Palestine Journal of Mathematics

Vol 15(2)(2026) , 328–338

1-Absorbing Primary Ideals of a Lattice

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Communicated by: Ayman Badawi

MSC 2020 Classifications: 06B10, 03G10.

Keywords and phrases: Lattice, Ideal, Primary ideal, 1-absorbing primary ideal.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract Let L be a lattice with the greatest element 1. In this paper, we introduce the concept of 1-absorbing primary ideals of L as a generalization of primary ideals. A proper ideal I of L is called 1-absorbing primary ideal if for all nonunit elements $a, b, c \in L$ such that $a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $c \in \sqrt{I}$. Some properties of 1-absorbing primary ideals are investigated. We show that a proper ideal I of L is a 1-absorbing primary ideal of L if and only if whenever $I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L, then either $I_1 I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$. Some properties of primary ideals and 1-absorbing primary ideals are studied under lattice homomorphisms. Finally, we have studied different properties of 1-absorbing primary ideals in product lattices.

1 Introduction

The study of prime ideals is an important aspect of ring theory. The concept of prime ideals in a ring was first studied by Krull [14] and Fitting [9] and later on Golan [10] introduced the term prime ideal.

After that, the notion of prime ideals has been extended and generalized by many researchers in many different ways. For instance, in 1978, Hedstrom and Houston [13] introduced the concept of strongly prime ideal using the notion of quotient field of a ring. For a ring R and the quotient field K of R, they defined a proper ideal I of R to be strongly prime if for $a, b \in K$ with $ab \in I$, then either $a \in I$ or $b \in I$. In 2003, Anderson and Smith [1] introduced the notion of weakly prime ideals of a commutative ring as a generalization of prime ideals. A proper ideal I of a commutative ring R is called weakly prime ideal if for $a, b \in R$ and $0 \neq ab \in I$, either $a \in I$ or $b \in I$. In 2005, Bhatwadekar and Sharma [6] introduced the notion of almost prime ideals which is also a generalisation of prime ideals. A proper ideal I of an integral domain R is said to be almost prime if for $a, b \in R$ with $ab \in I \setminus I^2$, then either $a \in I$ or $b \in I$. It is clear that every weakly prime ideal is an almost prime ideal. Another generalisation of prime ideal is 2-Prime ideal. The concept of 2-Prime ideals and their applications were introduced by Beddani and Messirdi[5] in 2016. A nonzero proper ideal I of R is called a 2-Prime ideal if for $a, b \in R$ and $ab \in I$, then either $a^2 \in I$ or $b^2 \in I$. The concept of 1-absorbing prime ideals of commutative rings were studied by Yassine, Nikmehr and Nikandish [18] in 2021. Following this, the concept of 1-absorbing primary ideals of commutative rings were studied by Badawi and Yetkin [4] in 2020. In this way, a significant amount of research work has been done by many researchers on various extensions and generalizations of prime ideals of commutative rings over the years.

The concept of primary ideals and their generalizations are also studied in ring theory . Badawi [2] introduced the concept of 2-absorbing ideal in a commutative ring. Later on, Payrovi and Babaei [16] extended the concept of 2-absorbing ideal in a commutative ring. Furthermore, Badawi et.al [3] introduced the notion of 2-absorbing primary ideals in commutative rings. After that, Mustafanasab and Darani [15] studied the concepts of 2-absorbing primary ideals and weakly 2-absorbing primary ideals in commutative rings.

The study of prime ideals, primary ideals and absorbing ideals have drawn attention in lattice theory also. Recently, Estaji and Haghdadi [8] studied the notion of n-absorbing ideals in a lattice. For a positive integer n, a proper ideal I of a lattice L is said to be an n-absorbing ideal of L whenever $a_1 \wedge a_2 \wedge ... \wedge a_{n+1} \in I$, then there are n of the a_i's whose meet is in I for all $a_1, a_2, ...a_{n+1} \in L$. Furthermore, Wasadikar and Gaikwad [17] studied the concept of 2-absorbing ideals and 2-absorbing primary ideals in lattices and discussed their various properties. The notion of 1-absorbing prime ideals in lattices has been introduced and studied in [7].

In this paper, we introduce the notion of 1-absorbing primary ideals of a lattice. We discuss several characterization properties of 1-absorbing primary ideals of a lattice. We show that a proper ideal I of a lattice L is a 1-absorbing primary ideal of L if and only if whenever$I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L, then either $I_1 I_2 \subseteq I$ or $I_3 \subseteq I$. We have also shown that if I is a 1-absorbing primary ideal of L, then $\sqrt{I}$ is a prime ideal of L. Furthermore, we have studied the properties of primary ideals and 1-absorbing primary ideals under lattice homomorphisms. Finally, we have studied various properties of 1-absorbing primary ideals in product lattices.

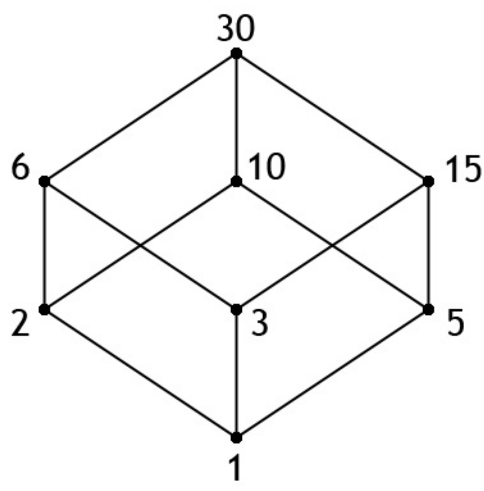
2 Preliminaries

In this section, we discuss some preliminary definitions and results of lattice theory which will be needed in the consequent sections. Throughout this paper we consider L to be a lattice with the greatest element 1. Any unreferred and undefined terminology related to lattice theory can be found in [11, 12]. We begin the section with the following definition.

