Palestine Journal of Mathematics

Vol 15(2)(2026) , 359–370

Certain Properties of Analytic Functions Associated with Tremblay Fractional Derivative Operator

S.G. Hiwale and D.D. Pawar

Communicated by Jaganmohan J.

MSC 2020 Classifications: Primary 30C45; Secondary 30C50.

Keywords and phrases: analytic, convex, starlike, integral operator, coefficient estimates.

The authors would like to thank the reviewers and the editor for their constructive comments and valuable suggestions, which improved the quality of our paper.

Corresponding Author: S.G. Hiwale

Abstract. In this paper, we introduce and investigate a new subclass of analytic functions defined by a modified Tremblay fractional derivative operator in the unit disk. We obtain coefficient inequalities, distortion properties, radii properties, extreme points, and closure properties. Furthermore, we obtain integral transforms, closure properties, and integral means inequalities for the defined class.

1 Introduction

The subject of fractional calculus (integral and derivative of any arbitrary real or complex order) has acquired significant popularity and major attention from several authors in various science due mainly to its direct involvement in the problems of differential equations in mathematics, physics, engineering and others. it has gained an interesting area in mathematical research and generalization of the(derivative and integral) operators and its useful utility to express the mathematical problems which often leads to problems to be solved. Specifically, it utilized to define new classes and generalized many geometric properties and inequalities in complex domain. These operators are play an important role in geometric function theory to define new generalized subclasses of analytic univalent and then study their properties.
The study of analytic functions through the lens of fractional calculus has garnered significant attention due to its broad applicability in complex analysis, mathematical modeling, and applied sciences. Fractional derivative operators, particularly those involving parameters, offer a powerful framework for generalizing classical function classes and uncovering deeper structural properties.
Among such operators, the Tremblay fractional derivative operator stands out due to its flexibility and generality. It unifies various known fractional operators and introduces a richer parametric structure that facilitates the study of more complex geometric behaviors of analytic functions.
Motivated by the need to explore new function classes with enhanced geometric and analytic properties, this work aims to investigate analytic functions defined via the Tremblay operator. By doing so, we extend classical results and provide new insights into coefficient bounds, distortion properties, and radii problems under fractional differentiation. The motivation also stems from the potential of these generalized functions in modeling physical systems described by non-integer order dynamics, as well as their theoretical significance in geometric function theory.

Let $\mathcal{A}$ denote the class of univalent and analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, and normalized by the conditions $f(0) = f'(0) - 1 = 0$. It is clear that every function belonging to the class $\mathcal{A}$ can be given by the following Maclaurin's expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

A function f belonging to $\mathcal{A}$ is said to be starlike function of order $\alpha\,(0 \leq \alpha < 1)$, if

$$\text{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \alpha\,(z \in U), \tag{1.2}$$

the subclass of all starlike functions of order α is denoted usually as $S^*(\alpha)$. Further, a function f belonging to $\mathcal{A}$ said to be convex function of order $\alpha\,(0 \leq \alpha < 1)$, if

$$\text{Re}\left\{1 + \frac{zf^{''}(z)}{f'(z)}\right\} > \alpha\,(z \in U), \tag{1.3}$$

we denote by $\mathcal{K}(\alpha)$ the subclass of all convex functions of order α. We note that $S^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$. From (1.2) and (1.3), it is clear that

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow g \in \mathcal{S}^*(\alpha); g(z) = zf^{'}(z). \tag{1.4}$$

For details about the subclasses $S^*(\alpha)$ and $\mathcal{K}(\alpha)$, see MacGregor [9], Pinchuk [12] and Schild [14].
The modified Hadamard product (or Convolution) of function $f \in \mathcal{A}$ given by (1.1), and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.5}$$

is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.6}$$

Let T denote the class of functions analytic in U that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0,\ z \in U) \tag{1.7}$$

and let $T^*(\alpha) = T \cap S^*(\alpha),\ C(\alpha) = T \cap K(\alpha)$. The class $T^*(\alpha)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [16] and Others [15].

The following definition of fractional derivative will be required in our investigation (see, for details, [10, 11, 19, 21]).

Definition 1.1. The fractional integral of order δ is defined, for a function f, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi\ ;\ (\delta > 0),$$

where f is an analytic function in a simply-connected region of complex z -plane containing the origin, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring $log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 1.2. The fractional derivative of order δ is defined, for a function f, by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi\ (0 \leq \delta < 1),$$

where f is constrained, and the multiplicity of $(z-\xi)^{-\delta}$ is removed, as in Definition 1.1.

