, by the last relations (3.8) – (3.10), for all $u \in \mathcal{M}$ with $\|u\| = \Theta$, we get:

$$\begin{aligned}\mathcal{J}_{\lambda,\eta}(u) =&\mathcal{Q}_1(u) - \lambda\mathcal{Q}_2(u) + \eta\mathcal{Q}_3(u)\\ \geq&\frac{M_1}{\rho_1(p^+)^{\rho_1}}\|u\|^{\rho_1p^+} - \lambda c_5|f(x)|_{\beta(x)}\|u\|^{q^-} - \eta c_6\int_{\partial\Omega}|g(\varsigma, .)|_{s(x)}d\varsigma\ \|u\|^{\gamma^-}\\ \geq&\frac{M_1}{\rho_1(p^+)^{\rho_1}}\|u\|^{\rho_1p^+} - \|u\|^{\gamma^-}\left(\lambda c_5|f(x)|_{\beta(x)} + \eta c_6\int_{\partial\Omega}|g(\varsigma, .)|_{s(x)}d\varsigma\right) \text{(because } \|u\|^{\gamma^-} > \|u\|^{q^-}\text{)}\\ =&\|u\|^{\rho_1p^+}\left[\frac{M_1}{\rho_1(p^+)^{\rho_1}} - \|u\|^{\gamma^- - \rho_1p^+}\left(\lambda c_5|f(x)|_{\beta(x)} + \eta c_6\int_{\partial\Omega}|g(\varsigma, .)|_{s(x)}d\varsigma\right)\right].\end{aligned} \quad (3.11)$$

It is easy to observe that

$$\frac{M_1}{\rho_1(p^+)^{\rho_1}} - \sigma^{\gamma^- - \rho_1p^+}\left(\lambda c_5|f(x)|_{\beta(x)} + \eta c_6\int_{\partial\Omega}|g(\varsigma, .)|_{s(x)}d\varsigma\right) > 0, \quad (3.12)$$

for all $\sigma \in (0, \sigma^\star)$, where

$$\sigma^\star < \left[\frac{M_1}{\rho_1(p^+)^{\rho_1}\left(\lambda c_5|f(x)|_{\beta(x)} + \eta c_6\int_{\partial\Omega}|g(\varsigma, .)|_{s(x)}d\varsigma\right)}\right]^{\dfrac{1}{\gamma^- - \rho_1p^+}}.$$

Finally, from (3.11) – (3.12), we deduce for all $\lambda, \eta > 0$, there exist $\Theta, \delta > 0$ such that

$$\mathcal{J}_{\lambda,\eta}(u) \geq \delta > 0,\ \forall u \in \mathcal{M} \text{ with } \|u\| = \Theta.$$

The proof of Lemma 3.7 is complete. □

Lemma 3.8. *Assume that the assumptions* $(\mathcal{A}_1)-(\mathcal{A}_4)$ *hold. Then, there exists* $\xi \in \mathcal{M}$ *with* $\|\xi\| > \Theta$*, such that* $\mathcal{J}_{\lambda,\eta}(\xi) < 0$*, where* Θ *is given in Lemma 3.7.*

Proof. Let $\mathcal{G} \in C_0^\infty(\Omega)$, such that $\mathcal{G} \geq 0$ and $\mathcal{G} \neq 0$. For $t > 1$, we have