Definition 2.1: A non-empty subset I of L is called an ideal of L if (i) $a, b \in I$ implies $a \vee b \in I$ and (ii) $a \in I, l \in L$ implies $a \wedge l \in I$. An ideal I of L is said to be proper if $I \neq L$.

Definition 2.2: Let L be a lattice. A proper ideal I of L is called prime if $a, b \in L$ such that $a \wedge b \in I$, then either $a \in I$ or $b \in I$.

Example 2.3: Let us consider the lattice of divisors of 30. In this lattice, $I = \{1, 2, 5, 10\}$ is a prime ideal.



L

Fig. 1.

Definition 2.4: Let L be a lattice and $a \in L$ be any element. Then the principal ideal generated by a denoted by $(a]$, is defined as $(a] = \{x \in L \mid x \leq a\}$.

Definition 2.5: The radical of an ideal I of L denoted by $\sqrt{I}$ is defined as the intersection of all prime ideals of L containing I.

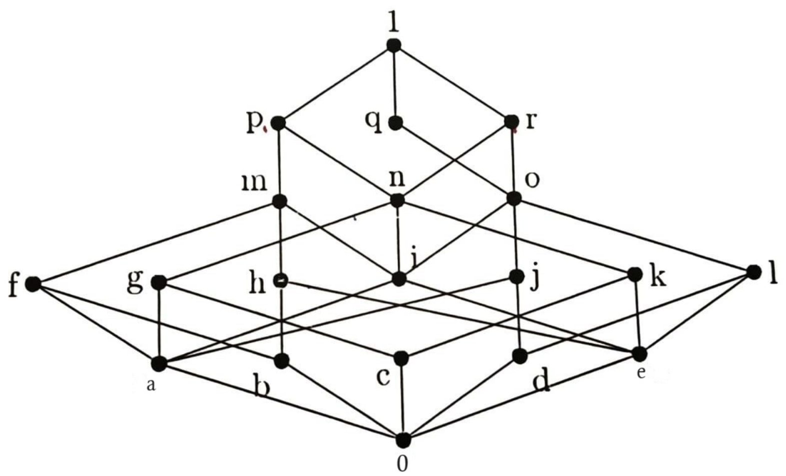
Remark 2.6: If there does not exist a prime ideal containing an ideal I in a lattice L, then

$\sqrt{I} = L$.

Definition 2.7: A lattice L is said to be a distributive lattice if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.

Remark 2.8: Following Gratzer[11, p.75] in a distributive lattice L, an ideal I is the intersection of all prime ideals containing it, that is, $I = \sqrt{I}$.
However, this may or may not hold in case of a non distributive lattice which can be observed from the following example.

Example 2.9: Let us consider the lattice L as shown in Fig. 2 below. Clearly, the lattice L here is non distributive. Now, let us consider the ideal $I = (n]$ of L. Then, $\sqrt{I} = (p] \cap (r] = (n]$ which shows that $\sqrt{I} = I$.



L

Fig. 2.

Again, let us consider the ideal $I = (l]$ of the lattice L as shown in Fig. 2 above. Then we have, $\sqrt{I} = (q] \cap (r] = (o]$ which shows that $I \subset \sqrt{I}$.

Definition 2.10:[17] Let L be a lattice. A proper ideal I of L is called primary if $a, b \in L$ such that $a \wedge b \in I$, then either $a \in I$ or $b \in \sqrt{I}$.

Example 2.11: Let us consider the ideal $I = (m]$ of the lattice L as shown in Fig. 2 above. Here, $\sqrt{I} = (p]$. Clearly, I is a primary ideal of L.
Again, let us consider the ideal $I = (j]$ of L. Here, $\sqrt{I} = (q] \cap (r] = (o]$. The ideal I is not a primary ideal since $f \wedge g = a \in I$ but neither $f \in I$ nor $g \in \sqrt{I}$.

Lemma 2.12: Let L be a lattice. If I is a prime ideal of L, then $\sqrt{I} = I$.
However, the converse of this lemma need not hold always which can be seen from the following example.

Example 2.13: Let us consider the ideal $I = (i]$ of the lattice L as shown in Fig. 2 above. It is clear that $\sqrt{I} = I$.
However $h \wedge k = e \in I$ but neither $h \in I$ nor $k \in I$. Hence, I is not a prime ideal of L.

Lemma 2.14:[17] If I is a prime ideal of a lattice L, then I is a primary ideal of L.
However, the converse of this lemma does not hold in general as shown in the following example.

Example 2.15: Let us consider the ideal $I = (m]$ of the lattice L as shown in Fig. 2 above. Here, $\sqrt{I} = (p]$ as $(p]$ is the only prime ideal containing I. Clearly, I is a primary ideal of L. However $k \wedge l = e \in I$ but neither $k \in I$ nor $l \in I$. Thus, I is not a prime ideal L.

Definition 2.16: An element a of a lattice L is called a unit if there exist $b \in L$ such that $a \wedge b = 1$.

Definition 2.17: A nonzero nonunit element a of a lattice L is called irreducible if $a = b \wedge c$ for some $b, c \in L$, then b is a unit of L or c is a unit of L.

Definition 2.18:[7] Let L be a lattice. A proper ideal I of L is called a 1-absorbing prime ideal if for all nonunit elements $a, b, c \in L$ such that $a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $c \in I$.

Definition 2.19:[17] Let L be a lattice. A proper ideal I of L is called 2-absorbing if for $a, b, c \in L$ such that $a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.

Definition 2.20:[17] Let L be a lattice. A proper ideal I of L is called 2-absorbing primary if for $a, b, c \in L$ such that $a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$.

Lemma 2.21:[17] If I is a primary ideal of a lattice L, then I is a 2-absorbing primary ideal of L.
However, the following example shows that the converse of this lemma does not hold.