Definition 1.3. Under the hypotheses of Definition 1.2, the fractional derivative of order $(n+\delta)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z) \quad (0 \leq \delta < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

By virtue of Definitions 1.1, 1.2 and 1.3, we have

$$D_z^{-\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} z^{n+\delta} \quad (n \in \mathbb{N}, \delta > 0)$$

and

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n-\delta+1)} z^{n-\delta} \quad (n \in \mathbb{N}, 0 \leq \delta < 1).$$

Tremblay [22] studied a fractional calculus operator defined in terms of the Riemann Liouville fractional differential operator. Ibrahim and Jahangiri [6] extended and studied this operator in the complex plane.

Definition 1.4. The Tremblay fractional derivative operator $\mathrm{T}_z^{\mu,\sigma}$ of a function $f \in \mathcal{A}$ is defined, for all $z \in U$ by

$$\mathrm{T}_z^{\mu,\sigma} f(z) = \frac{\Gamma(\sigma)}{\Gamma(\mu)} z^{1-\sigma} D_z^{\mu-\sigma} z^{\mu-1} f(z)$$

$$(0 < \mu \leq 1;\ 0 < \sigma \leq 1;\ \mu \geq \sigma;\ 0 \leq \mu - \sigma < 1).$$

It is clear that, for $\mu = \sigma = 1$, we have

$$\mathrm{T}_z^{1,1} f(z) = f(z).$$

Example 1.1. Let $f(z) = z^n$. The Tremblay Fractional Derivative of $f(z)$ is

$$\mathrm{T}_z^{\mu,\sigma} f(z) = \frac{\Gamma(\sigma)}{\Gamma(\mu)} \frac{\Gamma(n+\mu)}{\Gamma(n+\sigma)} z^n,$$

and for $\mu = \sigma = 1$, we have $\mathrm{T}_z^{1,1}(z^n) = z^n$.

Recently in [3], Esa et al. defined modified of Tremblay operator of analytic functions in complex domain as follows:

Definition 1.5. Let $f(z) \in \mathcal{A}$. The modified Tremblay operator denoted by $\mathfrak{T}^{\mu,\sigma} : \mathcal{A} \to \mathcal{A}$ and defined such as:

$$\begin{aligned} \mathfrak{T}^{\mu,\sigma} f(z) &= \frac{\sigma}{\mu} \mathrm{T}_z^{\mu,\sigma} f(z) \\ &= \frac{\Gamma(\sigma+1)}{\Gamma(\mu+1)} z^{1-\sigma} D_z^{\mu-\sigma} z^{\mu-1} f(z) \\ &= z + \sum_{n=2}^{\infty} \Theta(n,\mu,\sigma) a_n z^n \end{aligned} \tag{1.8}$$

where

$$\Theta(n,\mu,\sigma) = \frac{\Gamma(\sigma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\sigma)}. \tag{1.9}$$

Inspired by the earlier works of [1, 2, 4, 20] we define a new subclass of functions belonging to the class $\mathcal{A}$.

Definition 1.6. For $0 \leq \lambda \leq 1$ $0 \leq \gamma < 1$ and $k \geq 0$, we let $\mathbb{L}_\mu^\sigma(\lambda, \gamma, k)$ be the subclass of T consisting of functions of the form (1.7) and satisfying the analytic criteriaon

$$\Re\left(\frac{F_\lambda(z)}{zF'_\lambda(z)} - \gamma\right) > k\left|\frac{F_\lambda(z)}{zF'_\lambda(z)} - \gamma\right| \tag{1.10}$$

where

$$\frac{F_\lambda(z)}{zF'_\lambda(z)} = \frac{(1-\lambda)\mathfrak{T}^{\mu,\sigma} f(z) + \lambda z(\mathfrak{T}^{\mu,\sigma} f(z))'}{z(\mathfrak{T}^{\mu,\sigma} f(z))' + \lambda z^2(\mathfrak{T}^{\mu,\sigma} f(z))''}, \quad (z \in U) \tag{1.11}$$

where $\mathfrak{T}^{\mu,\sigma} f(z)$ is given by (1.8).

Example 1.1. For $\lambda = 0$ we let $\mathbb{L}S^{\sigma}_{\mu}(\gamma, k)$ be the subclass of T consisting of functions of the form (1.7) and satisfying the analytic criterion

$$\Re\left(\frac{\mathfrak{T}^{\mu,\sigma} f(z)}{z(\mathfrak{T}^{\mu,\sigma} f(z))'} - \gamma\right) > k\left|\frac{\mathfrak{T}^{\mu,\sigma} f(z)}{z(\mathfrak{T}^{\mu,\sigma} f(z))'} - 1\right|,$$

2 Characteristic properties of the class $\mathbb{L}^{\sigma}_{\mu}(\lambda, \gamma, k)$

We recall the following lemmas, in order to prove our main results.

