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A GENERALIZATION OF GAMMA, BETA, AND HYPERGEOMETRIC FUNCTIONS VIA MULTI-INDEX $(\lambda, \mu)_n$ MITTAG-LEFFLER KERNEL

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Abstract. We develop $2n$-parametric extensions of the gamma, beta, and Gauss hypergeometric functions using the $(\lambda, \mu)_n$ Mittag-Leffler kernel. These generalized functions preserve key properties of the classical forms while admitting new integral representations, transformation identities, summation formulas, differentiation rules, and Mellin transforms. Statistical applications are also explored through generalized distributions with explicit moment expressions. Theoretical developments are supported by numerical results that confirm the effectiveness of the proposed framework in modeling fractional systems. This work provides enhanced analytical tools for the theory of special functions and their applications.

1 Introduction and Preliminaries

Special functions are fundamental tools in diverse areas such as applied mathematics, physics, engineering, and statistics. Among the most significant functions are the gamma $\Gamma(z)$, beta $B(z,w)$, Gauss hypergeometric $F(a,b;c;z)$, and confluent hypergeometric function $\Phi(b;c;z)$. These functions and their classical forms are extensively detailed in [1] and are defined as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0, \tag{1.1}$$

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt, \quad \Re(z) > 0, \quad \text{and} \quad \Re(w) > 0 \tag{1.2}$$

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1, \tag{1.3}$$

$$\Phi(b;c;z) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1. \tag{1.4}$$

In the above series, $(a)_n$ represents the Pochhammer symbol (also known as the rising factorial), defined as:

$$(a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)(a+2)\cdots(a+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

This notation appears frequently in the theory of special functions and hypergeometric series (see [1]).

Alternatively, the Pochhammer symbol can be written in terms of the gamma function:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

provided that a is not a non-positive integer (i.e., $a \notin \{0, -1, -2, \ldots\}$), since the gamma function has singularities at those points.

Alternative forms of the Gauss and confluent hypergeometric functions can be written using the classical beta function as:

$$F(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \cdot \frac{B(b+n, c-b)}{B(b, c-b)} \cdot \frac{z^n}{n!}, \tag{1.5}$$

$$(|z| < 1,\ \Re(c) > \Re(b) > 0),$$

where the above conditions ensure convergence of the series.

$$\Phi(b; c; z) = \sum_{n=0}^{\infty} \frac{B(b+n, c-b)}{B(b, c-b)} \cdot \frac{z^n}{n!}, \tag{1.6}$$

with $z \in \mathbb{C}$ satisfying $|z| < 1,$ and $\Re(c) > \Re(b) > 0$.

In addition to their series forms, these functions also admit integral representations involving the beta function (see [1]):

$$F(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \tag{1.7}$$

$$(|\arg(1-z)| < \pi,\ \text{and } \Re(c) > \Re(b) > 0),$$

$$\Phi(b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{tz} dt, \tag{1.8}$$

$$(\Re(c) > \Re(b) > 0).$$

In recent years, a wide variety of generalizations of classical special functions have been proposed by numerous researchers [2–12] driven by their broad applicability across mathematics, physics, and engineering. A selection of these generalizations is reviewed below.

For example, Chaudhry and co-authors proposed a p-generalization of the Gamma function [2] and later extended this approach to define a corresponding Beta function [3] as given below:

$$\Gamma_p(z) = \int_0^\infty t^{z-1} \exp\left(-t - \frac{p}{t}\right) dt, \tag{1.9}$$

$$(\Re(p) \geq 0,\ \Re(z) > 0),$$

$$B_p(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} \exp\left(-\frac{p}{t(1-t)}\right) dt, \tag{1.10}$$

$$(\Re(p) \geq 0,\ \Re(z) > 0,\ \Re(w) > 0).$$

Furthermore, Chaudhry et al. [4] employed the p-extended Beta function $B_p(z, w)$, given in Eq. (1.10), to formulate generalized versions of the Gauss hypergeometric and confluent hypergeometric functions incorporating the parameter p, as follows:

$$F_p(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.11}$$

$$(p\geq 0,\ |z|<1,\ \text{and}\ \Re(c)>\Re(b)>0)\,,$$

$$\Phi_p(b;c;z)=\sum_{n=0}^{\infty}\frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.12}$$

$$(p\geq 0,\ |z|<1,\ \text{and}\ \Re(c)>\Re(b)>0)\,.$$

The corresponding integral representations for these functions are expressed as:

$$F_p(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\exp\left(-\frac{p}{t(1-t)}\right)dt, \tag{1.13}$$

$$\text{where } p\geq 0,\ |\arg(1-z)|<\pi,\ \Re(c)>\Re(b)>0,$$

$$\Phi_p(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp\left(zt-\frac{p}{t(1-t)}\right)dt, \tag{1.14}$$

$$\text{where } p\geq 0,\ \Re(c)>\Re(b)>0.$$

Additionally, Ata [5] introduced generalizations in terms of the Gamma and Beta functions by incorporating the Wright function. These variants, known as the Ψ-Gamma and Ψ-Beta functions, are expressed below:

$${}^{\Psi}\Gamma_p^{(\lambda,\mu)}(z)=\int_0^{\infty}t^{z-1}\,{}_1\Psi_1\left(\lambda,\mu;-t-\frac{p}{t}\right)dt, \tag{1.15}$$

$$(\Re(p)\geq 0,\ \Re(\lambda)>-1,\ \Re(z)>0)\,,$$

$${}^{\Psi}B_p^{(\lambda,\mu)}(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,{}_1\Psi_1\left(\lambda,\mu;-\frac{p}{t(1-t)}\right)dt, \tag{1.16}$$

$$(\Re(p)\geq 0,\ \Re(\lambda)>-1,\ \Re(z)>0,\ \Re(w)>0)\,.$$

The function ${}^{\Psi}B_p^{(\lambda,\mu)}(z,w)$ given in Eq. (1.16) can be used to define the Ψ-generalized Gauss and confluent hypergeometric functions [5]:

$${}^{\Psi}F_p^{(\lambda,\mu)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{{}^{\Psi}B_p^{(\lambda,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.17}$$

$${}^{\Psi}\Phi_p^{(\lambda,\mu)}(b;c;z)=\sum_{n=0}^{\infty}\frac{{}^{\Psi}B_p^{(\lambda,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}. \tag{1.18}$$

Here, ${}^{\Psi}F_p^{(\lambda,\mu)}(a,b;c;z)$ is referred to as the Ψ-Gauss hypergeometric function, while ${}^{\Psi}\Phi_p^{(\lambda,\mu)}(b;c;z)$ denotes the Ψ-confluent hypergeometric function.
In a related development, Shadab et al. [6] introduced novel extensions of the Beta, Gauss hypergeometric, and confluent hypergeometric functions based on the classical Mittag-Leffler function:

$$B_{\alpha}^{p}(z,w)=\int_{0}^{1} t^{z-1}(1-t)^{w-1}E_{\alpha}\left(-\frac{p}{t(1-t)}\right)dt, \tag{1.19}$$

$$\left(\alpha\in\mathbb{R}_0^+,\Re(p)\geq 0,\Re(z)>0,\Re(w)>0\right),$$

$$F_{p,\alpha}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.20}$$

$$\text{where }\left(\alpha\in\mathbb{R}^+,p\in\mathbb{R}_0^+,|z|<1,\text{ and }\Re(c)>\Re(b)>0\right),$$

$$\Phi_{p,\alpha}(b;c;z)=\sum_{n=0}^{\infty}\frac{B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.21}$$

$$\text{where }\left(\alpha\in\mathbb{R}^+,p\in\mathbb{R}_0^+,\text{ and }\Re(c)>\Re(b)>0\right),$$

where $E_{\alpha}(z)$ denotes the one-parameter Mittag-Leffler function, defined by:

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad\left(z\in\mathbb{C},\alpha\in\mathbb{R}_0^+\right). \tag{1.22}$$

More recently, Nabiullah *et al.* [10] further generalized this class of Beta-type functions by coupling the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ with variable exponents μ and ν:

$$B_{\alpha,\beta}^{p,\mu,\nu}(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}E_{\alpha,\beta}\left(-\frac{p}{t^{\mu}(1-t)^{\nu}}\right)dt, \tag{1.23}$$

$$\text{where }\Re(z)>0,\ \Re(w)>0,\ \alpha,\beta,\mu,\nu>0,\ p\geq 0.$$

The corresponding extended hypergeometric and confluent hypergeometric functions were defined as:

$$F_{\alpha,\beta}^{p,\mu,\nu}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{\alpha,\beta}^{p,\mu,\nu}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.24}$$

$$\text{where }p\geq 0,\ \alpha,\beta,\mu,\nu>0,\ |z|<1,\ \Re(c)>\Re(b)>0,$$

$$\Phi_{\alpha,\beta}^{p,\mu,\nu}(b;c;z)=\sum_{n=0}^{\infty}\frac{B_{\alpha,\beta}^{p,\mu,\nu}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.25}$$

$$\text{where }p\geq 0,\ \alpha,\beta,\mu,\nu>0,\ \Re(c)>\Re(b)>0.$$

Here, $E_{\alpha,\beta}(z)$ denotes the classical two parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\qquad(z\in\mathbb{C},\ \alpha,\beta>0). \tag{1.26}$$

Many researchers have developed new classes of Beta-type functions by employing various forms of the Mittag-Leffler kernel.

Likewise, Ghayasuddin *et al.* [15] proposed a multi-index extension of the Beta function using the generalized (multi-index) Mittag-Leffler function $E_{(1/a_i),(b_i)}(z)$ defined as

$$B_p^{(a_1,\ldots,a_s,\,b_1,\ldots,b_s)}(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}E_{\left(\frac{1}{a_i}\right),\,(b_i)}\left(-\frac{p}{x(1-x)}\right)dx, \tag{1.27}$$

which converges absolutely for $\Re(m)>0$, $\Re(n)>0$, and $p\geq 0$, provided that $a_i,b_i>0$ for all $i=1,2,\ldots,s$.

Also, Goyal, Momani, Agarwal, and Rassias [16] extended this framework by employing the two-parameter Mittag-Leffler (Wiman) function $E_{\rho_1,\rho_2}(z)$:

$$B^{(r)}_{(\rho_1,\rho_2)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}E_{\rho_1,\rho_2}\left(-\frac{r}{t(1-t)}\right)dt, \qquad \Re(x),\Re(y)>0,\ \rho_1,\rho_2>0,\ r\geq 0. \tag{1.28}$$

In another direction, Rasheeed [13] constructed extensions of the beta and hypergeometric functions using fractional integral operators involving Mittag-Leffler kernels. Related investigations along similar lines were carried out by Khan et al. [19] and Ali [20], whose works appeared in the Palestine Journal of Mathematics. Choi et al. [14] also developed generalized Mittag-Leffler-type functions linked with fractional operators and Wright-type functions. These studies highlight the flexibility of Mittag-Leffler-based kernels in the theory of generalized special functions.

Remark 1.1. Several authors have introduced extensions of the Beta, Gauss hypergeometric, and confluent hypergeometric functions using exponential kernels, Wright functions, or one-parameter Mittag-Leffler functions [3, 5, 6, 10]. However, to the best of our knowledge, no generalization of these functions has yet been proposed using the multi-index $2n$-parameter Mittag-Leffler function introduced by Gorenflo et al. [17]. Motivated by this, we propose new extensions based on the following definition:

$$E_{(\lambda,\mu)_n}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\prod\limits_{j=1}^{n}\Gamma(\lambda_j k+\mu_j)}, \tag{1.29}$$

$$\left(z\in\mathbb{C},\ \lambda_j\in\mathbb{R},\ \lambda_1^2+\cdots+\lambda_n^2\neq 0,\ \mu_j\in\mathbb{C},\ j=1,2,\ldots,n\right).$$

2 The $(\lambda,\mu)_n$-Gamma and Beta Functions

We now introduce novel generalizations of the gamma and beta functions with the $2n$-parameter Mittag-Leffler function $E_{(\lambda,\mu)_n}(z)$, as defined in (1.29). These new variants are given as follows:

$$\Gamma^{(p,x,y)}_{(\lambda,\mu)_n}(z) = \int_0^{\infty} t^{z-1}E_{(\lambda,\mu)_n}\left(-t^x-\frac{p}{t^y}\right)dt, \tag{2.1}$$

valid for $x,y,\lambda_j,\mu_j\in\mathbb{R}^+$, $j=1,2,\ldots,n$, $p\geq 0$, and $\Re(z)>0$.

$$B^{(p,x,y)}_{(\lambda,\mu)_n}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right)dt, \tag{2.2}$$

where the parameters again satisfy $x,y,\lambda_j,\mu_j\in\mathbb{R}^+$, $p\geq 0$, and $\Re(z),\Re(w)>0$.

These definitions broaden the classical gamma and beta frameworks by incorporating the rich structure of the generalized Mittag-Leffler function with multiple parameters. Depending on specific parameter choices, they reduce to known extensions such as the p-gamma function, the Ψ-gamma and beta functions, and the Mittag-Leffler-based variants by Chaudhry, Ata, and others.

We refer to these expressions as the generalized $(\lambda,\mu)_n$-gamma and beta functions, respectively.

Remark 2.1. By choosing parameter values $x=y=1$, $\lambda=\mu=1$, and $n=2$, Eqs. (2.1) and (2.2) reduce to the Ψ-gamma and Ψ-beta functions as described in Eqs. (1.15) and (1.16):

$$\Gamma^{p,1,1}_{(1,1)_2}(z) = {}^{\Psi}\Gamma^{(1,1)}_p(z),$$
$$B^{p,1,1}_{(1,1)_2}(z,w) = {}^{\Psi}B^{(1,1)}_p(z,w).$$

Remark 2.2. Setting $x = y = 1$ and $n = 1$, Eqs. (2.1) and (2.2) reduce to the extended gamma and beta functions proposed by Chaudhry *et al.* in Eqs. (1.9) and (1.10):

$$\Gamma^{p,1,1}_{(1,1)_1}(z) = \Gamma_p(z),$$
$$B^{p,1,1}_{(1,1)_1}(z,w) = B_p(z,w).$$

Remark 2.3. When $x = y = 1$, $\lambda = \mu = 1$, $n = 1$, and $p = 0$, Eqs. (2.1) and (2.2) reduce to the classical Euler gamma and beta functions:

$$\Gamma^{0,1,1}_{(1,1)_1}(z) = \Gamma(z),$$
$$B^{0,1,1}_{(1,1)_1}(z,w) = B(z,w).$$

Remark 2.4. The functions $\Gamma^{p,x,y}_{(\lambda,\mu)_n}(z)$ and $B^{p,x,y}_{(\lambda,\mu)_n}(z,w)$ encompass a wide variety of previously proposed models. For instance, Eqs. (2.1) and (2.2) reduce to the following special cases:

- When $x = y = 1$, $\lambda_j = \mu_j = \alpha$, and $n = 1$, they reduce to the Mittag-Leffler-based beta function given in Eq. (1.19) of Shadab *et al.* [6]:

$$B^{p,1,1}_{(\alpha,\alpha)_1}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} E_\alpha\left(-\frac{p}{t(1-t)}\right) dt.$$

- If $x = \mu$, $y = \nu$, $\lambda_j = \alpha$, $\mu_j = \beta$, and $n = 1$, they coincide with the two-parameter Mittag-Leffler-based extension given in Eq. (1.23) of Nabiullah *et al.* [10]:

$$B^{p,\mu,\nu}_{(\alpha,\beta)_1}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} E_{\alpha,\beta}\left(-\frac{p}{t^\mu(1-t)^\nu}\right) dt.$$

Remark 2.5. With suitable selections of the sequences λ_j and μ_j, Eqs. (2.1) and (2.2) are reducible to several well-known Mittag-Leffler–based kernels previously reported in the literature:

- For $\lambda_j = a_i^{-1}$ and $\mu_j = b_i$ $(i = 1, 2, \ldots, s)$, the formulation coincides with the multi–index Mittag-Leffler kernel introduced by Ghayasuddin *et al.* [15], which leads to their generalized multi–index Beta-type function.
- For $n = 1$ and $\lambda_j = \rho_1$, $\mu_j = \rho_2$, the expression reduces to the two–parameter (Wiman) Mittag-Leffler kernel employed by Goyal *et al.* [16] in the construction of their Wiman based extension of the Beta function.