Example 2.22: Let us consider the ideal $I = (l]$ of the lattice L as shown in Fig.2 above. Then, $\sqrt{I} = (q] \cap (r] = (o]$. Clearly, I is a 2-absorbing primary ideal of L. Now, $g \wedge h = 0 \in I$, but neither $g \in I$ nor $h \in \sqrt{I}$. Thus, I is not a primary ideal of L.

Remark 2.23: Radicalization of an ideal of a lattice is an idempotent operation. For a proper ideal I of a lattice L, we have $\sqrt{\sqrt{I}} = \sqrt{I}$. Moreover, $\sqrt{I}$ is the smallest radical ideal containing I.

Definition 2.24: Let L_1 and L_2 be two lattices. A mapping $f : L_1 \to L_2$ is called a lattice homomorphism if $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$, for all $a, b \in L_1$.

Lemma 2.25:[17] Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) If I' is an ideal of L', then $f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$.

(2) If f is an isomorphism and I is an ideal of L, then $f(\sqrt{I}) = \sqrt{f(I)}$.

Definition 2.26: Let $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ be two lattices. Then $(L, \wedge, \vee)$ is the direct product of lattices L_1 and L_2, where $L = L_1 \times L_2$ and the binary operation $\vee(join)$ and $\wedge(meet)$ on L are defined in such a way that for any (a_1, b_1) and (a_2, b_2) in L, we have $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$ and $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$.

Lemma 2.27:[17] Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices with 1. Then the following hold:

(1) If I_1 is an ideal of L_1, then $\sqrt{I_1 \times L_2} = \sqrt{I_1} \times L_2$.

(2) If I_2 is an ideal of L_2, then $\sqrt{L_1 \times I_2} = L_1 \times \sqrt{I_2}$.

Definition 2.28: A lattice L is called a bounded lattice if it has the greatest element 1 and a least element 0.

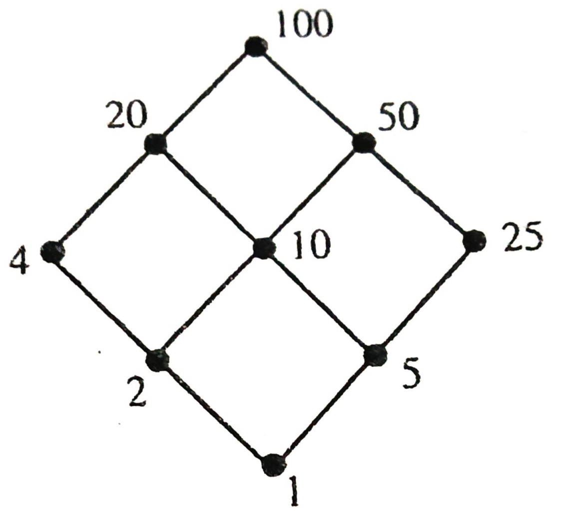
3 1-absorbing primary ideals of a lattice

In this section, we begin by introducing the concept of 1-Absorbing primary ideals of a lattice L in order to define an 1-absorbing version of well-known results regarding primary ideals.

Definition 3.1: Let L be a lattice. A proper ideal I of L is called a 1-absorbing primary ideal if for all nonunit elements $a, b, c \in L$ such that $a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $c \in \sqrt{I}$.

Clearly, every primary ideal of L is 1-absorbing primary but the converse is not true which can be observed from the following example.

Example 3.2: Let us consider the lattice L of divisors of 100 as shown below in Fig. 3 . Let us take the ideal $I = (2]$ of L. Then $\sqrt{I} = (4] \cap (50] = (2]$. Clearly, I is a 1-absorbing primary ideal of L. However, $4 \wedge 10 = 2 \in I$ but neither $4 \in I$ nor $10 \in \sqrt{I}$. Hence, I is not a primary ideal of L.



L

Fig.3.

Proposition 3.3: Let I be a proper ideal of a lattice L. If I is a 1-absorbing primary ideal of L, then I is a 2-absorbing primary ideal of L.

Proof: Let I be a proper ideal of L. Let us suppose that $a \wedge b \wedge c \in I$ for $a, b, c \in L$. If at least one of a, b and c is unit, say $a = 1$, then $b \wedge c \in I$ and we are done. Therefore, let us assume that a, b and c are all nonunit elements of L. Since I is a 1-absorbing primary ideal of L, so we have either $a \wedge b \in I$ or $c \in \sqrt{I}$. This gives either $a \wedge b \in I$ or $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$. Hence, I is 2-absorbing primary ideal of L.

The converse of the Proposition 3.3 does not hold always which can be shown by the following example.

Example 3.4: Let us consider the ideal $I = (0]$ of the lattice shown in Fig. 2. Then, $\sqrt{I} = (p] \cap (q] \cap (r] = (i]$. It is clear that I is a 2-absorbing primary ideal of L. However, $i \wedge j \wedge l = 0 \in I$, but neither $i \wedge j = a \in I$ nor $l \in \sqrt{I}$. Hence, I is not a 1-absorbing primary ideal of L.

Lemma 3.5: Let L be a lattice. If I and J are two ideals of L, then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

The above lemma says that the radical of an intersection of ideals of a lattice is equal to the intersection of their radicals.

Definition 3.6 Let I be a 1-absorbing primary ideal of a lattice L such that $\sqrt{I} = P$ is a prime ideal of L. Then, we call I a P-1-absorbing primary ideal of L.

Proposition 3.7: Let $I_1, I_2, ..., I_n$ be P-1-absorbing primary ideal of L, then $I = \cap_{i=1}^{n} I_i$ is a P-1-absorbing primary ideal of L.