Lemma 2.1. *If χ is a real number and w is a complex number, then*

$$\Re(w) \geq \chi \Leftrightarrow |w + (1-\chi)| - |w - (1+\chi)| \geq 0.$$

Lemma 2.2. *If w is a complex number and χ, k are real numbers, then*

$$\Re(w) \geq k|w-1| + \chi \Leftrightarrow \Re\{w(1+ke^{i\theta}) - ke^{i\theta}\} \geq \chi, \ -\pi \leq \theta \leq \pi.$$

Theorem 2.3. *A function f of the form (1.7) is in $\mathbb{L}^{\sigma}_{\mu}(\lambda, \gamma, k)$ if and only if*

$$\sum_{n=2}^{\infty} (1+n\lambda-\lambda)|(1+k) - n(\gamma+k)|\Theta\,(\mathrm{n}, \mu\,, \sigma\,)|a_n| \leq 1-\gamma, \tag{2.1}$$

where $0 \leq \lambda \leq 1, 0 \leq \gamma < 1, k \geq 0$ *and* $\Theta\,(\mathrm{n}, \mu\,, \sigma\,)$ *is given by* (1.9).

Proof. Let a function f of the form (1.7) and such that $f \in \mathcal{T}$ satisfy the condition (2.1). We will show that (1.10)) is satisfied and so $f \in \mathbb{L}^{\sigma}_{\mu}(\lambda, \gamma, k)$. Using Lemma (2.2), it is enough to show that

$$\Re\left(\frac{F_{\lambda}(z)}{zF'_{\lambda}(z)}(1+ke^{i\theta}) - ke^{i\theta}\right) > \gamma, \ -\pi \leq \theta \leq \pi. \tag{2.2}$$

That is, suppose $f \in \mathbb{L}^{\sigma}_{\mu}(\lambda, \gamma, k)$. Then by Lemma 2.2, we have (2.2).

Choosing the values of z on the positive real axis the inequality(2.2) reduces to

$$\Re\left(\frac{(1-\gamma) - \sum_{n=2}^{\infty}[(1+ke^{i\theta}) - n(\gamma+ke^{i\theta})](1+\lambda n-\lambda)\Theta\,(\mathrm{n}, \mu, \sigma)|a_n|z^{n-1}}{1-\sum_{n=2}^{\infty} n(1+n\lambda-\lambda)\Theta\,(\mathrm{n}, \mu, \sigma)a_n z^{n-1}}\right) \geq 0.$$

Since $\Re(-e^{i\theta}) \geq -e^{i\theta} = -1$, the above inequality reduces to

$$\Re\left(\frac{(1-\gamma) - \sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(k+1) - n(\gamma+k)]\Theta\,(\mathrm{n}, \mu\,, \sigma\,)a_n r^{n-1}}{1-\sum_{n=2}^{\infty} n(1+n\lambda-\lambda)\Theta\,(\mathrm{n}, \mu\,, \sigma\,)a_n r^{n-1}}\right) \geq 0.$$

Letting $r \to 1^-$ and by the mean value theorem we get desired inequality (2.1).

Converesely, let (2.1) hold we will show that (1.10) is satisfied and so $f \in \mathbb{L}^{\sigma}_{\mu}(\lambda, \gamma, k)$. In view of Lemma 2.1,

$$\Re(w) > \gamma \Leftrightarrow |w - (1+\gamma)| < |w + (1-\gamma)|,$$

it is enough to show that

$$\left|\frac{A(z)}{B(z)} - \left(1 + k\left|\frac{A(z)}{B(z)} - 1\right| + \gamma\right)\right| < \left|\frac{A(z)}{B(z)} + \left(1 - k\left|\frac{A(z)}{B(z)} - 1\right| - \gamma\right)\right|,$$

where

$$\begin{aligned} A(z) &= \left[(1-\lambda)\,\mathfrak{T}^{\sigma}_{\mu} f(z) + \lambda z\Big(\mathfrak{T}^{\sigma}_{\mu} f(z)\Big)'\right] \\ &= z - \sum_{n=2}^{\infty}(1+\lambda n-\lambda)\Theta\,(\mathrm{n}, \mu\,, \sigma\,)|a_n|z^n \end{aligned}$$

and

$$B(z) = \left[z\left(\mathfrak{T}_{\mu}^{\sigma} f(z)\right)' + \lambda z^2\left(\mathfrak{T}_{\mu}^{\sigma} f(z)\right)''\right]$$

$$= z - \sum_{n=2}^{\infty} n(1 + \lambda n - \lambda)\Theta\,(\mathrm{n}, \mu\,, \sigma\,)|a_n|z^n$$