$$\begin{aligned}
\mathcal{J}_{\lambda,\eta}(t\mathcal{G}) &= \mathcal{L}\left(\int_\Omega \frac{1}{p(x)}|\Delta t\mathcal{G}|^{p(x)}dx\right) - \lambda\int_\Omega \frac{1}{q(x)} f(x)|t\mathcal{G}|^{q(x)}dx \\
&+ \eta\int_{\partial\Omega}\int_\Omega \frac{1}{\gamma(x)} g(\varsigma,x)|t\mathcal{G}|^{\gamma(x)}dxd\varsigma \\
&\leq \frac{M_2}{\rho_2}\left(\int_\Omega \frac{1}{p(x)}|\Delta t\mathcal{G}|^{p(x)}dx\right)^{\rho_2} - \lambda\frac{t^{q^-}}{q^+}\int_\Omega f(x)|\mathcal{G}|^{q(x)}dx \\
&+ \eta\frac{t^{\gamma^+}}{\gamma^-}\int_{\partial\Omega}\int_\Omega g(\varsigma,x)|\mathcal{G}|^{\gamma(x)}dxd\varsigma \\
&\leq \frac{M_2}{\rho_2(p^-)^{\rho_2}} t^{\rho_2 p^+}\left(\int_\Omega |\Delta\mathcal{G}|^{p(x)}dx\right)^{\rho_2} - \frac{\lambda}{q^+}t^{q^-}\int_\Omega f(x)|\mathcal{G}|^{q(x)}dx \\
&+ \frac{\eta}{\gamma^-}t^{\gamma^+}\int_{\partial\Omega}\int_\Omega g(\varsigma,x)|\mathcal{G}|^{\gamma(x)}dxd\varsigma := \mathcal{D}(t)
\end{aligned} \tag{3.13}$$

We have $\lim_{t\to+\infty}\mathcal{D}(t) = -\infty$, due to $q^- > \gamma^+ > \rho_2 p^+ > 1$. Hence, by (3.13), we obtain

$$\lim_{t\to+\infty}\mathcal{J}_{\lambda,\eta}(t\mathcal{G}) = -\infty.$$

Ultimately, for $t > 1$ large enough, we can choose $\xi = t\mathcal{G}$ such that:

$$\|\xi\| > \Theta \text{ and } \mathcal{J}_{\lambda,\eta}(\xi) < 0.$$

The proof of Lemma 3.8 is complete. □

Proof of Theorem 3.4. By Lemmas 3.5 – 3.8, and the fact that $\mathcal{J}_{\lambda,\eta}(0) = 0$, we deduce that $\mathcal{J}_{\lambda,\eta}$ satisfies all the conditions of Proposition 2.8. Therefore, we conclude the existence of a sequence $(\mathcal{Z}_n) \subset \mathcal{M}$ such that

$$\mathcal{J}_{\lambda,\eta}(\mathcal{Z}_n) \to \mathcal{C} > 0,\ \ \mathcal{J}'_{\lambda,\eta}(\mathcal{Z}_n) \to 0 \text{ in } \mathcal{M}^* \text{ as } n \to +\infty, \tag{3.14}$$

where

$$\mathcal{C} = \inf_{\varphi\in\Gamma}\max_{t\in[0,1]}\mathcal{J}_{\lambda,\eta}(\varphi(t)),$$

and

$$\Gamma = \{\varphi \in C([0,1],\mathcal{M}) \mid \varphi(0) = 0, \varphi(1) = \xi\}.$$

Step 1: We show that $(\mathcal{Z}_n)$ is bounded in $\mathcal{M}$.
By contradiction, we assume $\|\mathcal{Z}_n\| \to \infty$ as $n \to +\infty$ and $\|\mathcal{Z}_n\| > 1$.