Validation: For $\lambda_j = a_i^{-1}$ and $\mu_j = b_i$,Eq. (2.2) reproduces the multi–index Mittag-Leffler structure proposed by Ghayasuddin *et al.* [15], whereas for $n = 1$ and $\lambda_j = \rho_1$, $\mu_j = \rho_2$, Eq. (2.2) transforms into the Wiman type Mittag-Leffler kernel discussed by Goyal *et al.* [16].

3 Integral Properties of the $(\lambda, \mu)_n$-Gamma and Beta Functions

Now,we proceed to investigate several essential properties associated with the $(\lambda, \mu)_n$-gamma and beta functions. introduced earlier, including alternative integral representations, recurrence relations, series identities, and transform formulas.

Theorem 3.1. *An alternative integral expression of the $(\lambda, \mu)_n$-gamma function can be expressed as:*

$$\Gamma^{p,x,y}_{(\lambda,\mu)_n}(z) = 2\int_0^\infty u^{2z-1} E_{(\lambda,\mu)_n}\left(-u^{2x} - \frac{p}{u^{2y}}\right) du, \tag{3.1}$$

with the condition, $x, y, \lambda_j, \mu_j \in \mathbb{R}^+$ for $j = 1, 2, \ldots, n$, $p \geq 0$, and $\Re(z) > 0$.

Proof. Starting from the definition of $(\lambda,\mu)_n$-gamma function in Eq. (2.1), we write:

$$\Gamma^{p,x,y}_{(\lambda,\mu)_n}(z) = \int_0^\infty t^{z-1} E_{(\lambda,\mu)_n}\left(-t^x - \frac{p}{t^y}\right) dt.$$

Applying the substitution $t = u^2$, which implies $dt = 2u\,du$, transforms the integral into:

$$\begin{aligned}\Gamma^{p,x,y}_{(\lambda,\mu)_n}(z) &= \int_0^\infty (u^2)^{z-1} E_{(\lambda,\mu)_n}\left(-u^{2x} - \frac{p}{u^{2y}}\right)(2u)\,du \\ &= 2\int_0^\infty u^{2z-1} E_{(\lambda,\mu)_n}\left(-u^{2x} - \frac{p}{u^{2y}}\right) du.\end{aligned}$$

□

Theorem 3.2. *For two independent variables z and w, the product $(\lambda,\mu)_n$-gamma functions is expressed as:*

$$\begin{aligned}\Gamma^{p,x,y}_{(\lambda,\mu)_n}(z)\,\Gamma^{p,x,y}_{(\lambda,\mu)_n}(w) = 4\int_0^\infty \int_0^{\pi/2} r^{2(z+w)-1}\, E_{(\lambda,\mu)_n}\left(-r^{2x}\cos^{2x}\phi - \frac{p}{r^{2y}\cos^{2y}\phi}\right) \\ \times E_{(\lambda,\mu)_n}\left(-r^{2x}\sin^{2x}\phi - \frac{p}{r^{2y}\sin^{2y}\phi}\right) d\phi\,dr,\end{aligned} \tag{3.2}$$

where $x, y, \lambda_j, \mu_j \in \mathbb{R}^+$ for $j = 1, 2, \ldots, n$, $p \geq 0$, and $\Re(z), \Re(w) > 0$.

Proof. We proceed by applying the alternative form of the $(\lambda,\mu)_n$-gamma function as stated in Eq. (3.1):

$$\begin{aligned}\Gamma^{p,x,y}_{(\lambda,\mu)_n}(z)\,\Gamma^{p,x,y}_{(\lambda,\mu)_n}(w) &= 2\int_0^\infty u^{2z-1} E_{(\lambda,\mu)_n}\left(-u^{2x} - \frac{p}{u^{2y}}\right) du \times 2\int_0^\infty v^{2w-1} E_{(\lambda,\mu)_n}\left(-v^{2x} - \frac{p}{v^{2y}}\right) dv \\ &= 4\int_0^\infty \int_0^\infty u^{2z-1} v^{2w-1} E_{(\lambda,\mu)_n}\left(-u^{2x} - \frac{p}{u^{2y}}\right) E_{(\lambda,\mu)_n}\left(-v^{2x} - \frac{p}{v^{2y}}\right) du\,dv.\end{aligned}$$

Now, we apply the polar coordinate transformation $u = r\cos\phi$, $v = r\sin\phi$, which leads to:

$$\begin{aligned}\Gamma^{p,x,y}_{(\lambda,\mu)_n}(z)\,\Gamma^{p,x,y}_{(\lambda,\mu)_n}(w) = 4\int_0^\infty \int_0^{\pi/2} r^{2(z+w)-1} E_{(\lambda,\mu)_n}\left(-r^{2x}\cos^{2x}\phi - \frac{p}{r^{2y}\cos^{2y}\phi}\right) \\ \times E_{(\lambda,\mu)_n}\left(-r^{2x}\sin^{2x}\phi - \frac{p}{r^{2y}\sin^{2y}\phi}\right) d\phi\,dr.\end{aligned}$$

□

Theorem 3.3. *We present the following integral forms associated with the $(\lambda,\mu)_n$-beta function:*

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = 2\int_0^{\pi/2} \cos^{2z-1}\phi\,\sin^{2w-1}\phi\,E_{(\lambda,\mu)_n}\left(-p\sec^{2x}\phi\cdot\csc^{2y}\phi\right) d\phi, \tag{3.3}$$

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = \int_0^\infty u^{z-1}(1+u)^{-z-w} E_{(\lambda,\mu)_n}\left(-p\frac{(1+u)^{x+y}}{u^y}\right) du, \tag{3.4}$$

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = \frac{1}{2^{z+w-1}}\int_{-1}^1 (1+u)^{z-1}(1-u)^{w-1} E_{(\lambda,\mu)_n}\left(-\frac{2^{x+y}p}{(1+u)^x(1-u)^y}\right) du, \tag{3.5}$$

where $x, y, \lambda_j, \mu_j \in \mathbb{R}^+$ for $j = 1, 2, \ldots, n$, $p \geq 0$, and $\Re(z), \Re(w) > 0$.