Proof: First we observe that $\sqrt{I} = P$. Let us suppose that $a \wedge b \wedge c \in I$ for some nonunit elements $a, b, c \in L$ and $a \wedge b \notin I$. Without loss of generality, we may assume that $a \wedge b \notin I_1$. Since, I_1 is a P-1-absorbing primary ideal of L and $a \wedge b \notin I_1$, we must have $c \in P$. Thus, I is a

P-1-absorbing primary ideal of L.

Proposition 3.8: Let I be an ideal of a lattice L. Then $\sqrt{I}$ is a prime ideal of L if and only if $\sqrt{I}$ is a primary ideal of L.

Proof: Let us suppose that $\sqrt{I}$ is a prime ideal of L. If $a \wedge b \in \sqrt{I}$ for $a, b \in L$, then either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. This gives either $a \in \sqrt{I}$ or $b \in \sqrt{\sqrt{I}}$ (by Remark 2.23). Hence, $\sqrt{I}$ is a primary ideal of L.
Conversely, let us suppose that $\sqrt{I}$ is a primary ideal of L. Let us assume that $a \wedge b \in \sqrt{I}$ for $a, b \in L$. Since $\sqrt{I}$ is a primary ideal, so we have either $a \in \sqrt{I}$ or $b \in \sqrt{\sqrt{I}}$. This gives either $a \in \sqrt{I}$ or $b \in \sqrt{I}$ (by Remark 2.23). Thus, $\sqrt{I}$ is a prime ideal of L.

In the similar fashion we establish the following result.

Proposition 3.9: Let I be an ideal of a lattice L. Then $\sqrt{I}$ is a 1-absorbing prime ideal of L if and only if $\sqrt{I}$ is a 1-absorbing primary ideal of L.

Proof: Let us suppose that $\sqrt{I}$ is a 1-absorbing prime ideal of L and $a \wedge b \wedge c \in \sqrt{I}$ for some nonunit elements $a, b, c \in L$. Then, we have either $a \wedge b \in \sqrt{I}$ or $c \in \sqrt{I}$. This gives either $a \wedge b \in \sqrt{I}$ or $c \in \sqrt{\sqrt{I}}$ (by Remark 2.23). Hence, $\sqrt{I}$ is a 1-absorbing primary ideal of L.
Conversely, let us assume that $\sqrt{I}$ is a 1-absorbing primary ideal of L and $a \wedge b \wedge c \in \sqrt{I}$ for some nonunit elements $a, b, c \in L$. Then, we have either $a \wedge b \in \sqrt{I}$ or $c \in \sqrt{\sqrt{I}}$. This gives either $a \wedge b \in \sqrt{I}$ or $c \in \sqrt{I}$ (by Remark 2.23). Hence, $\sqrt{I}$ is a 1-absorbing prime ideal of L.

Lemma 3.10: Let I be a 1-absorbing primary ideal of a lattice L. If $a \wedge b \wedge J \subseteq I$ for all proper ideals J of L and nonunit elements $a, b \in L$, then $a \wedge b \in I$ or $J \subseteq \sqrt{I}$.

Proof: Let us suppose to the contrary that $a \wedge b \wedge J \subseteq I$, but $a \wedge b \notin I$ and $J \nsubseteq \sqrt{I}$. Then there exists a nonunit element $j \in J$ such that $j \notin \sqrt{I}$. This implies $a \wedge b \wedge j \in I$. Since I is a 1-absorbing primary ideal of L, so we have $a \wedge b \in I$ and $j \in \sqrt{I}$, a contradiction. Thus, $a \wedge b \in I$ or $J \subseteq \sqrt{I}$.

Proposition 3.11: Let I be a proper ideal of a lattice L. Then the following statements are equivalent:

(1) I is a 1-absorbing primary ideal of L.

(2) For any proper ideals I_1, I_2, I_3 of L such that $I_1 I_2 I_3 \subseteq I$ implies that either $I_1 I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.

Proof:

$(1) \Rightarrow (2)$ Let I be a 1-absorbing primary ideal of L. We assume that $I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L and $I_1 I_2 \nsubseteq I$. Then there exist nonunit elements $a \in I_1$ and $b \in I_2$ such that $a \wedge b \notin I$. Since, $a \wedge b \wedge I_3 \subseteq I$ and $a \wedge b \notin I$. so we can conclude that $I_3 \subseteq \sqrt{I}$ (by Lemma 3.10).

$(2) \Rightarrow (1)$ Let us suppose that I be a proper ideal of L such that if $I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L then $I_1 I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$. We show that I is a 1-absorbing primary ideal of L. Let $a \wedge b \wedge c \in I$ for nonunit elements $a, b, c \in L$. This implies that $(a] \wedge (b] \wedge (c] \subseteq I$. Let $I_1 = (a]$, $I_2 = (b]$ and $I_3 = (c]$. By hypothesis, either $(a] \wedge (b] \subseteq I$ or $(c] \subseteq \sqrt{I}$. Thus, we have either $a \wedge b \in I$ or $c \in \sqrt{I}$. Hence, I is a 1-absorbing primary ideal of L.

Proposition 3.12: Let P_1 and P_2 be two distinct primary ideals of a lattice L, then $P_1 \cap P_2$ is a 1-absorbing primary ideal of L.

Proof: Let $a, b, c \in L$ be nonunit elements such that $a \wedge b \wedge c \in P_1 \cap P_2$. Then $a \wedge b \wedge c \in P_1$ and $a \wedge b \wedge c \in P_2$. Since P_1 and P_2 are distinct primary ideals of L and we know that every primary ideal of L is a 1-absorbing primary ideal of L, so we have $a \wedge b \in P_1$ or $c \in \sqrt{P_1}$ and $a \wedge b \in P_2$ or $c \in \sqrt{P_2}$. Therefore, we have ($a \wedge b \in P_1$ and $a \wedge b \in P_2$) or ($c \in \sqrt{P_1}$ and $c \in \sqrt{P_2}$). This implies that $a \wedge b \in P_1 \cap P_2$ or $c \in \sqrt{P_1} \cap \sqrt{P_2}$. Thus, we have $a \wedge b \in P_1 \cap P_2$ or $c \in \sqrt{P_1 \cap P_2}$ (by Lemma 3.5). Hence, $P_1 \cap P_2$ is a 1-absorbing primary ideal of L.