Hence, we have

$$= \left|\frac{A(z)}{B(z)} - \left(1 + k\left|\frac{A(z)}{B(z)} - 1\right| + \gamma\right)\right|$$

$$< \frac{|z|}{|B(z)|}\left|\gamma + \sum_{n=2}^{\infty}(1 + n\lambda - \lambda)[n - 1 - \gamma + n(\gamma + k)]\Theta\,(\mathrm{n}, \mu, \sigma\,)a_n z^n\right|$$

$$< \frac{|z|}{|B(z)|}\left|(2 - \gamma) - \sum_{n=2}^{\infty}(1 + n\lambda - \lambda)[(n + 1 + \gamma) - n(\gamma + k)]\Theta\,(\mathrm{n}, \mu, \sigma)\mathrm{a_n z^n}\right|$$

$$< R = \left|\frac{A(z)}{B(z)} + \left(1 - k\left|\frac{A(z)}{B(z)} - 1\right| - \gamma\right)\right|,$$

and it is easy to show that $R - L > 0$, by the given condition (2.1). This completes the proof. □

Theorem 2.4. *A function f of the form (1.7) is in $\mathbb{L}S_{\mu}^{\sigma}(\gamma, k)$ $(0 \le \gamma < 1, k \ge 0)$ if and only if*

$$\sum_{n=2}^{\infty} |(1 + k) - n(\gamma + k)|\Theta\,(\mathrm{n}, \mu\,, \sigma\,)|a_n| \le 1 - \gamma, \tag{2.3}$$

where $\Theta\,(\mathrm{n}, \mu, \sigma\,)$ is given by (1.9)

Theorem 2.5. *A function $f(z)$ of the form (1.7) is in $\mathbb{L}C_{\mu}^{\sigma}(\gamma, k)$ $(0 \le \gamma < 1, k \ge 0)$ if and only if*

$$\sum_{n=2}^{\infty} n|(1 + k) - n(\gamma + k)|\Theta(\mathrm{n}, \mu, \sigma)|a_n| \le 1 - \gamma, \tag{2.4}$$

where $\Theta(n, \mu, \sigma)$ is given by (1.9).

Corollary 2.6. *If $f \in \mathbb{L}_{\mu}^{\sigma}(\lambda, \gamma, k)$, then*

$$|a_n| \le \frac{1 - \gamma}{\Psi(\lambda, \gamma, k, n)}, \quad 0 \le \lambda \le 1, 0 \le \gamma < 1, k \ge 0,$$

where

$$\Psi(\lambda, \gamma, k, n) = (1 + n\lambda - \lambda)|(1 + k) - n(\gamma + k)|\Theta\,(\mathrm{n}, \mu, \sigma) \tag{2.5}$$

and $\Theta(n, \mu, \sigma)$ is given by (1.9). *Equality holds for the function*

$$f(z) = z - \frac{1 - \gamma}{\Psi(\lambda, \gamma, k, n)} z^n.$$

For the sake of brevity we let,

$$\Psi(\lambda, \gamma, k, 2) = (1 + \lambda)|1 - k - 2\gamma|\Theta(2, \mu, \sigma) \tag{2.6}$$

and

$$\Theta(2, \mu, \sigma) = \frac{1 + \mu}{1 + \gamma} \tag{2.7}$$

unless otherwise stated.

3 Distrotion Bounds and Extreme Points

By a routine procedure one can find the distortion property and extreme points for function $\mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma,k)$ so we state the results without proof.

Theorem 3.1. *Let the function f defined by (1.7) belong to $\mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma,k)$. Then we have*

$$r-\frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)}r^{2}\leq|f(z)|\leq r+\frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)}r^{2},|z|=r \tag{3.1}$$

and

$$1-\frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)}r\leq|f'(z)|\leq 1+\frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)}r,\quad |z|=r. \tag{3.2}$$

Equalities are sharp for the function
$f(z)=z-\frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)}z^{2}$ *, where $\Psi(\lambda,\gamma,k,2)$ is given by (2.6).*

Next we discuss the extreme points for the class.

Theorem 3.2. *The extreme points of $\mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma,k)$ are*

$$f_1(z)=z \text{ and } f_n(z)=z-\frac{1-\gamma}{\Psi(\lambda,\gamma,k,n)}z^{n},\ \text{for}\ \ n=2,3,4,\ldots \tag{3.3}$$

where $\Psi(\lambda,\gamma,k,n)$ is defined in (2.5). Then $f\in\mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma,k)$ if and only if $f(z)=\sum_{n=1}^{\infty}\omega_n f_n(z)$, for $\omega_n\geq 0$ and $\sum_{n=1}^{\infty}\omega_n=1$.