For n large enough, we have

$$\begin{aligned}
& 1+\mathcal{C}+\|\mathcal{Z}_n\| \\
\geq\ & \mathcal{J}_{\lambda,\eta}(\mathcal{Z}_n)-\frac{1}{q^-}\left\langle \mathcal{J}'_{\lambda,\eta}(\mathcal{Z}_n),\mathcal{Z}_n\right\rangle \\
=\ & \mathcal{L}\left(\int_\Omega \frac{1}{p(x)}|\Delta\mathcal{Z}_n|^{p(x)}dx\right)-\frac{1}{q^-}\mathcal{K}\left(\int_\Omega \frac{1}{p(x)}|\Delta\mathcal{Z}_n|^{p(x)}dx\right)\int_\Omega |\Delta\mathcal{Z}_n|^{p(x)}dx \\
+\ & \eta\left(\int_{\partial\Omega}\int_\Omega \frac{1}{\gamma(x)}g(\varsigma,x)|\mathcal{Z}_n|^{\gamma(x)}dxd\varsigma-\frac{1}{q^-}\int_{\partial\Omega}\int_\Omega g(\varsigma,x)|\mathcal{Z}_n|^{\gamma(x)}dxd\varsigma\right) \\
+\ & \lambda\left(\frac{1}{q^-}\int_\Omega f(x)|\mathcal{Z}_n|^{q(x)}dx-\int_\Omega \frac{1}{q(x)}f(x)|\mathcal{Z}_n|^{q(x)}dx\right). \\
\geq\ & \mathcal{K}\left(\int_\Omega \frac{1}{p(x)}|\Delta\mathcal{Z}_n|^{p(x)}dx\right)\int_\Omega \frac{1}{p(x)}|\Delta\mathcal{Z}_n|^{p(x)}dx-\frac{1}{q^-}\mathcal{K}\left(\int_\Omega \frac{1}{p(x)}|\Delta\mathcal{Z}_n|^{p(x)}dx\right)\int_\Omega |\Delta\mathcal{Z}_n|^{p(x)}dx \\
+\ & \eta\left(\frac{1}{\gamma^+}\int_{\partial\Omega}\int_\Omega g(\varsigma,x)|\mathcal{Z}_n|^{\gamma(x)}dxd\varsigma-\frac{1}{q^-}\int_{\partial\Omega}\int_\Omega g(\varsigma,x)|\mathcal{Z}_n|^{\gamma(x)}dxd\varsigma\right) \\
+\ & \lambda\left(\frac{1}{q^-}\int_\Omega f(x)|\mathcal{Z}_n|^{q(x)}dx-\frac{1}{q^-}\int_\Omega f(x)|\mathcal{Z}_n|^{q(x)}dx\right). \\
\geq\ & \frac{M_1}{(p^+)^{\rho_1-1}}\left(\int_\Omega |\Delta\mathcal{Z}_n|^{p(x)}dx\right)^{\rho_1-1}\left[\frac{1}{p^+}\int_\Omega |\Delta\mathcal{Z}_n|^{p(x)}dx-\frac{1}{q^-}\int_\Omega |\Delta\mathcal{Z}_n|^{p(x)}dx\right] \\
+\ & \eta\left(\frac{1}{\gamma^+}-\frac{1}{q^-}\right)\int_{\partial\Omega}\int_\Omega g(\varsigma,x)|\mathcal{Z}_n|^{\gamma(x)}dxd\varsigma \\
\geq\ & \frac{M_1}{(p^+)^{\rho_1-1}}\|\mathcal{Z}_n\|^{\rho_1 p^-}\left(\frac{1}{p^+}-\frac{1}{q^-}\right)+\eta\left(\frac{1}{\gamma^+}-\frac{1}{q^-}\right)\int_{\partial\Omega}\int_\Omega g(\varsigma,x)|\mathcal{Z}_n|^{\gamma(x)}dxd\varsigma.
\end{aligned}$$

Since $q^- > \gamma^+$ and $g(\varsigma,x)>0$ for all $(\varsigma,x)\in\partial\Omega\times\Omega$, we deduce that

$$1+\mathcal{C}+\|\mathcal{Z}_n\| \geq \frac{M_1}{(p^+)^{\rho_1-1}}\|\mathcal{Z}_n\|^{\rho_1 p^-}\left(\frac{1}{p^+}-\frac{1}{q^-}\right), \tag{3.15}$$

Dividing the last relation by $\|\mathcal{Z}_n\|^{\rho_1 p^-}$, we get

$$\frac{1+\mathcal{C}}{\|\mathcal{Z}_n\|^{\rho_1 p^-}}+\frac{1}{\|\mathcal{Z}_n\|^{\rho_1 p^- -1}} \geq \frac{M_1}{(p^+)^{\rho_1-1}}\left(\frac{1}{p^+}-\frac{1}{q^-}\right),$$

Passing to the limit as $n\to+\infty$, we obtain a contradiction since $\rho_1 p^- > 1$, $q^- > p^+$ and $\frac{M_1}{(p^+)^{\rho_1-1}}>0$. Finally, $(\mathcal{Z}_n)$ is bounded in $\mathcal{M}$.