In the following result we state the generalization of the above Proposition 3.15 .

Proposition 3.13: Let $P_1, P_2, ..., P_n$ be any primary ideals of a lattice L, then $I = \cap_{i=1}^{n} P_i$ is a 1-absorbing primary ideal of L.

The following result establishes a relationship of 1-absorbing primary ideal with irreducible elements of a lattice.

Proposition 3.14: Suppose that I is a 1-absorbing primary ideal of L that is not a primary ideal. Then there exists an irreducible element $x \in L$ and a nonunit element $y \in L$ such that $x \wedge y \in I$, but neither $x \in I$ nor $y \in \sqrt{I}$. Furthermore, if $a \wedge b \in I$ for some nonunit elements $a, b \in L$ such that neither $a \in I$ nor $b \in \sqrt{I}$, then a is an irreducible element of L.

Proof: Since I is not a primary ideal of L, so there exist nonunit elements $x, y \in L$ and $x \wedge y \in I$ such that neither $x \in I$ nor $y \in \sqrt{I}$. Suppose that x is not an irreducible element of L. Then $x = c \wedge d$ for some nonunit elements $c, d \in L$. Since $x \wedge y = c \wedge d \wedge y \in I$ with $y \notin \sqrt{I}$ and I is a 1-absorbing primary ideal of L, therefore we can conclude that $c \wedge d = x \in I$, a contradiction. Hence, x is an irreducible element.

Proposition 3.15: Let I be a 1-absorbing primary ideal of a lattice L. Then $(I : c) = \{x \in L : c \wedge x \in I\}$ is a primary ideal of L for every element $c \in L \setminus I$.

Proof: Let us suppose that $a \wedge b \in (I : c)$ for some elements $a, b \in L$ and nonunit element $c \in L \setminus I$ such that $a \notin (I : c)$. Then we have $a \wedge b \wedge c \in I$ and $a \wedge c \notin I$. Let us now assume that a, b are nonunit elements of L. Since I is a 1-absorbing primary ideal of L, so we have $b \in \sqrt{I} \subseteq \sqrt{(I : c)}$. Thus, $(I : c)$ is a primary ideal of L.

4 Some properties of 1-absorbing primary ideals under lattice homomorphism

In this section, we study some properties of primary ideals and 1-absorbing primary ideals of a lattice under homomorphism of lattices. We begin with the following proposition.

Proposition 4.1: Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) If P' is a primary ideal of L', then $f^{-1}(P')$ is a primary ideal of L.

(2) If f is an isomorphism and P is a primary ideal of L, then $f(P)$ is a primary ideal of L'.

Proof:

(1) Let $a \wedge b \in f^{-1}(P')$ for $a, b \in L$. Then $f(a \wedge b) \in P'$. This implies $f(a) \wedge f(b) \in P'$. Since P' is a primary ideal of L', so we have either $f(a) \in P'$ or $f(b) \in \sqrt{P'}$. This gives either $a \in f^{-1}(P')$ or $b \in f^{-1}(\sqrt{P'})$. Therefore, we have either $a \in f^{-1}(P')$ or $b \in \sqrt{f^{-1}(P')}$ (by Lemma 2.25(1)). Thus, $f^{-1}(P')$ is a primary ideal of L.

(2) Let $a' \wedge b' \in f(P)$ for some $a', b' \in L'$. Then there exists some $a, b \in L$ such that $f(a) = a'$ and $f(b) = b'$. Now, $f(a) \wedge f(b) = a' \wedge b' \in f(P)$ which implies $f(a \wedge b) \in f(P)$. Since f is an isomorphism, so we have $a \wedge b \in P$. As P is a primary ideal of L, therefore we have either $a \in P$ or $b \in \sqrt{P}$. This yields either $f^{-1}(a') \in P$ or $f^{-1}(b') \in \sqrt{P}$. This implies either $a' \in f(P)$ or $b' \in f(\sqrt{P})$. Therefore, we have $a' \in f(P)$ or $b' \in \sqrt{f(P)}$ (by Lemma 2.25(2)). Hence, $f(P)$ is a primary ideal of L'.

Proposition 4.2: Let L_1 and L_2 be two lattices and $f : L_1 \to L_2$ be a lattice homomorphism such that $f(1) = 1$ and $f(a)$ is nonunit in L_2 for every nonunit element a in L_1. Then the following statements hold:

(1) If I is a 1-absorbing primary ideal of L_2, then $f^{-1}(I)$ is a 1-absorbing primary ideal of L_1.

(2) If f is onto and I is a 1-absorbing primary ideal of L_1 with $ker(f) \subseteq I$, then $f(I)$ is a 1-absorbing primary ideal of L_2.

Proof:

(1) Let us suppose that I is a 1-absorbing primary ideal of L_2 and $a \wedge b \wedge c \in f^{-1}(I)$ for some nonunit elements $a, b, c \in L_1$. Then $f(a \wedge b \wedge c) = f(a) \wedge f(b) \wedge f(c) \in I$. Since I is a 1-absorbing primary ideal of L_2, so we have either $f(a) \wedge f(b) \in I$ or $f(c) \in \sqrt{I}$. This implies that either $a \wedge b \in f^{-1}(I)$ or $c \in f^{-1}(\sqrt{I})$. Therefore,we have either $a \wedge b \in f^{-1}(I)$ or $c \in \sqrt{f^{-1}(I)}$ (by Lemma 2.25(1)). Hence, $f^{-1}(I)$ is a 1-absorbing primary ideal of L_1.

