Palestine Journal of Mathematics

Vol 15(2)(2026) , 371–384

ON ϕ-n-ABSORBING PRIMARY SUBMODULES

Nilofer I. Shaikh and Rajendra P. Deore

Communicated by: Ayman Badawi

MSC 2020 Classifications: 13A15, 13C05, 13C13.

Keywords and phrases: ϕ-n-absorbing primary submodule, n-absorbing primary submodule, weakly n-absorbing primary submodule, almost n-absorbing primary submodule.

Corresponding Author: Nilofer I. Shaikh

Abstract Let R be a commutative ring with identity, M be a unitary R-module and n be a positive integer. In this paper, we introduce the concept of ϕ-n-absorbing primary submodule generalizing the concept of n-absorbing primary submodule. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function, where $\mathcal{S}(M)$ denotes the set of all submodules of M. A proper submodule N of M is called a ϕ-n-absorbing primary submodule if whenever $a_1 \cdots a_n m \in N - \phi(N)$ for $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or there are $n-1$ of the $a_i's$ whose product with m is in N. We prove several results concerning ϕ-n-absorbing primary submodule. We also investigate ϕ-n-absorbing primary submodule of some well-known modules.

1 Introduction

Throughout this paper, all rings are commutative with $1 \neq 0$ and all modules are unital. Let R be a ring and M be an R-module. We denote the set of ideals of R by $\mathcal{S}(R)$ and the set of submodules of M by $\mathcal{S}(M)$. For an ideal I of R, the radical of I is denoted by $\sqrt{I}$ and is given by $\sqrt{I} = \{r \in R : r^k \in I \text{ for some } k \in \mathbb{N}\}$. For a submodule N of M, $(N : M)$ denotes the residual of N over M and is given by $(N : M) = \{r \in R : rM \subseteq N\}$. If $\sqrt{(N : M)} = (N : M)$, then we call N to be a radical submodule.

As prime ideals play an important role in commutative ring theory, many authors have generalized prime ideals in different ways. The concept of weakly prime ideal was introduced by Anderson and Smith in [1]. Bhatwadekar and Sharma defined almost prime ideal in [2]. Anderson and Batanieh in [3] defined ϕ-prime ideal. For a function $\phi : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$, they called a proper ideal I of R a ϕ-prime ideal if $a, b \in R$ with $ab \in I - \phi(I)$ implies $a \in I$ or $b \in I$. A different generalization of prime ideal, called, 2-absorbing ideal, was introduced by Badawi in [4].

As a generalization of primary ideal, Darani introduced the concept of ϕ-primary ideal in [5]. He called a proper ideal I of R a ϕ-primary ideal if for $a, b \in R$, $ab \in I - \phi(I)$ implies $a \in I$ or $b \in \sqrt{I}$. Badawi et al. defined 2-absorbing primary ideal and weakly 2-absorbing primary ideal in [6] and [7] respectively. In [8], Badawi et al. defined a proper ideal I of R to be a ϕ-2-absorbing primary ideal if whenever $a, b, c \in R$ with $abc \in I - \phi(I)$ implies $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

In [9], Anderson and Badawi expanded the definition of 2-absorbing ideal to n-absorbing ideal for a positive integer n. Becker extended the definition of 2-absorbing primary ideal to n-absorbing primary ideal in [10]. Ebrahimpour and Nekooei generalized the concept of n-absorbing ideal to ϕ-n-absorbing ideal in [11]. They defined a proper ideal I of R to be a ϕ-n-absorbing ideal if $a_1 a_2 \cdots a_{n+1} \in I - \phi(I)$ for $a_1, a_2, \dots, a_{n+1} \in R$ implies that there are n of the $a_i's$ whose product is in I. Later, Mostafanasab and Darani in [12] defined a proper ideal I of R to be a ϕ-n-absorbing primary ideal if $a_1 a_2 \cdots a_{n+1} \in I - \phi(I)$ for elements $a_1, a_2, \dots, a_{n+1} \in R$ implies that either $a_1 a_2 \cdots a_n \in I$ or the product of a_{n+1} with $(n-1)$ of $a_1, a_2, \dots, a_n$ is in $\sqrt{I}$.

A natural generalization of prime ideal is the notion of prime submodule. In [13], Zamani generalized the concept of prime submodule to ϕ-prime submodule. For a function $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$, he called a proper submodule N of an R-module M to be a ϕ-prime submodule if $a \in R$ and $m \in M$ with $am \in N - \phi(N)$ implies that $a \in (N : M)$ or $m \in N$. Darani and Soheilnia introduced the concepts of 2-absorbing and weakly 2-absorbing submodules in [14] as generalizations of prime and weak prime submodules.