Proof. Each representation follows by making an appropriate substitution in the original definition of the beta function Eq. (2.2) specifically:

$$t = \cos^2\phi, \quad t = \frac{u}{1+u}, \quad \text{and} \quad t = \frac{1+u}{2}.$$

□

Theorem 3.4. *We now present a functional relation satisfied by the generalized beta function:*

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = B^{p,x,y}_{(\lambda,\mu)_n}(z+1,w) + B^{p,x,y}_{(\lambda,\mu)_n}(z,w+1). \tag{3.6}$$

Proof. Starting with the integral form of the beta function:

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt,$$

we use the identity $t + (1-t) = 1$ to write:

$$\begin{aligned} &= \int_0^1 t^{z}(1-t)^{w-1} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt + \int_0^1 t^{z-1}(1-t)^{w} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt \\ &= B^{p,x,y}_{(\lambda,\mu)_n}(z+1,w) + B^{p,x,y}_{(\lambda,\mu)_n}(z,w+1). \end{aligned}$$

□

From this, it follows directly that:

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = B^{p,x,y}_{(\lambda,\mu)_n}(z+2,w) + 2B^{p,x,y}_{(\lambda,\mu)_n}(z+1,w+1) + B^{p,x,y}_{(\lambda,\mu)_n}(z,w+2), \tag{3.7}$$

and more generally, for any $r \in \mathbb{N}$:

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = \sum_{k=0}^{r} \binom{r}{k} B^{p,x,y}_{(\lambda,\mu)_n}(z+k,w-k+r). \tag{3.8}$$

Theorem 3.5. *We now present a summation formula satisfied by the $(\lambda,\mu)_n$-beta function:*

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,1-w) = \sum_{k=0}^{\infty} \frac{(w)_k}{k!} B^{p,x,y}_{(\lambda,\mu)_n}(z+k,1), \tag{3.9}$$

where $(w)_k$ denotes the Pochhammer symbol.

Proof. We consider of the integral form:

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,1-w) = \int_0^1 t^{z-1}(1-t)^{-w} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt.$$

Expanding $(1-t)^{-w}$ as a binomial series:

$$(1-t)^{-w} = \sum_{k=0}^{\infty} \frac{(w)_k}{k!} t^k, \quad (|t|<1),$$

and interchanging the sum and integral, we obtain:

$$\begin{aligned} B^{p,x,y}_{(\lambda,\mu)_n}(z,1-w) &= \sum_{k=0}^{\infty} \frac{(w)_k}{k!} \int_0^1 t^{z+k-1} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt \\ &= \sum_{k=0}^{\infty} \frac{(w)_k}{k!} B^{p,x,y}_{(\lambda,\mu)_n}(z+k,1). \end{aligned}$$

□

Likewise, we can deduce:

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = \sum_{k=0}^{\infty} B^{p,x,y}_{(\lambda,\mu)_n}(z+k,w+1). \tag{3.10}$$

Theorem 3.6. *We now derive Mellin transform of the $(\lambda, \mu)_n$-beta function with respect to p is expressed by:*

$$\mathcal{M}\left[B^{p,x,y}_{(\lambda,\mu)_n}(z,w)\right] = B(z+sx, w+sy)\,\Gamma_{(\lambda,\mu)_n}(s), \tag{3.11}$$

for $\Re(s) > 0$, $\Re(z+sx) > 0$, $\Re(w+sy) > 0$.

Proof. By definition of the Mellin transform:

$$\mathcal{M}\left[B^{p,x,y}_{(\lambda,\mu)_n}(z,w); s\right] = \int_0^\infty p^{s-1} B^{p,x,y}_{(\lambda,\mu)_n}(z,w)dp.$$

Substituting from Eq. (2.2) and switching the order of integration:

$$= \int_0^1 t^{z-1}(1-t)^{w-1}\left\{\int_0^\infty p^{s-1} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right)dp\right\}dt.$$

Using the substitution $u = \frac{p}{t^x(1-t)^y}$, we obtain:

$$\mathcal{M}[B^{p,x,y}_{(\lambda,\mu)_n}(z,w)] = B(z+sx, w+sy)\,\Gamma_{(\lambda,\mu)_n}(s).$$

□

Corollary 3.7. *Using Mellin inversion, the $(\lambda, \mu)_n$-beta function can be represented by the contour integral:*

$$B^{p,x,y}_{(\lambda,\mu)_n}(z,w) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} B(z+sx, w+sy)\Gamma_{(\lambda,\mu)_n}(s)p^{-s(x+y)}ds. \tag{3.12}$$

Remark 3.8. The contour of integration is taken along the vertical Bromwich line $\Re(s) = C$, oriented upward from $C - i\infty$ to $C + i\infty$, following the standard definition of the inverse Mellin transform [18].

4 Some Applications of the $(\lambda, \mu)_n$- based extension of Gamma and Beta Functions

We present statistical models derived from the $(\lambda, \mu)_n$-based extensions of the classical gamma and beta functions and present their associated density functions. We also establish expressions for moments, moment generating functions, and cumulative distributions based on these models.

The probability density function (PDF) corresponding to the generalized gamma distribution is defined as:

$$g(t) = \begin{cases} \frac{1}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)} t^{\alpha-1} E_{(\lambda,\mu)_n}\left(-t^x - \frac{p}{t^y}\right), & t > 0, \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

Similarly, the PDF associated with the generalized beta distribution takes the form:

$$b(t) = \begin{cases} \frac{1}{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right), & 0 < t < 1, \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

where $x, y, \lambda_j, \mu_j \in \mathbb{R}^+$ for $j = 1, 2, \ldots, n$, $p \geq 0$, and $\alpha, \beta \in \mathbb{R}$.

Theorem 4.1. *The r^{th} raw moment (i.e., about the origin) for the $(\lambda, \mu)_n$- based extension of gamma and beta distributions can be represented as:*

$$^{g}\mu_r' = \frac{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha+r)}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)}, \tag{4.3}$$

and

$$^{b}\mu_r' = \frac{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha+r,\beta)}{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha,\beta)}. \tag{4.4}$$

Proof. The r^{th} moment of a continuous distribution is defined as $\mathbb{E}(X^r)$. For the gamma-type model, we compute:

$$\begin{aligned}
{}^{g}\mu_r^{'} &= \int_0^\infty t^r g(t)dt \\
&= \int_0^\infty t^r \left\{ \frac{1}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)} t^{\alpha-1} E_{(\lambda,\mu)_n}\left(-t^x - \frac{p}{t^y}\right) \right\} dt \\
&= \frac{1}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)} \int_0^\infty t^{\alpha+r-1} E_{(\lambda,\mu)_n}\left(-t^x - \frac{p}{t^y}\right) dt \\
&= \frac{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha+r)}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)}.
\end{aligned}$$

The result for the beta-type model Eq. (4.4) follows similarly by substituting $g(t)$ with $b(t)$. □

Corollary 4.2. *The mean and variance for both distributions are obtained as:*

$$\textit{Mean:} \quad \overline{X}_g = \frac{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha+1)}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)},$$

$$\textit{Variance:} \quad \sigma_g^2 = \frac{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha+2) - \left[\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha+1)\right]^2}{\left[\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)\right]^2},$$

and

$$\overline{X}_b = \frac{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha+1,\beta)}{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha,\beta)},$$

$$\sigma_b^2 = \frac{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha,\beta)B^{p,x,y}_{(\lambda,\mu)_n}(\alpha+2,\beta) - \left[B^{p,x,y}_{(\lambda,\mu)_n}(\alpha+1,\beta)\right]^2}{\left[B^{p,x,y}_{(\lambda,\mu)_n}(\alpha,\beta)\right]^2}.$$

Theorem 4.3. *The MGFs of the $(\lambda,\mu)_n$-gamma and beta distributions can be expressed as:*

$$M_g(u) = \sum_{r=0}^\infty \frac{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha+r)}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)} \frac{u^r}{r!}, \tag{4.5}$$

and

$$M_b(u) = \sum_{r=0}^\infty \frac{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha+r,\beta)}{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha,\beta)} \frac{u^r}{r!}. \tag{4.6}$$

Proof. Starting with the gamma distribution, we write:

$$\begin{aligned}
M_g(u) &= \int_0^\infty e^{ut} g(t)dt \\
&= \frac{1}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)} \int_0^\infty e^{ut} t^{\alpha-1} E_{(\lambda,\mu)_n}\left(-t^x - \frac{p}{t^y}\right) dt.
\end{aligned}$$