(2) Let us suppose that f is onto and I is a 1-absorbing primary ideal of L_1 with $ker(f) \subseteq I$. Let $x \wedge y \wedge z \in f(I)$ for some nonunit elements $x, y, z \in L_2$. Since f is onto, there exist nonunit elements $a, b, c \in L_1$ such that $x = f(a), y = f(b), z = f(c)$. Therefore, we have $f(a \wedge b \wedge c) = f(a) \wedge f(b) \wedge f(c) = x \wedge y \wedge z \in f(I)$. Since $ker(f) \subseteq I$, so we can conclude that $a \wedge b \wedge c \in I$. Given I is a 1-absorbing primary ideal of L_1, we have either $a \wedge b \in I$ or $c \in \sqrt{I}$. This implies that either $x \wedge y \in f(I)$ or $z \in f(\sqrt{I})$. Therefore, we have either $x \wedge y \in f(I)$ or $z \in \sqrt{f(I)})$ (by Lemma 2.25(2)). Thus, $f(I)$ is a 1-absorbing primary ideal of L_2.

Proposition 4.3: Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) If I' is a 1-absorbing primary ideal of L', then $f^{-1}(I')$ is a 1-absorbing primary ideal of L.

(2) If f is an isomorphism and I is a 1-absorbing primary ideal of L, then $f(I)$ is a 1-absorbing primary ideal of L'.

Proof:

(1) Let $a \wedge b \wedge c \in f^{-1}(I')$ for some nonunit elements $a, b, c \in L$. Then $f(a \wedge b \wedge c) = f(a) \wedge f(b) \wedge f(c) \in I'$. As I' is a 1-absorbing primary ideal of L', so we have either $f(a) \wedge f(b) \in I'$ or $f(c) \in \sqrt{I'}$. This implies either $f(a \wedge b) \in I'$ or $f(c) \in \sqrt{I'}$. This gives either $a \wedge b \in f^{-1}(I')$ or $c \in f^{-1}(\sqrt{I'})$. Therefore, we have either $a \wedge b \in f^{-1}(I')$ or $c \in \sqrt{f^{-1}(I')}$ (by Lemma 2.25(1)). Thus, $f^{-1}(I')$ is a 1-absorbing primary ideal of L.

(2) Let $a' \wedge b' \wedge c' \in f(I)$ for some nonunit elements $a', b', c' \in L'$. Then there exists nonunit elements $a, b, c \in L$ such that $f(a) = a', f(b) = b', f(c) = c'$. This implies that $f(a) \wedge f(b) \wedge f(c) = f(a \wedge b \wedge c) \in f(I)$ which gives $a \wedge b \wedge c \in I$. As I is a 1-absorbing primary ideal of L, so we have either $a \wedge b \in I$ or $c \in \sqrt{I}$. This yields either $f^{-1}(a') \wedge f^{-1}(b') \in I$ or $f^{-1}(c') \in \sqrt{I}$. Therefore, we have either $f^{-1}(a' \wedge b') \in I$ or $f^{-1}(c') \in \sqrt{I}$. Thus, we have either $a' \wedge b' \in f(I)$ or $c' \in f(\sqrt{I})$. Therefore, we have either $a' \wedge b' \in f(I)$ or $c' \in \sqrt{f(I)})$ (by Lemma 2.25(2)). Hence, $f(I)$ is a 1-absorbing primary ideal of L'.

5 1-absorbing primary ideals in product lattices

In this section, we prove some results on 1-absorbing primary ideals in product lattices. We start the section with the following proposition.

Proposition 5.1: Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let I be a proper ideal of L_1. Then $I \times L_2$ is a 1-absorbing primary ideal of L if and only if I is a 1-absorbing primary ideal of L_1.

Proof: Let us suppose that $I \times L_2$ is a 1-absorbing primary ideal of L. Let $a \wedge b \wedge c \in I$ for nonunit elements $a, b, c \in L_1$. Then $(a \wedge b \wedge c, x) \in I \times L_2$ for $x \in L_2$. Since $I \times L_2$ is a 1-absorbing primary ideal of L, so we have either $(a \wedge b, x) \in I \times L_2$ or $(c, x) \in \sqrt{I \times L_2}$. This implies we have either $(a \wedge b, x) \in I \times L_2$ or $(c, x) \in \sqrt{I} \times L_2$ (by Lemma 2.27(1)). This gives either $a \wedge b \in I$ or $c \in \sqrt{I}$. Hence, I is a 1-absorbing primary ideal of L_1.

Conversely, let us suppose that I is a 1-absorbing primary ideal of L_1. Let $(a \wedge b \wedge c, x) \in I \times L_2$ for nonunit elements $a, b, c \in L_1$ and $x \in L_2$. As I is a 1-absorbing primary ideal of L_1, we have either $a \wedge b, \in I$ or $c \in \sqrt{I}$. Thus, we have either $(a \wedge b, x) \in I \times L_2$ or $(c, x) \in \sqrt{I} \times L_2 = \sqrt{I \times L_2}$ (by Lemma 2.27(1)). This gives that $I \times L_2$ is a 1-absorbing primary ideal of L.

The following proposition is similar to Proposition 5.1 .

Proposition 5.2: Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let J be a proper ideal of L_2. Then $L_1 \times J$ is a 1-absorbing primary ideal of L if and only if J is a 1-absorbing primary ideal of L_2.

Proposition 5.3: Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let I_1 and I_2 be proper ideals of L_1 and L_2 respectively. If $I = I_1 \times I_2$ is a 1-absorbing primary ideal of L then I_1 and I_2 are 1-absorbing primary ideals of L_1 and L_2 respectively.