As a generalization of primary submodule, Dubey and Aggarwal introduced the concept of 2-absorbing primary submodule in [15]. In [16], Moradi and Ebrahimpour called a proper submodule N of an R-module M to be a ϕ-2-absorbing submodule if $abm \in N - \phi(N)$ implies $am \in N$ or $bm \in N$ or $ab \in (N : M)$, where $a, b \in R$ and $m \in M$. They also defined a proper submodule N of an R-module M to be a ϕ-2-absorbing primary submodule if $abm \in N - \phi(N)$ implies $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N : M)}$, where $a, b \in R$ and $m \in M$. These two concepts are also studied by Mostafanasab et al. in [17].

For a positive integer n, the concept of n-absorbing submodule was introduced by Darani and Soheilnia in [18]. They defined a proper submodule N of an R-module M to be an n-absorbing submodule if whenever $a_1 \cdots a_n m \in N$ for $a_1, \ldots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in (N : M)$ or there are $n - 1$ of the $a_i's$ whose product with m is in N. In [19], Shaikh and Deore introduced the concept of n-absorbing primary submodule. They defined a proper submodule N of an R-module M to be an n-absorbing primary submodule if whenever $a_1, \ldots, a_n \in R,\ m \in M$ and $a_1 \cdots a_n m \in N$, then either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or there are $n - 1$ of the $a_i's$ whose product with m is in N. In [20], Yiarayong and Siripitukdet called a proper submodule N of an R-module M to be a weakly n-absorbing primary submodule if for each $m \in M$ and $a_1, a_2, \ldots, a_n \in R$, $0 \neq a_1 a_2 \cdots a_n m \in N$, then $a_1 a_2 \cdots a_n \in \sqrt{(N : M)}$ or $a_2 a_3 \cdots a_n m \in N$ or $a_1 a_2 \cdots a_i a_{i+2} \cdots a_n m \in N$ for some $i \in \{1, 2, \ldots, n - 1\}$. The concept of almost n-absorbing primary submodule was introduced by Shaikh and Deore in [21]. They defined a proper submodule N of an R-module M to be an almost n-absorbing primary submodule if whenever $a_1 \cdots a_n m \in N - (N : M)N$ for $a_1, \ldots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or there are $n - 1$ of the $a_i's$ whose product with m is in N.

We call a proper submodule N of an R-module M to be a ϕ-n-absorbing submodule of M if whenever $a_1 \cdots a_n m \in N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in (N : M)$ or there are $n - 1$ of the $a_i's$ whose product with m is in N. Note that ϕ-n-absorbing submodule has been investigated by Ebrahimpour and Nekooei in [22] as $(n, n + 1)$-ϕ-prime submodule. In this paper, we define and study the concept of ϕ-n-absorbing primary submodule. A proper submodule N of an R-module M is called a ϕ-n-absorbing primary submodule of M if whenever $a_1 \cdots a_n m \in N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or there are $n - 1$ of the $a_i's$ whose product with m is in N.

In section 2, we discuss the relations among n-absorbing primary, weakly n-absorbing primary, almost n-absorbing primary, ϕ-n-absorbing and ϕ-n-absorbing primary submodules. For a ϕ-n-absorbing primary submodule N of an R-module M, we define the notion of ϕ-primary-$(n + 1)$-tuple-zero of N. Using this notion, we determine a few conditions under which a ϕ-n-absorbing primary submodule will be an n-absorbing primary submodule. In Theorem 2.10, we show that if N is a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary, then $(N : M)^n N \subseteq \phi(N)$. We also prove several results concerning characterizations of ϕ-n-absorbing primary submodule.

In section 3, we investigate ϕ-n-absorbing primary submodule of some well-known modules such as quotient modules, localization of modules and direct product of modules. In Corollary 3.5, we prove that K is a ϕ-n-absorbing primary submodule of an R-module M if and only if $K/\phi(K)$ is a weakly n-absorbing primary submodule of $M/\phi(K)$. In Theorem 3.12, we take an R_1-module M_1, an R_2-module M_2 and show that if N_1 is a weakly n-absorbing primary submodule of M_1, then $N_1 \times M_2$ is a ϕ-n-absorbing primary submodule of $M_1 \times M_2$ as an $(R_1 \times R_2)$-module for some ϕ with $\phi_\omega \leq \phi \leq \phi_1$.

2 On ϕ-n-absorbing primary submodule

In this section, we define ϕ-n-absorbing primary submodule and prove several results concerning its characterizations. We also define ϕ-primary-$(n + 1)$-tuple-zero of N for a ϕ-n-absorbing primary submodule N of an R-module M and prove some results using the same.

Definition 2.1. Let M be an R-module, $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function where $\mathcal{S}(M)$ is the set of submodules of M and n be a positive integer. A proper submodule N of M is called a

ϕ-n-absorbing primary submodule if whenever $a_1 \cdots a_n m \in N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or there are $n-1$ of the $a_i's$ whose product with m is in N.