4 Closure Theorem

Let the functions $f_j(z)(j=1,2)$ be defined by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^{n}\ \text{for}\ a_{n,j}\geq 0, z\in U. \tag{4.1}$$

Theorem 4.1. *Let the functions $f_j(z)(j=1,2,...m)$ defined by (4.1) be in the classes $\mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma_j,k)(j=1,2,...m)$ respectively. Then the function $h(z)$ defined by*

$$h(z)=z-\frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m}a_{n,j}\right)z^{n}$$

is in the class $\mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma,k)$, where $\gamma=\min_{1\leq j\leq m}\{\gamma_j\}$ with $-1\leq\gamma_j<1$

Proof. Since $f_j(z)\in\mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma_j,k)$ $(j=1,2,3,...m)$ $(j=1,2,3,...m)$ $(j=1,2,3,...m)$, by applying Theorem 2.3 to 4.1, we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\Psi(\lambda,\gamma,k,n)\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\right)\\
&=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}\Psi(\lambda,\gamma,k,n)a_{n,j}\right)\\
&\leq\frac{1}{m}\sum_{j=1}^{m}(1-\gamma_j)\leq 1-\gamma
\end{aligned}$$

where and $\Psi(\lambda,\gamma,k,n)$ is defined in (2.5) which, in view of Theorem 2.3, again implies that $h(z)\in\mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma,k)$. So, the proof is complete. □

5 Integral Transform of the class $\mathbb{L}_{\mu}^{\sigma}(\lambda, \gamma, k)$

In this section, we prove that the class $\mathbb{L}_{\mu}^{\sigma}(\lambda, \gamma, k)$ is closed under integral transform.

For $f \in \mathcal{A}$ we define the integral transform

$$\mathcal{V}_{\nu}(f)(z) = \int_0^1 \nu(t)\frac{f(tz)}{t}dt,$$

where ν is a real valued, non-negative weight function normalized so that $\int_0^1 \nu(t)dt = 1$. Since special cases of $\nu(t)$ are particularly interesting, such as $\nu(t) = (1+c)t^c, c > -1$, for which $\mathcal{V}_{\nu}$ is known as the Bernardi operator and

$$\nu(t) = \frac{(c+1)^{\delta}}{\nu(\delta)}t^c\left(\log\frac{1}{t}\right)^{\delta-1}, \; c > -1, \; \delta \geq 0,$$

which gives the Komatu operator. For more details, see [7].

First we show that the class $\mathbb{L}_{\mu}^{\sigma}(\lambda, \gamma, k)$ is closed under $\mathcal{V}_{\nu}(f)(z)$.

Theorem 5.1. *Let* $f \in \mathbb{L}_{\mu}^{\sigma}(\lambda, \gamma, k)$ *. Then* $\mathcal{V}_{\nu}(f)(z) \in \mathbb{L}_{\mu}^{\sigma}(\lambda, \gamma, k)$

Proof. By definition, we have

$$\begin{aligned}\mathcal{V}_{\nu}(f)(z) &= \frac{(c+1)^{\delta}}{\nu(\delta)}\int_0^1 (-1)^{\delta-1}t^c(\log t)^{\delta-1}\left(z - \sum_{n=2}^{\infty}|a_n|z^n t^{n-1}\right)dt \\ &= \frac{(-1)^{\delta-1}(c+1)^{\delta}}{\nu(\delta)}\lim_{r\to 0^+}\left[\int_r^1 t^c(\log t)^{\delta-1}\left(z - \sum_{n=2}^{\infty}|a_n|z^n t^{n-1}\right)\right].\end{aligned}$$

A simple calculation gives

$$\mathcal{V}_{\nu}(f)(z) = z - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta}|a_n|z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty}\frac{\Psi(\lambda, \gamma, k, n)}{1-\gamma}\left(\frac{c+1}{c+n}\right)^{\delta}|a_n| \leq 1. \tag{5.1}$$

On the other hand by Theorem 2.3, $f \in \mathbb{L}_{\mu}^{\sigma}(\lambda, \gamma, k)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\Psi(\lambda, \gamma, k, n)}{1-\gamma}|a_n| \leq 1,$$

where $\Psi(\lambda, \gamma, k, n)$ is defined in (2.5). Hence $\frac{c+1}{c+n} < 1$. Therefore (5.1) holds and the proof is complete.

□

The above theorem yields the following two special cases.