Step 2: We prove that $(\mathcal{Z}_n)$ converge strongly to $\mathcal{Z}$ in $\mathcal{M}$.
According to the reflexivity of $\mathcal{M}$, for a sub-sequence still noted $(\mathcal{Z}_n)_n$, we have:

$$\mathcal{Z}_n \rightharpoonup \mathcal{Z} \ \ in \ \ \mathcal{M}. \tag{3.16}$$

From Propositions 2.1 and 2.3, we get:

$$\begin{aligned}
\left|\int_\Omega f(x)|\mathcal{Z}_n|^{q(x)-2}\mathcal{Z}_n(\mathcal{Z}_n-\mathcal{Z})dx\right| &\leq 3|f(x)|_{\beta(x)}\left||\mathcal{Z}_n|^{q(x)-1}\right|_{\frac{q(x)}{q(x)-1}}|\mathcal{Z}_n-\mathcal{Z}|_{\frac{\beta(x)q(x)}{\beta(x)-q(x)}} \\
&\leq 3|f(x)|_{\beta(x)}|\mathcal{Z}_n|_{q(x)}^{q^i-1}|\mathcal{Z}_n-\mathcal{Z}|_{\tau(x)}.
\end{aligned} \tag{3.17}$$

Also,

$$\begin{aligned}
\left|\int_\Omega g(\varsigma,x)|\mathcal{Z}_n|^{\gamma(x)-2}\mathcal{Z}_n(\mathcal{Z}_n-\mathcal{Z})dx\right| &\leq 3|g(\varsigma,.)|_{s(x)}\left||\mathcal{Z}_n|^{\gamma(x)-1}\right|_{\frac{\gamma(x)}{\gamma(x)-1}}|\mathcal{Z}_n-\mathcal{Z}|_{\frac{s(x)\gamma(x)}{s(x)-\gamma(x)}} \\
&\leq 3|g(\varsigma,.)|_{s(x)}|\mathcal{Z}_n|_{\gamma(x)}^{\gamma^i-1}|\mathcal{Z}_n-\mathcal{Z}|_{\omega(x)}.
\end{aligned}$$

The previous inequality implies that

$$\left| \int_{\partial\Omega} \int_{\Omega} g(\varsigma, x) |\mathcal{Z}_n|^{\gamma(x)-2}\, \mathcal{Z}_n(\, \mathcal{Z}_n - \mathcal{Z}) dx d\varsigma \right| \leq 3 |\mathcal{Z}_n - \mathcal{Z}|_{\omega(x)} |\mathcal{Z}_n|_{\gamma(x)}^{\gamma^i - 1} \int_{\partial\Omega} |g(\varsigma, .)|_{s(x)} d\varsigma. \tag{3.18}$$

Since:

- $(\mathcal{Z}_n)$ is bounded in $L^{\gamma(x)}(\Omega)$ and in $L^{q(x)}(\Omega)$ because it is bounded in $\mathcal{M}$, which is continuously embedded into both $L^{\gamma(x)}(\Omega)$ and $L^{q(x)}(\Omega)$.
- $\lim\limits_{n\to+\infty} |\mathcal{Z}_n - \mathcal{Z}|_{\omega(x)} = 0$ and $\lim\limits_{n\to+\infty} |\mathcal{Z}_n - \mathcal{Z}|_{\tau(x)} = 0$, as a consequence of (3.16) and the compact embeddings $\mathcal{M} \hookrightarrow L^{\omega(x)}(\Omega)$ and $\mathcal{M} \hookrightarrow L^{\tau(x)}(\Omega)$.