Proof: Let us suppose that $a \wedge b \wedge c \in I_1$ for some nonunit elements $a, b, c \in L_1$. Then $(a \wedge b \wedge c, x) \in I_1 \times I_2$ for $x \in I_2$. As $I = I_1 \times I_2$ is a 1-absorbing primary ideal of L, we have either $(a \wedge b, x) \in I_1 \times I_2$ or $(c, x) \in \sqrt{I_1 \times I_2}$. Therefore, we have either $(a \wedge b, x) \in I_1 \times I_2$ or $(c, x) \in \sqrt{I_1} \times \sqrt{I_2}$ (by Lemma 2.27). This gives that either $a \wedge b \in I_1$ or $c \in \sqrt{I_1}$. Thus, I_1 is a 1-absorbing primary ideal of L_1.
Again, let us assume that $a \wedge b \wedge c \in I_2$ for some nonunit elements $a, b, c \in L_2$. Then $(y, a \wedge b \wedge c) \in I_1 \times I_2$ for $y \in I_1$. Since $I = I_1 \times I_2$ is a 1-absorbing primary ideal of L, so we have either $(y, a \wedge b) \in I_1 \times I_2$ or $(y, c) \in \sqrt{I_1 \times I_2}$. Thus, we have either $(y, a \wedge b) \in I_1 \times I_2$ or $(y, c) \in \sqrt{I_1} \times \sqrt{I_2}$ (by Lemma 2.27). This gives that either $a \wedge b \in I_2$ or $c \in \sqrt{I_2}$. Hence, I_2 is a 1-absorbing primary ideal of L_2.

The converse of the above Proposition 5.3 need not hold. The following example illustrates that if I_1 and I_2 are 1-absorbing primary ideals of L_1 and L_2 respectively, then $I = I_1 \times I_2$ may not be a 1-absorbing primary ideal of $L = L_1 \times L_2$.

Example 5.4: Let us consider the lattices L_1, L_2 and $L = L_1 \times L_2$ as shown in Fig. 4 below. Let us take the ideals $I_1 = \{0\}, I_2 = \{0\}$ of the lattices L_1 and L_2 respectively. Then $I_1 \times I_2 = \{(0,0)\}$ and thus $\sqrt{I_1 \times I_2} = \{(0,0)\}$. The ideals I_1 and I_2 are 1-absorbing primary ideals of L_1 and L_2 respectively. But for $(a,1) \wedge (1,0) \wedge (b,0) = (0,0) \in I_1 \times I_2$, neither $(a,1) \wedge (1,0) = (a,0) \in I_1 \times I_2$ nor $(b,0) \in \sqrt{I_1 \times I_2}$. Thus, $I_1 \times I_2$ is not a 1-absorbing primary ideal of L.

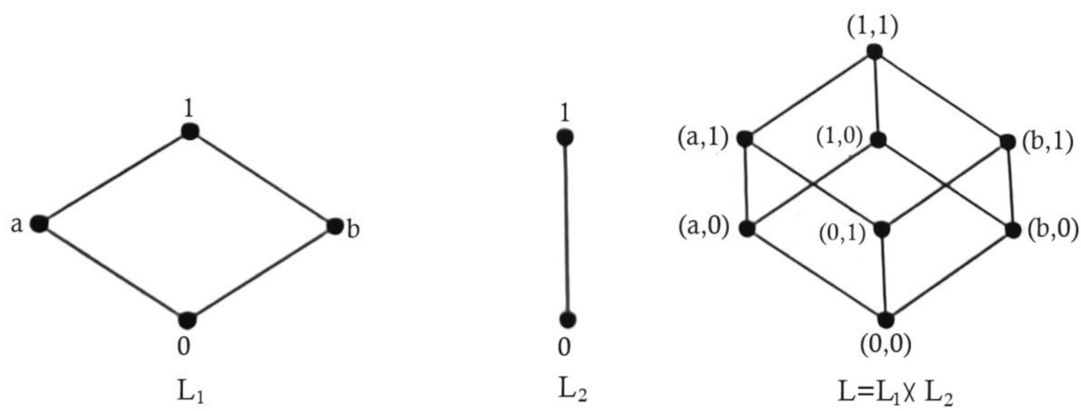


Fig. 4.

Proposition 5.5: Let $L = L_1 \times L_2$, where L_1 and L_2 are bounded lattices. Let J be a proper ideal of L. Then the following statements are equivalent:

(1) J is a 1-absorbing primary ideal of L.

(2) Either $J = I_1 \times L_2$ for some 1-absorbing primary ideal I_1 of L_1 or $J = L_1 \times I_2$ for some 1-absorbing primary ideal I_2 of L_2 or $J = I_1 \times I_2$ for some primary ideal I_1 of L_1 and some primary ideal I_2 of L_2.

Proof: $(1) \Rightarrow (2)$ Suppose that J is a 1-absorbing primary ideal of L. Then $J = I_1 \times I_2$ for some ideal I_1 of L_1 and some ideal I_2 of L_2.

Case 1: If $I_2 = L_2$, then $I_1 \neq L_1$. Thus $J = I_1 \times L_2$. Let $a \wedge b \wedge c \in I_1$ for some nonunit elements $a, b, c \in L_1$. Then $(a \wedge b \wedge c, x \wedge y \wedge z) \in I_1 \times L_2$, where $x, y, z \in L_2$. As J is a 1-absorbing primary ideal of L, we have either $(a \wedge b, x \wedge y) \in I_1 \times L_2$ or $(c, z) \in \sqrt{I_1 \times L_2}$. This implies either $(a \wedge b, x \wedge y) \in I_1 \times L_2$ or $(c, z) \in \sqrt{I_1} \times L_2$ (by Lemma 2.27(1)). This gives either $a \wedge b \in I_1$ or $c \in \sqrt{I_1}$. Hence, I_1 is a 1-absorbing primary ideal of L_1.