Theorem 5.2. *If $f(z)$ is starlike of order γ then $\mathcal{V}_{\nu}(f)(z)$ is also starlike of order γ.*

Theorem 5.3. *If $f(z)$ is convex of order γ then $\mathcal{V}_{\nu}(f)(z)$ is also convex of order γ.*

Theorem 5.4. *Let* $f \in \mathbb{L}_{\mu}^{\sigma}(\lambda, \gamma, k)$ *. Then* $\mathcal{V}_{\nu}(f)(z)$ *is starlike of order* $0 \leq \xi < 1$ *in* $|z| < R_1$ *where*

$$R_1 = \inf_n\left[\left(\frac{c+n}{c+1}\right)^{\delta}\frac{(1-\xi)\Psi(\lambda, \gamma, k, n)}{(n-\xi)(1-\gamma)}\right]^{\frac{1}{n-1}}, (n \geq 2)$$

where $\Psi(\lambda, \gamma, k, n)$ *is defined in (2.5).*

Proof. It is sufficient to prove

$$\left|\frac{z(\mathcal{V}_\nu(f)(z))'}{\mathcal{V}_\nu(f)(z)}\right| < 1-\xi. \tag{5.2}$$

For the left hand side of(5.2), we have

$$\left|\frac{z(\mathcal{V}_\nu(f)(z))'}{\mathcal{V}_\nu(f)(z)}-1\right| = \left|\frac{\sum_{n=2}^{\infty}(1-n)\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^{n-1}}\right|$$
$$\leq \frac{\sum_{n=2}^{\infty}(1-n)\left(\frac{c+1}{c+n}\right)^{\delta} |a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} |a_n||z|^{n-1}}.$$

The last expression is less than $(1-\xi)$, since

$$|z|^{n-1} < \left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\Psi(\lambda,\gamma,k,n)}{(n-\xi)(1-\gamma)}.$$

Therefore the proof is complete. □

Using the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike, we obtain the following.

Theorem 5.5. *Let* $f \in \mathbb{L}_\mu^\sigma(\lambda,\gamma,k)$. *Then* $\mathcal{V}_\nu(f)(z)$ *is convex of order* $0 \leq \xi < 1$ *in* $|z| < R_2$ *where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\Psi(\lambda,\gamma,k,n)}{n(n-\xi)(1-\gamma)}\right]^{\frac{1}{n-1}}, (n \geq 2)$$

where $\Psi(\lambda,\gamma,k,n)$ *is defined in (2.5).*

6 Neighbourhood Results

In this section, we discuss neighbourhood results of the class $\mathbb{L}_\mu^\sigma(\lambda,\gamma,k)$. Following [5, 13], we define the δ - neighbourhood of function $f \in T$ by

$$N_\delta(f) := \left\{h \in T\colon h(z) = z - \sum_{n=2}^{\infty}|d_n|z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta\right\}. \tag{6.1}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{h \in T\colon g(z) = z - \sum_{n=2}^{\infty}|d_n|z^n \text{and} \sum_{n=2}^{\infty} n|d_n| \leq \delta.\right\} \tag{6.2}$$

Theorem 6.1. *If*

$$\delta = \frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)},$$

then

$$L_\mu^\sigma(\lambda,\gamma,k) \subset N_\delta(e),$$

where

$$\Psi(\lambda,\gamma,k,2) = (1+\lambda)|1-k-2\gamma|\Theta(2,\mu,\sigma).$$

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

be a member of the class $L_\mu^\sigma(\lambda,\gamma,k)$.

By Theorem 2.3, we have

$$\sum_{n=2}^{\infty} \Psi(\lambda,\gamma,k,n)|a_n| \le 1-\gamma, \tag{6.3}$$

where

$$\Psi(\lambda,\gamma,k,n) = (1+n\lambda-\lambda)\,|(1+k)-n(\gamma+k)|\,\Theta(n,\mu,\sigma).$$

Assume that

$$\Psi(\lambda,\gamma,k,n) \ge \frac{n}{2}\Psi(\lambda,\gamma,k,2), \qquad n \ge 2. \tag{6.4}$$

Using (6.4) in (6.3), we obtain

$$\sum_{n=2}^{\infty} \frac{n}{2}\Psi(\lambda,\gamma,k,2)|a_n| \le \sum_{n=2}^{\infty} \Psi(\lambda,\gamma,k,n)|a_n| \le 1-\gamma.$$

Hence

$$\frac{\Psi(\lambda,\gamma,k,2)}{2}\sum_{n=2}^{\infty} n|a_n| \le 1-\gamma.$$

Therefore

$$\sum_{n=2}^{\infty} n|a_n| \le \frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)}. \tag{6.5}$$

Define

$$\delta = \frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)}.$$

Then (6.5) becomes

$$\sum_{n=2}^{\infty} n|a_n| \le \delta.$$

By the definition of the δ-neighbourhood of the identity function

$$N_\delta(e) = \left\{ h(z) = z - \sum_{n=2}^{\infty} |d_n| z^n : \sum_{n=2}^{\infty} n|d_n| \le \delta \right\},$$

it follows that

$$f \in N_\delta(e).$$

Since f was arbitrary,

$$L_\mu^\sigma(\lambda,\gamma,k) \subset N_\delta(e).$$

Hence the proof is complete. □

Now we determine the neighborhood for the class $\mathbb{L}_\mu^\sigma(\rho,\lambda,\gamma,k)$ which we define as follows.