Expanding e^{ut} as a power series and by reversing the order of integration and summation, the expression becomes:

$$\begin{aligned}
M_g(u) &= \sum_{k=0}^\infty \frac{u^k}{k!} \cdot \frac{1}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)} \int_0^\infty t^{k+\alpha-1} E_{(\lambda,\mu)_n}\left(-t^x - \frac{p}{t^y}\right) dt \\
&= \sum_{k=0}^\infty \frac{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(k+\alpha)}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)} \frac{u^k}{k!}.
\end{aligned}$$

Similarly, the beta model Eq. (4.6) follow by replacing $g(t)$ with $b(t)$. □

Theorem 4.4. *The cumulative distribution functions (CDFs) of the generalized gamma and beta distributions are:*

$$G(z) = \frac{\Gamma^{p,x,y}_{(\lambda,\mu)_n,z}(\alpha)}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)}, \tag{4.7}$$

$$B(z) = \frac{B^{p,x,y}_{(\lambda,\mu)_n,z}(\alpha,\beta)}{B^{p,x,y}_{(\lambda,\mu)_n}(\alpha,\beta)}, \tag{4.8}$$

where the incomplete versions of gamma and beta functions are expressed as:

$$\Gamma^{p,x,y}_{(\lambda,\mu)_n,z}(\alpha) = \int_0^z t^{\alpha-1} E_{(\lambda,\mu)_n}\left(-t^x - \frac{p}{t^y}\right) dt,$$

$$B^{p,x,y}_{(\lambda,\mu)_n,z}(\alpha,\beta) = \int_0^z t^{\alpha-1}(1-t)^{\beta-1} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt.$$

Proof. An explicit expression for the cumulative distribution function is given by:

$$G(z) = \mathbb{P}(X \le z) = \int_0^z g(t)dt,$$

which simplifies to:

$$G(z) = \frac{1}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)} \int_0^z t^{\alpha-1} E_{(\lambda,\mu)_n}\left(-t^x - \frac{p}{t^y}\right) dt = \frac{\Gamma^{p,x,y}_{(\lambda,\mu)_n,z}(\alpha)}{\Gamma^{p,x,y}_{(\lambda,\mu)_n}(\alpha)}.$$

The beta-type Eq. (4.8) CDF follows by replacing the gamma expression with the beta function and its incomplete form. □

5 On the $(\lambda, \mu)_n$-Extensions of Gauss and Confluent Hypergeometric Functions

This part introduces two classes of generalized special functions derived via the $(\lambda,\mu)_n$-beta function, defined in Eq. (2.2). These functions serve as extensions of classical Gauss and confluent hypergeometric functions.

$$F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z) = \sum_{k=0}^{\infty} (a)_k \frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+k,c-b)}{B(b,c-b)} \frac{z^k}{k!}, \tag{5.1}$$

$$\big(p \ge 0;\ x,y,\lambda_j,\mu_j > 0;\ j = 1,2,\ldots,n;\ |z| < 1;\ \Re(c) > \Re(b) > 0\big),$$

$$\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z) = \sum_{k=0}^{\infty} \frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+k,c-b)}{B(b,c-b)} \frac{z^k}{k!}, \tag{5.2}$$

$$\big(p \ge 0;\ x,y,\lambda_j,\mu_j > 0;\ j = 1,2,\ldots,n;\ \Re(c) > \Re(b) > 0\big).$$

5.1 Representation of Integrals from

The series representations above can be equivalently written in terms of integrals involving the Mittag-Leffler-type kernel $E_{(\lambda,\mu)_n}$.

Theorem 5.1. *The expressions below represent valid integral formulations:*

$$F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt, \tag{5.3}$$

$$\textit{where}\big(p \ge 0;\ x,y,\lambda_j,\mu_j > 0;\ |\arg(1-z)| < \pi;\ \textit{and}\ \Re(c) > \Re(b) > 0\big),$$

$$\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt, \tag{5.4}$$

$$\big(p \ge 0;\ x,y,\lambda_j,\mu_j > 0;\ \textit{and}\ \Re(c) > \Re(b) > 0\big).$$

Proof. To establish (5.3), we start from the series representation (5.1):

$$F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z) = \sum_{k=0}^{\infty}(a)_k \frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+k,c-b)}{B(b,c-b)}\frac{z^k}{k!}$$
$$= \frac{1}{B(b,c-b)}\sum_{k=0}^{\infty}(a)_k\left[\int_0^1 t^{b+k-1}(1-t)^{c-b-1}E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right)dt\right]\frac{z^k}{k!}.$$

Swapping the sum and the integral (justified by uniform convergence for small $|z|$), we obtain:

$$F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right)\sum_{k=0}^{\infty}(a)_k\frac{(zt)^k}{k!}dt$$
$$= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right)dt.$$

The proof of (5.4) follows by the same steps, replacing the series $\sum z^k/k!$ with the exponential function. □

6 Derivative Formulae

We derive order-r differential identities for the generalized $(\lambda,\mu)_n$-Gauss and confluent hypergeometric functions in this section.

Theorem 6.1. *The r^{th} order derivatives of the respective functions satisfy the relations:*

$$\frac{d^r}{dz^r}\left\{F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z)\right\} = \frac{(a)_r(b)_r}{(c)_r}F^{p,x,y}_{(\lambda,\mu)_n}(a+r,b+r;c+r;z), \tag{6.1}$$
$$\big(p\geq 0,\ x,y,\lambda_j,\mu_j>0,\ j=1,\dots,n;\ \Re(c)>\Re(b)>0,\ n,r\in\mathbb{N}\big),$$

$$\frac{d^r}{dz^r}\left\{\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z)\right\} = \frac{(b)_r}{(c)_r}\Phi^{p,x,y}_{(\lambda,\mu)_n}(b+r;c+r;z), \tag{6.2}$$
$$\big(p\geq 0,\ x,y,\lambda_j,\mu_j>0,\ j=1,\dots,n;\ \Re(c)>\Re(b)>0,\ n,r\in\mathbb{N}\big).$$

Proof. We begin by differentiating the function $F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z)$ with respect to z:

$$\frac{d}{dz}\left\{F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z)\right\} = \frac{d}{dz}\sum_{k=0}^{\infty}(a)_k\frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+k,c-b)}{B(b,c-b)}\frac{z^k}{k!}$$
$$= \sum_{k=1}^{\infty}(a)_k\frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+k,c-b)}{B(b,c-b)}\frac{z^{k-1}}{(k-1)!}.$$

Changing index via $k\mapsto k+1$ gives:

$$\frac{d}{dz}\left\{F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z)\right\} = \sum_{k=0}^{\infty}(a)_{k+1}\frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+k+1,c-b)}{B(b,c-b)}\frac{z^k}{k!}.$$

Now by identities $(a)_{k+1} = a(a+1)_k$ and the beta function recurrence $B(b,c-b) = \frac{c}{b}B(b+1,c-b)$, we rewrite:

$$\frac{d}{dz}\left\{F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z)\right\} = \frac{ab}{c}\sum_{k=0}^{\infty}(a+1)_k\frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+k+1,c-b)}{B(b+1,c-b)}\frac{z^k}{k!}$$
$$= \frac{ab}{c}F^{p,x,y}_{(\lambda,\mu)_n}(a+1,b+1;c+1;z).$$

Repeated application of this process up to order r yields the formula in (6.1). A similar technique proves the formula for the confluent case given in (6.2). □

7 Transformation and Summation Identities

In this subsection, we present identities that transform the argument of the generalized $(\lambda, \mu)_n$-hypergeometric functions. These formulas extend known classical transformations to the present generalized setting.