Hence, from the relations (3.17) – (3.18), we deduce that:

$$\lim_{n\to+\infty} \int_{\partial\Omega} \int_{\Omega} g(\varsigma, x) |\mathcal{Z}_n|^{\gamma(x)-2} \mathcal{Z}_n(\mathcal{Z}_n - \mathcal{Z}) dx d\varsigma = 0. \tag{3.19}$$

Moreover,

$$\lim_{n\to+\infty} \int_{\Omega} f(x) |\mathcal{Z}_n|^{q(x)-2} \mathcal{Z}_n(\mathcal{Z}_n - \mathcal{Z}) dx = 0. \tag{3.20}$$

According to (3.14) and the fact that $(\mathcal{Z}_n)_n$ is bounded in $\mathcal{M}$, one has:

$$\begin{aligned} \left|\left\langle \mathcal{J}'_{\lambda,\eta}(\mathcal{Z}_n), \mathcal{Z}_n - \mathcal{Z} \right\rangle\right| &\leq \left|\left\langle \mathcal{J}'_{\lambda,\eta}(\mathcal{Z}_n), \mathcal{Z}_n \right\rangle\right| + \left|\left\langle \mathcal{J}'_{\lambda,\eta}(\mathcal{Z}_n), \mathcal{Z} \right\rangle\right| \\ &\leq \|\mathcal{J}'_{\lambda,\eta}(\mathcal{Z}_n)\| \|\mathcal{Z}_n\| + \|\mathcal{J}'_{\lambda,\eta}(\mathcal{Z}_n)\| \|\mathcal{Z}\| \\ &\Longrightarrow \left\langle \mathcal{J}'_{\lambda,\eta}(\mathcal{Z}_n), \mathcal{Z}_n - \mathcal{Z} \right\rangle \to 0. \end{aligned} \tag{3.21}$$

Then, by combining (3.19) – (3.21), we obtain:

$$\lim_{n\to+\infty} \mathcal{K}\left(\int_{\Omega} \frac{1}{p(x)} |\Delta\mathcal{Z}_n|^{p(x)} dx \right) \int_{\Omega} |\Delta\mathcal{Z}_n|^{p(x)-2} \Delta\mathcal{Z}_n . \Delta(\mathcal{Z}_n - \mathcal{Z}) dx = 0.$$

Passing to a subsequence, if necessary, we may assume that

$$\lim_{n\to+\infty} \int_{\Omega} \frac{1}{p(x)} |\Delta\mathcal{Z}_n|^{p(x)} dx = \mathcal{E} \geq 0.$$

If $\mathcal{E} = 0$, then $\mathcal{Z}_n \to 0$ in $\mathcal{M}$ as $n \to +\infty$ and the proof is finished.
If $\mathcal{E} > 0$, then by continuity of $\mathcal{K}$, we have

$$\lim_{n\to+\infty} \mathcal{K}\left(\int_{\Omega} \frac{1}{p(x)} |\Delta\mathcal{Z}_n|^{p(x)} dx \right) = \mathcal{K}(\mathcal{E}) > 0.$$

From assumption $(\mathcal{A}_4)$, for n large enough, we obtain

$$0 < c_7 \leq \mathcal{K}\left(\int_{\Omega} \frac{1}{p(x)} |\Delta\mathcal{Z}_n|^{p(x)} dx \right) \leq c_8.$$

Thus, it follows that

$$\lim_{n\to+\infty} \langle \mathcal{B}(\mathcal{Z}_n), \mathcal{Z}_n - \mathcal{Z} \rangle = 0,$$

where $\langle \mathcal{B}(\mathcal{Z}_n), \mathcal{Z}_n - \mathcal{Z} \rangle = \int_{\Omega} |\Delta\mathcal{Z}_n|^{p(x)-2} \Delta\mathcal{Z}_n . \Delta(\mathcal{Z}_n - \mathcal{Z}) dx = 0$.
Eventually, by Proposition 2.7, we deduce that $\mathcal{Z}_n \to \mathcal{Z}$ in $\mathcal{M}$.
Finally, using (3.14) and the fact that $\mathcal{J}_{\lambda,\eta} \in C^1(\mathcal{M}, \mathbb{R})$, we conclude that

$$\mathcal{J}_{\lambda,\eta}(\mathcal{Z}) = \mathcal{C} > 0 \quad \text{and} \quad \mathcal{J}'_{\lambda,\eta}(\mathcal{Z}) = 0.$$

That is, $\mathcal{Z}$ is a nontrivial weak solution of the problem $(\mathcal{P})$ for all $\lambda, \eta > 0$.
The proof of Theorem 3.4 is complete. □

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