Definition 6.2. A function $f \in T$ is said to be in the class $\mathbb{L}_\mu^\sigma(\rho,\lambda,\gamma,k)$ if there exists a function $h \in \mathbb{L}_\mu^\sigma(\rho,\lambda,\gamma,k)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1-\rho, \ (z \in U,\ 0 \le \rho < 1). \tag{6.6}$$

Theorem 6.3. *If* $h \in \mathbb{L}_\mu^\sigma(\rho,\lambda,\gamma,k)$ *and*

$$\rho = 1 - \frac{\delta\Psi(\lambda,\gamma,k,2)}{2[(\Psi(\lambda,\gamma,k,2)-(1-\gamma)]} \tag{6.7}$$

then

$$N_\delta(h) \subset \mathbb{L}_\mu^\sigma(\rho,\lambda,\gamma,k) \tag{6.8}$$

where $\Psi(\lambda,\gamma,k,2)$ *is defined in (2.6).*

Proof. Suppose that $f \in N_\delta(h)$. Then we find from (6.1) that

$$\sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - d_n| \leq \frac{\delta}{2}.$$

Next, since $h \in \mathbb{L}_\mu^\sigma(\lambda, \gamma, k)$, we have

$$\sum_{n=2}^{\infty} d_n = \frac{1-\gamma}{\Psi(\lambda, \gamma, k, 2)}$$

so that

$$\begin{aligned} \left|\frac{f(z)}{h(z)} - 1\right| &< \frac{\sum_{n=2}^{\infty} |a_n - d_n|}{1 - \sum_{n=2}^{\infty} d_n} \\ &\leq \frac{\delta}{2}\left[\frac{\Psi(\lambda, \gamma, k, 2)}{\Psi(\lambda, \gamma, k, 2) - (1-\gamma)}\right] \\ &= 1 - \rho \end{aligned}$$

provided that ρ is given precisely by (6.7).Thus by Definition 6.2, we have $f \in \mathbb{L}_\mu^\sigma(\rho, \lambda, \gamma, k)$ for ρ given by (6.7). This completes the proof. □

7 Integral Means Inequality

In [16], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means in equality, conjectured in [17] and settled in [18], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [18], he also proved his conjecture for the subclasses $T^*(\gamma)$, the class of starlike functions, and $\mathcal{C}(\gamma)$, the class of convex functions with negative coefficients.

We recall the following definition and lemma to prove our result on integral means inequality.

Definition 7.1. (Subordination Principle) [8].

For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$ if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in U$

Lemma 7.2. *[8] If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{7.1}$$

Applying Lemma 7.2, Theorem 2.3 and Theorem 3.2, we prove Silverman's conjecture for the functions in the family $\mathbb{L}_\mu^\sigma(\lambda, \gamma, k)$.

Theorem 7.3. *Suppose* $f \in \mathbb{L}_{\mu}^{\sigma}(\lambda,\gamma,k)$, $\eta > 0$, $0 \le \lambda \le 1$, $0 \le \gamma < 1$, $k \ge 0$ *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{1-\gamma}{\psi(\lambda,\gamma,k,2)} z^2,$$

where $\Psi(\lambda,\gamma,k,2)$ *is defined in (2.6). Then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^{\eta}\, d\theta \le \int_0^{2\pi} |f_2(z)|^{\eta}\, d\theta. \tag{7.2}$$

Proof. For $f \in T$, (7.2) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n\, z^{n-1}\right|^{\eta} d\theta \le \int_0^{2\pi} \left|1 - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} z\right|^{\eta} d\theta.$$

By Lemma 7.2, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n|\, z^{n-1} \prec 1 - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n|\, z^{n-1} = 1 - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} w(z), \tag{7.3}$$

and using (2.1), we obtain

$$\begin{aligned} |w(z)| &= \left|\sum_{n=2}^{\infty} \frac{\Psi(\lambda,\gamma,k,n)}{1-\gamma} a_n z^{n-1}\right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{\Psi(\lambda,\gamma,k,n)}{1-\gamma} |a_n| \\ &\le |z|. \end{aligned}$$

This completes the proof.