Theorem 7.1. *The following transformation relationships hold:*

$$F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; z) = (1 - z)^{-a} F^{p,x,y}_{(\lambda,\mu)_n}\left(a, c - b; c; -\frac{z}{1 - z}\right), \tag{7.1}$$

$$F^{p,x,y}_{(\lambda,\mu)_n}\left(a, b; c; 1 - \frac{1}{z}\right) = z^a F^{p,x,y}_{(\lambda,\mu)_n}(a, c - b; c; 1 - z), \tag{7.2}$$

$$F^{p,x,y}_{(\lambda,\mu)_n}\left(a, b; c; 1 + \frac{1}{z}\right) = (1 + z)^a F^{p,x,y}_{(\lambda,\mu)_n}(a, c - b; c; -z), \tag{7.3}$$

$$\Phi^{p,x,y}_{(\lambda,\mu)_n}(b; c; z) = e^z \Phi^{p,x,y}_{(\lambda,\mu)_n}(c - b; c; -z), \tag{7.4}$$

$$\big(p \geq 0,\ x, y, \lambda_j, \mu_j > 0,\ j = 1, \ldots, n; \textit{and}\ \Re(c) > \Re(b) > 0\big).$$

Proof. Now derive identity (7.1) by making substitution $t \mapsto 1 - t$ in integral representation (5.3). Then:

$$(1 - z(1 - t))^{-a} = (1 - z)^{-a}\left(1 + \frac{z}{1 - z}t\right)^{-a}.$$

By inserting the expression into Eq. (5.3), the integral becomes:

$$\begin{aligned} F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; z) &= \frac{(1 - z)^{-a}}{B(b, c - b)} \int_0^1 (1 - t)^{b-1} t^{c-b-1} \left(1 + \frac{zt}{1 - z}\right)^{-a} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x (1 - t)^y}\right) dt \\ &= (1 - z)^{-a} F^{p,x,y}_{(\lambda,\mu)_n}\left(a, c - b; c; -\frac{z}{1 - z}\right). \end{aligned}$$

The other identities (7.2) and (7.3) follow from applying variable substitutions to the result of (7.1), namely $z \mapsto 1 - \frac{1}{z}$ and $z \mapsto 1 + \frac{1}{z}$, respectively. Lastly, identity (7.4) follows similarly from the exponential integral form of $\Phi^{p,x,y}_{(\lambda,\mu)_n}(b; c; z)$ using the property $e^{zt} = e^z e^{-z(1-t)}$ under the same substitution $t \mapsto 1 - t$. □

Theorem 7.2. *The value of the $(\lambda, \mu)_n$-Gauss hypergeometric function at $z = 1$ can be expressed as:*

$$F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; 1) = \frac{B^{p,x,y}_{(\lambda,\mu)_n}(b, c - a - b)}{B(b, c - b)}, \tag{7.5}$$

where $p \geq 0$, $x, y, \lambda_j, \mu_j > 0$ $(j = 1, 2, \ldots, n)$, and $\Re(c) > \Re(b) > 0$.

Proof. From the integral form (5.3) of $F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; z)$, we write:

$$F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - zt)^{-a} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x (1 - t)^y}\right) dt.$$

Setting $z = 1$ yields:

$$\begin{aligned} F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; 1) &= \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-a-b-1} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x (1 - t)^y}\right) dt \\ &= \frac{B^{p,x,y}_{(\lambda,\mu)_n}(b, c - a - b)}{B(b, c - b)}. \end{aligned}$$

□

Remark 7.3. Substituting $n = 1$, and letting $\lambda = \mu = x = y = 1$, and $p = 0$, the generalized formula (7.5) reduces to the classical Gauss summation identity:

$$ {}_2F_1(a, b; c; 1) = \frac{B^{0,1,1}_{(1,1)_1}(b, c-a-b)}{B(b, c-b)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

8 A Generating Function for $(\lambda, \mu)_n$-Extended Gauss Hypergeometric Function

Theorem 8.1. *A generating function involving the family* $F^{p,x,y}_{(\lambda,\mu)_n}(a+k, b; c; z)$ *satisfies:*

$$ \sum_{k=0}^{\infty}(a)_k F^{p,x,y}_{(\lambda,\mu)_n}(a+k, b; c; z)\frac{t^k}{k!} = (1-t)^{-a} F^{p,x,y}_{(\lambda,\mu)_n}\left(a, b; c; \frac{z}{1-t}\right), \tag{8.1} $$

valid for $p \geq 0$, $x, y, \lambda_j, \mu_j > 0$ $(j = 1, \dots, n)$, *and* $\Re(c) > \Re(b) > 0$.

Proof. We begin by substituting the definition of $F^{p,x,y}_{(\lambda,\mu)_n}(a+k, b; c; z)$ into the left-hand side of (8.1):

$$ \sum_{k=0}^{\infty}(a)_k \left[\sum_{r=0}^{\infty}(a+k)_r \frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+r, c-b)}{B(b, c-b)}\frac{z^r}{r!}\right]\frac{t^k}{k!} $$

$$ = \sum_{r=0}^{\infty}\left[\sum_{k=0}^{\infty}(a)_k(a+k)_r\frac{t^k}{k!}\right]\frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+r, c-b)}{B(b, c-b)}\frac{z^r}{r!}. $$

Applying the identity $(a)_k(a+k)_r = (a)_r(a+r)_k$, this becomes:

$$ \sum_{r=0}^{\infty}(a)_r\frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+r, c-b)}{B(b, c-b)}\left[\sum_{k=0}^{\infty}(a+r)_k\frac{t^k}{k!}\right]\frac{z^r}{r!}. $$

Since the inner sum is the expansion of $(1-t)^{-a-r}$, we get:

$$ \sum_{k=0}^{\infty}(a)_k F^{p,x,y}_{(\lambda,\mu)_n}(a+k, b; c; z)\frac{t^k}{k!} = \sum_{r=0}^{\infty}(a)_r\frac{B^{p,x,y}_{(\lambda,\mu)_n}(b+r, c-b)}{B(b, c-b)}\frac{z^r}{r!}(1-t)^{-a-r}. $$

This series is equivalent to:

$$ (1-t)^{-a} F^{p,x,y}_{(\lambda,\mu)_n}\left(a, b; c; \frac{z}{1-t}\right). $$

□

9 Mellin Transform Representation

Theorem 9.1. *Let* $s \in \mathbb{C}$ *satisfy* $\Re(s) > 0$, *and suppose* $p \geq 0$, $x, y, \lambda_j, \mu_j > 0$ $(j = 1, \dots, n)$, *with* $\Re(c) > \Re(b) > 0$ *and* $\Re(b+sx), \Re(c+sy) > 0$. *Then the Mellin transform of the generalized hypergeometric function satisfies [3, 18]:*

$$ \mathcal{M}\left[F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; z); s\right] = \frac{\Gamma_{(\lambda,\mu)_n}(s)\, B(b+sx,\, c+sy-b)}{B(b,\, c-b)} F(a,\, b+s;\, c+2s;\, z). \tag{9.1} $$

Proof. The Mellin transform is defined as:

$$ \mathcal{M}\left[F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; z); s\right] = \int_0^{\infty} p^{s-1} F^{p,x,y}_{(\lambda,\mu)_n}(a, b; c; z)\, dp. $$

Using the integral form (5.3), we obtain:

$$= \int_0^\infty p^{s-1} \left[\frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt \right] dp$$

$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \left[\int_0^\infty p^{s-1} E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dp \right] dt.$$

Applying the change of variable $u = \frac{p}{t^x(1-t)^y}$, we get:

$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b+sx-1}(1-t)^{c+sy-b-1}(1-zt)^{-a} dt \cdot \int_0^\infty u^{s-1} E_{(\lambda,\mu)_n}(-u)\, du$$

$$= \frac{\Gamma_{(\lambda,\mu)_n}(s)\, B(b+sx,\, c+sy-b)}{B(b,\, c-b)} F(a,\, b+s;\, c+2s;\, z).$$

□

Corollary 9.2. *Using the inverse Mellin transform [18], we obtain the complex-contour integral representation:*

$$F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\Gamma_{(\lambda,\mu)_n}(s)\, B(b+sx,\, c+sy-b)}{B(b,\, c-b)} F(a,\, b+s;\, c+2s;\, z)\, p^{-s}\, ds. \tag{9.2}$$

Theorem 9.3. *The Mellin transform of the $(\lambda,\mu)_n$-confluent hypergeometric function is given by [18]:*

$$\mathcal{M}\left[\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z); s\right] = \frac{\Gamma_{(\lambda,\mu)_n}(s)\, B(b+sx,\, c+sy-b)}{B(b,\, c-b)} \Phi(b+s;\, c+2s;\, z), \tag{9.3}$$

where $p \geq 0$, $x, y, \lambda_j, \mu_j > 0$ $(j = 1, 2, \ldots, n)$, $\Re(c) > \Re(b) > 0$, and $\Re(s), \Re(b+sx), \Re(c+sy) > 0$.