Case 2: If $I_1 = L_1$, then $I_2 \neq L_2$. Thus $J = L_1 \times I_2$. Let $a \wedge b \wedge c \in I_2$ for some nonunit elements $a, b, c \in L_2$. Then $(x \wedge y \wedge z, a \wedge b \wedge c) \in L_1 \times I_2$, where $x, y, z \in L_1$. As J is a 1-absorbing primary ideal of L, we have either $(x \wedge y, a \wedge b) \in L_1 \times I_2$ or $(z, c) \in \sqrt{L_1 \times I_2}$. This implies either $(x \wedge y, a \wedge b) \in L_1 \times I_2$ or $(z, c) \in L_1 \times \sqrt{I_2}$. This gives either $a \wedge b \in I_2$ or $c \in \sqrt{I_2}$ (by Lemma 2.27(2). Hence, I_2 is a 1-absorbing primary ideal of L_2.

Case 3: Now if $I_1 \neq L_1$ and $I_2 \neq L_2$ then $J = I_1 \times I_2$. On the contrary, let us suppose that I_1 is not a primary ideal of L_1. Then there exist $a, b \in L_1$ such that $a \wedge b \in I_1$ but neither $a \in I_1$ nor $b \in \sqrt{I_1}$. Let $x = (a, 1), y = (1, 0), z = (b, 1)$. Then $x \wedge y \wedge z = (a \wedge b, 0) \in J$ but neither $x \wedge y = (a, 0) \in J$ nor $z = (b, 1) \in \sqrt{J}$, which is a contradiction. Thus, I_1 is a primary ideal of L_1.
Let us suppose that I_2 is not a primary ideal of L_2.Then there exist $d, e \in L_2$ such that $d \wedge e \in I_2$ but neither $d \in I_2$ nor $e \in \sqrt{I_2}$. Let $x = (1, d), y = (0, 1), z = (1, e)$. Then $x \wedge y \wedge z = (0, d \wedge e) \in J$ but neither $x \wedge y = (0, d) \in J$ nor $z = (1, e) \in \sqrt{J}$, which is a contradiction. Hence, I_2 is a primary ideal of L_2.

$(2) \Rightarrow (1)$ Let us suppose that $J = I_1 \times L_2$ for some 1-absorbing primary ideal I_1 of L_1. Let $(a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I_1 \times L_2$. Then $a_1 \wedge a_2 \wedge a_3 \in I_1$. Since I_1 is a 1-absorbing primary ideal of L_1, so we have either $a_1 \wedge a_2 \in I_1$ or $a_3 \in \sqrt{I_1}$. This implies either $(a_1, b_1) \wedge (a_2, b_2) \in I_1 \times L_2$ or $(a_3, b_3) \in \sqrt{I_1} \times L_2$. Thus either $(a_1, b_1) \wedge (a_2, b_2) \in I_1 \times L_2$ or $(a_3, b_3) \in \sqrt{I_1 \times L_2}$ (by Lemma 2.27(1)). Hence, $J = I_1 \times L_2$ is a 1-absorbing primary ideal of L.
Similarly, we can prove that $J = L_1 \times I_2$ is a 1-absorbing primary ideal of L.
Again, we assume that $J = I_1 \times I_2$ for some primary ideal I_1 of L_1 and for some primary ideal I_2 of L_2. Then we have $P = I_1 \times L_2$ and $Q = L_1 \times I_2$ are 1-absorbing primary ideal of L (by Proposition 5.1 and 5.2). Thus, we have $P \cap Q = I_1 \times I_2$ is also a 1-absorbing primary ideals of L (By proposition 3.15). Hence, $J = I_1 \times I_2$ is a 1-absorbing primary ideal of L.

The following proposition is a generalization of Proposition 5.5.

Proposition 5.6: Let $L = L_1 \times L_2 \times .... \times L_n$, where $2 \leq n < \infty$, and $L_1, L_2, ...L_n$ are lattices. Let J be a proper ideal of L. Then the following statements are equivalent:

(1) J is 1-absorbing primary ideal of L.

(2) Either $J = \prod_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, ..., n\}$, I_k is a 1-absorbing primary ideal of L_k, and $I_t = L_t$ for every $t \in \{1, 2, ..., n\} \setminus \{k\}$ or $J = \prod_{t=1}^{n} I_t$ such that for some $k, m \in \{1, 2, ..., n\}, I_k$ is a primary ideal of L_k, I_m is a primary ideal of L_m, and $I_t \neq L_t$ for every $t \in \{1, 2, ...n\} \setminus \{k, m\}$.

Proof: $(1) \Leftrightarrow (2)$ We prove this theorem by induction on n. Let us assume that $n = 2$. Then, the result holds by Proposition 5.5 . Again, let us suppose that $3 \leq n < \infty$ and assume that the result is valid when $K = L_1 \times L_2 \times .... \times L_{n-1}$. Now we prove the result when $L = K \times L_n$. By Proposition 5.5, J is a 1-absorbing primary ideal of L if and only if either $J = A \times L_n$ for some 1-absorbing primary ideal A of K or $J = K \times A_n$ for some 1-absorbing primary ideal A_n of L_n or $J = A \times A_n$ for some primary ideal A of K and some primary ideal of A_n of L_n.Thus we observe that a proper ideal B of K is a primary ideal of K if and only if $B = \prod_{t=1}^{n-1} I_t$ such that for some $k \in \{1, 2, ..., n-1\}$, I_k is a primary ideal of L_k, and $I_t \neq L_t$ for every $t \in \{1, 2, ..., n-1\} \setminus \{k, m\}$.

Conflict of Interest:
On behalf of all authors, the corresponding author declares that there is no conflict of interest.

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Received: 2025-07-18
Accepted: 2025-10-09