8 Conclusion remarks

In this study, we investigated a new subclass of analytic functions defined via the Tremblay fractional derivative operator. By employing tools from geometric function theory, we established several important properties of this class, including coefficient estimates, growth and distortion theorems, and radii results for starlikeness and convexity. The derived conditions offer a deeper insight into the geometric behavior of functions under the influence of the Tremblay operator. Additionally, convolution properties and closure theorems were examined, providing structural robustness to the function class. The results obtained generalize and extend various known outcomes for standard fractional derivative operators, thereby highlighting the versatility and effectiveness of the Tremblay-type operator in complex analysis. These findings open avenues for further research in the context of fractional calculus and its applications to univalent function theory. Future directions may include exploring analogous subclasses under Tremblay-type integral operators, studying subordination and superordination results, and applying the theory to problems in mathematical physics and engineering where such operators naturally arise.

References

[1] Awad, A. Ahmed, Sidky, F. Ismail, Mohamed, D. Shokry, On subclasses of analytic functions defined by using Tremblay fractional derivative operator, Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 3025-3035.

[2] S. S. Eker and B. Seker, On subclasses of bi-convex functions defined by Tremblay fractional derivative operator, Stud. Univ. Babes-Bolyai Math. 64(2019), No. 4, 467-476

[3] Z.Esa, A. Kilicman, R.W. Ibrahim, M.R. Ismail, S.K.S. Husain, Application of Modifed Complex Tremblay Operator, AIP Conference Proceedings 1739, 020059 (2016), http://doi.org/10.1063/1.4952539.

[4] Fawzan Ismail Sidky, Doaa Shorky Mohamad, Amina Ahmed Awad, Some inclusion properties of certain subclasses of analytic functions defined by using the tremblay fractional derivative operator, WSEAS Transctions on systems, 20(2020), 209-216.

[5] A.W.Goodman, Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc., 8, 1957, 598–601.

[6] R.W. Ibrahim, J.M. Jahangiri, Boundary fractional differential equation in a complex domain, Bound. Value Probl., 2014, Art. ID 66: 1-11.

[7] Y.C.Kim, F. Ronninig, Integral transform of certain subclass of analytic functions, J.Math. Anal. Appl., 258 (2001),466-489.

[8] Littlewood, J. E., On inequalities in theory of functions, Proc. London Math. Soc., 23 (1925), 481–519.

[9] T.H. MacGregor, The radius of convexity for starlike function of order α, Proc. Amer. Math. Sec. 14 (1963), 71-76.

[10] S. Owa, On the distortion theorems I, Kyungpook Math. J., 18(1978), 53-59.

[11] S.Owa, H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math., 39(1987), 1057-1077.

[12] B.Pinchuk, On the starlike and convex functions of order α, Duke Math. J. 35 (1968), 721-734.

[13] S.Ruscheweyh, Neighborhoods of univalent functions, Proc. Amer. Math. Soc., 81(4) (1981), 521–527.

[14] A.Schild, On starlike function of order α, Amer. J. Math. 87 (1965), 65-70.

[15] A.Schild, H. Silverman, Convolutions of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sk lodowska Sect. A-Math. 29 (1975), 99-107.

[16] H.Silverman, Univalent functions with negative coefficients. Proc. Amer. Math. Soc. 51 (1975), 109-116.

[17] H.Silverman, A survey with open problems on univalent functions whose coefficients are negative, Rocky Mt. J. Math., 21 (1991), 1099–1125.

[18] H.Silverman, Integral means for univalent functions with negative coefficients, Houston J. Math., 23 (1997), 169–174.

[19] H.M.Srivastava, S. Owa, Some characterization and distortion theorems involving fractional calculus, linear operators and certain subclasses of analytic functions, Nagoya Math. J., 106(1987), 1-28.

[20] H.M.Srivastava, S.S. Eker, S.G. Hamidi and J.M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Bull. Iran. Math. Soc. 44 (2018), no. 1, 149-157.

[21] H.M.Srivastava, S. Owa, Univalent Functions, Fractional Calculus, and Their Applications, Halsted Press, Ellis Horwood Limited, Chichester and John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[22] R.Tremblay, Une Contribution a la Theorie de la Derivative Fractionnaire, Ph. D. Thesis, Laval University, Québec, 1974.

Author information

S.G. Hiwale, Department of Basic Sciences and Humanities, Maharashtra Institute of Technology, Aurangabad - 431 010, Maharashtra., India.
E-mail: sandip.hiwale@mit.asia

D.D. Pawar, School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University,
Nanded - 431 606, Maharashtra., India.
E-mail: dnyaneshwarpawar@srtmun.ac.in

Received: 2025-07-21
Accepted: 2026-02-23