Proof. The derivation follows the same method used in the Gauss-type case (see Eq. (5.4)). Starting with the definition:

$$\mathcal{M}\left[\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z); s\right] = \int_0^\infty p^{s-1} \Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z)\, dp,$$

and using the integral representation:

$$\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt}\, E_{(\lambda,\mu)_n}\left(-\frac{p}{t^x(1-t)^y}\right) dt,$$

interchanging the order of integration and applying the substitution $u = \frac{p}{t^x(1-t)^y}$, we obtain:

$$\mathcal{M}\left[\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z); s\right] = \frac{\Gamma_{(\lambda,\mu)_n}(s)\, B(b+sx,\, c+sy-b)}{B(b,\, c-b)} \Phi(b+s;\, c+2s;\, z).$$

□

Corollary 9.4. *Applying the inverse Mellin transform [18] to (9.3) gives*

$$\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\Gamma_{(\lambda,\mu)_n}(s)\, B(b+sx,\, c+sy-b)}{B(b,\, c-b)} \Phi(b+s;\, c+2s;\, z)\, p^{-s}\, ds. \tag{9.4}$$

10 Numerical Study

In this section a numerical study is carried out to validate the obtained results of the paper for different parameters. Table 1 gives the numerical values of $B^{p,x,y}_{(\lambda,\mu)_n}(z,w)$, $F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z)$ and $\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c;z)$ for different values of p. In the next subsection graphs are plotted for the values given in the table:

z	$B^{p,x,y}_{(\lambda,\mu)_n}(z,w)$				$F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c,z)$				$\Phi^{p,x,y}_{(\lambda,\mu)_n}(b;c,z)$			
–	$p=0.0$	$p=0.5$	$p=1.0$	$p=1.5$	$p=0.0$	$p=0.5$	$p=1.0$	$p=1.5$	$p=0.0$	$p=0.5$	$p=1.0$	$p=1.5$
0.001	3.6734	3.4402	3.2061	2.9723	1.0007	0.9544	0.9079	0.8614	1.0007	0.9544	0.9079	0.8614
0.051	3.3608	3.1553	2.9488	2.7428	1.0399	0.9885	0.9399	0.8913	1.0390	0.9878	0.9392	0.8907
0.101	3.0821	2.9007	2.7185	2.5366	1.0827	1.0254	0.9745	0.9237	1.0790	1.0225	0.9718	0.9212
0.151	2.8335	2.6731	2.5118	2.3509	1.1297	1.0657	1.0122	0.9589	1.1207	1.0586	1.0056	0.9528
0.201	2.6113	2.4691	2.3262	2.1835	1.1814	1.1096	1.0534	0.9973	1.1642	1.0960	1.0408	0.9857
0.251	2.4123	2.2861	2.1591	2.0324	1.2388	1.1579	1.0986	1.0393	1.2095	1.1350	1.0774	1.0198
0.301	2.234	2.1216	2.0085	1.8956	1.3028	1.2114	1.1485	1.0857	1.2567	1.1755	1.1153	1.0552
0.351	2.0738	1.9734	1.8725	1.7717	1.3749	1.2708	1.2039	1.1371	1.3060	1.2176	1.1548	1.0921
0.401	1.9297	1.8399	1.7495	1.6592	1.4566	1.3374	1.2659	1.1946	1.3575	1.2615	1.1959	1.1304
0.451	1.7998	1.7192	1.6380	1.5570	1.5503	1.4127	1.3358	1.2592	1.4111	1.3070	1.2385	1.1701
0.501	1.6825	1.6099	1.5368	1.4638	1.6590	1.4985	1.4155	1.3326	1.4671	1.3544	1.2829	1.2114
0.551	1.5765	1.5109	1.4448	1.3789	1.7869	1.5977	1.5072	1.4169	1.5255	1.4037	1.3289	1.2544
0.601	1.4804	1.4209	1.3610	1.3012	1.9401	1.7137	1.6142	1.515	1.5864	1.4549	1.3769	1.2989
0.651	1.3931	1.3390	1.2845	1.2300	2.1276	1.8519	1.7413	1.6310	1.6501	1.5083	1.4268	1.3453
0.701	1.3138	1.2644	1.2145	1.1648	2.3637	2.0199	1.8958	1.7710	1.7164	1.5638	1.4786	1.3935
0.751	1.2415	1.1962	1.1505	1.1048	2.6722	2.2301	2.0871	1.9444	1.7857	1.6216	1.5325	1.4436
0.801	1.1754	1.1338	1.0917	1.0496	3.0973	2.5028	2.3346	2.1667	1.8580	1.6816	1.5886	1.4957
0.851	1.1150	1.0765	1.0376	0.9988	3.7313	2.8757	2.6707	2.4661	1.9334	1.7442	1.6469	1.5499
0.901	1.0597	1.0239	0.9878	0.9518	4.8082	3.4277	3.1638	2.9005	2.0123	1.8092	1.7076	1.6062
0.951	1.0088	0.9755	0.9418	0.9083	7.0972	4.3585	3.9857	3.6137	2.0945	1.8769	1.7707	1.6647

Table 1. Table of different functions discussed in the paper.

10.1 Results for $p = 0.5, 1.0, 1.5$

In Figs. (1–3) variations are plotted for different values of p. Here we take $n = 4, x = y = 2, w = 1.0, a = 1.0, b = 1.5, c = 2.0$.

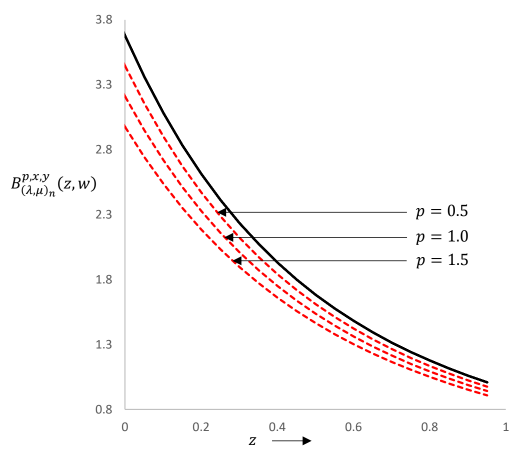

Figure 1. Beta function vs. increasing values of z

Fig. 1 displays the increase in the beta function with increasing values of z. Black solid line shows the variation in classical beta function (1.2) and red dotted lines show the variation in the $B^{p,x,y}_{(\lambda,\mu)_n}(z,w)$ for different values of p. It may be noted that, for small value of p ($viz.$ $p = 0.5$), values of $B^{p,x,y}_{(\lambda,\mu)_n}(z,w)$ approach to classical beta function.

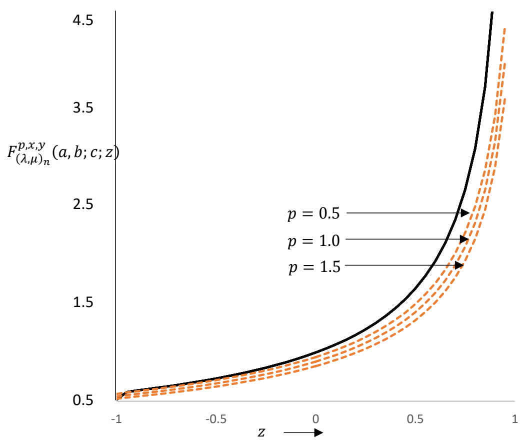

Figure 2. Gauss hypergeometric function vs. increasing values of z

Same variation for Gauss hypergeometric function is plotted in Fig. 2. Classical Gauss hypergeometric function (1.3) is represented by black line while the extended Gauss hypergeometric function is represented by red lines. An insignificant difference is seen in the values of $(\lambda,\mu)_n-$Gauss hypergeometric function and classical Gauss hypergeometric function when z and p are small $(i.e. z=-1, p=0.5)$ however a significant difference between the two is seen for large value of z and p.

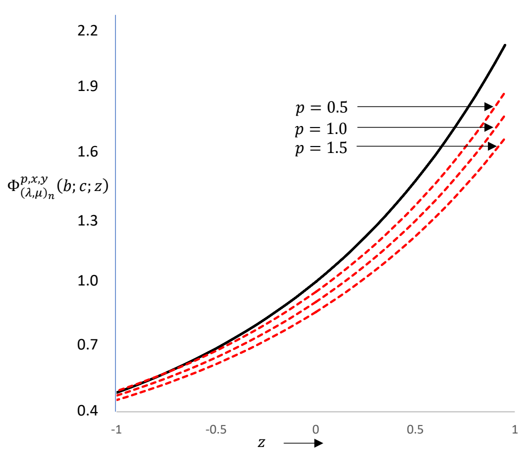

Figure 3. Confluent hypergeometric function vs. increasing values of z

Same variation for confluent hypergeometric function is plotted in Fig. 3. Classical confluent hypergeometric function (1.4) is represented by black line while extended confluent hypergeometric function is represented by red lines. It may be concluded from figure that small value of p gives the best result for extended confluent hypergeometric function.

10.2 Results for $n = 2, 4, 6$

In Figs. (4–6) variations are plotted for different number of parameters in generalized Mittag-Leffler function. Here we take $p = 2, x = y = 1, w = 2$.

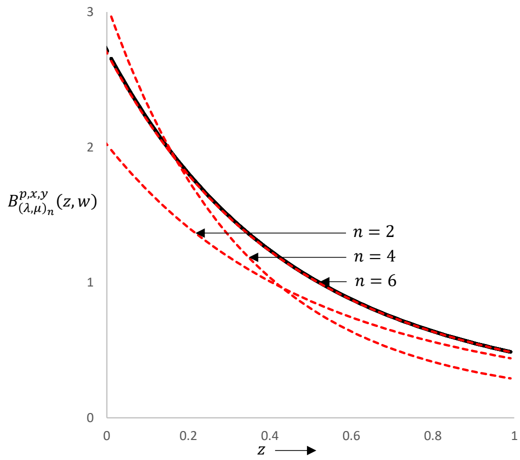

Figure 4. Beta function vs. increasing values of z

Fig. 4 shows the variation in beta function against the increasing values of argument z. Black solid line shows the variation in classical beta function (1.2) and red dotted lines show the variation in the $B^{p,x,y}_{(\lambda,\mu)_n}(z,w)$ for different values of n. It is seen that, when $n = 2$ the value of $B(z,w)$ function and $B^{p,x,y}_{(\lambda,\mu)_n}(z,w)$ function are significantly different. However as the value of n increases the difference between two function goes to zero i.e. when $n = 6$.

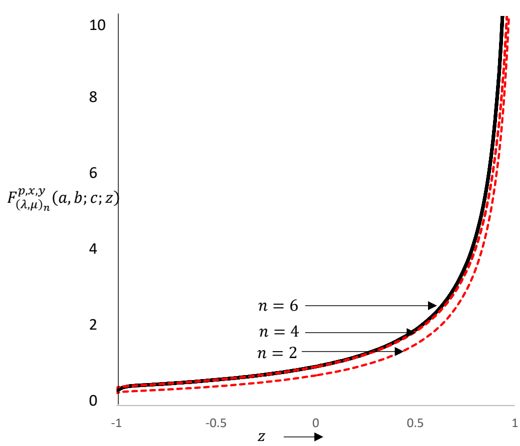

Figure 5. Gauss hypergeometric function vs. increasing values of z

Fig. 5 depicts the variation in Gauss hypergeometric function with the increasing values of z. Black solid line shows the variation in classical Gauss hypergeometric function (1.3) and red dotted lines show the variation in the $F^{p,x,y}_{(\lambda,\mu)_n}(a,b;c;z)$ for different values of n. It may be noted from the figure that, as number of parameters in the $2n-$parametric Mittag-Leffler function increases the proposed extension of Gauss hypergeometric function approaches classical Gauss hypergeometric function.

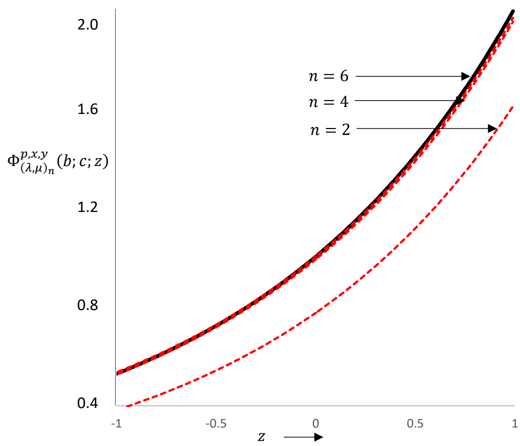

Figure 6. Confluent hypergeometric function vs. increasing values of z

Variation between z and confluent hypergeometric function is depicted in Fig. 6. Classical confluent hypergeometric function (1.4) is represented by black solid line while the $(\lambda, \mu)_n-$confluent hypergeometric function is represented by red dotted lines for different parameters n. It may be noted from the figure that, the large number of parameters in the Mittag-Leffler function makes the proposed function to approach the classical function.

11 Concluding Remarks

This work present a novel extension of the classical gamma and beta functions using the $2n$-parameter Mittag-Leffler function $E_{(\lambda,\mu)_n}(z)$. The proposed $(\lambda, \mu)_n$-extensions enrich the classical theory by adding parametric flexibility. Analytical properties such as integral representations and Mellin transforms have been established.Using the new $(\lambda, \mu)_n$-beta function, we constructed generalized Gauss and confluent hypergeometric functions. These extended functions exhibit key properties, including derivative formulas, transformation identities, summation theorems, generating functions, and Mellin-type integrals, offering a unified structure that connects classical and generalized functions.Applications include statistical modelling and areas of theoretical physics like quantum mechanics. Graphical results support the theoretical framework by illustrating parametric behaviour.Given the foundational role of hypergeometric functions, our results lay the groundwork for future studies involving further extensions, analytical investigations, and interdisciplinary applications.

Future work may explore the use of these generalized functions in solving fractional differential equations, as well as in constructing further extensions of Mittag-Leffler type functions. Such developments could provide new analytical tools for modeling complex systems in fractional calculus and applied mathematics.

Given the foundational role of hypergeometric functions, our results lay the groundwork for future studies involving further extensions, analytical investigations, and interdisciplinary applications.

Conflict of Interest

It is confirmed that no conflicts of interest are associated with this work.

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