Let $\widehat{a_i}$ denote the element of R obtained by deleting a_i from the product $a_1 \cdots a_n$. Then the above condition can be written as $a_1 \cdots a_n m \in N - \phi(N)$ implies either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or $\widehat{a_i} m \in N$ for some $1 \leq i \leq n$.

Remark 2.2. Let N be a ϕ-n-absorbing primary submodule of an R-module M where $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ is a function. Then

(1) If $\phi(N) = \emptyset$, then we say that $\phi = \phi_\emptyset$ and N is called a $\phi_\emptyset$-n-absorbing primary submodule of M, and hence N is an n-absorbing primary submodule of M.

(2) If $\phi(N) = 0$, then we say that $\phi = \phi_0$ and N is called a ϕ_0-n-absorbing primary submodule of M, and hence N is a weakly n-absorbing primary submodule of M.

(3) If $\phi(N) = N$, then we say that $\phi = \phi_1$ and N is called a ϕ_1-n-absorbing primary submodule of M.

(4) If $k \geq 2$ and $\phi(N) = (N : M)^{k-1}N$, then we say that $\phi = \phi_k$ and N is called a ϕ_k-n-absorbing primary submodule of M. In particular, if $k = 2$, then $\phi(N) = (N : M)N$ and we say that N is a ϕ_2-n-absorbing primary submodule of M, and hence N is an almost n-absorbing primary submodule of M.

(5) If $\phi(N) = \bigcap\limits_{i=1}^{\infty} (N : M)^i N$, then we say that $\phi = \phi_\omega$ and N is called a ϕ_ω-n-absorbing primary submodule of M.

Throughout this paper, ϕ denotes a function from $\mathcal{S}(M)$ to $\mathcal{S}(M) \cup \{\emptyset\}$. Since for any submodule N of M, $N - \phi(N) = N - (N \cap \phi(N))$, without loss of generality, we may assume that $\phi(N) \subseteq N$.

Remark 2.3. For any two functions $\psi_1, \psi_2 : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$, we say that $\psi_1 \leq \psi_2$ if $\psi_1(N) \subseteq \psi_2(N)$ for each $N \in \mathcal{S}(M)$. Hence it is clear that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

Lemma 2.4. *Let N be a proper submodule of an R-module M and* $\psi_1, \psi_2 : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ *be two functions with* $\psi_1 \leq \psi_2$. *If N is a* ψ_1-*n-absorbing primary submodule of M, then N is a* ψ_2-*n-absorbing primary submodule of M.*

Proof. Let $a_1, \ldots, a_n \in R$ and $m \in M$ such that $a_1 \cdots a_n m \in N - \psi_2(N)$. Then $a_1 \cdots a_n m \in N - \psi_1(N)$ as $\psi_1 \leq \psi_2$. Since N is a ψ_1-n-absorbing primary submodule of M, therefore either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or $\widehat{a_i} m \in N$ for some $1 \leq i \leq n$. Hence N is a ψ_2-n-absorbing primary submodule of M. □

Theorem 2.5. *Let N be a proper submodule of an R-module M. Then the following statements hold.*

(1) If N is a ϕ-*n-absorbing submodule of M, then N is a* ϕ-*n-absorbing primary submodule of M.*

(2) Let N be a radical submodule. Then N is a ϕ-*n-absorbing primary submodule of M if and only if N is a* ϕ-*n-absorbing submodule of M.*

(3) N is an n-absorbing primary submodule of M $\Rightarrow$ *N is a weakly n-absorbing primary submodule of M* $\Rightarrow$ *N is a* ϕ_ω-*n-absorbing primary submodule of M* $\Rightarrow$ *N is a* ϕ_{k+1}-*n-absorbing primary submodule of M for every* $k \geq 2 \Rightarrow$ *N is an almost n-absorbing primary submodule of M.*

(4) N is a ϕ_k-*n-absorbing primary submodule of M for every* $k \geq 2$ *if and only if N is a* ϕ_ω-*n-absorbing primary submodule of M.*

Proof. (1) It is clear from the definitions of ϕ-n-absorbing and ϕ-n-absorbing primary submodules.

(2) Since N is a radical submodule, $\sqrt{(N:M)} = (N:M)$. Hence the result.

(3) The result is clear from Remark 2.2, Remark 2.3 and Lemma 2.4.

(4) Assume that N is a ϕ_k-n-absorbing primary submodule of M for every $k \geq 2$. Let $a_1 \cdots a_n m \in N - \phi_\omega(N)$ for $a_1, \ldots, a_n \in R$ and $m \in M$, that is, $a_1 \cdots a_n m \in N - \bigcap\limits_{k=1}^{\infty} (N : M)^k N$. Then $a_1 \cdots a_n m \in N - (N:M)^{k-1} N$ for some $k \geq 2$, that is, $a_1 \cdots a_n m \in N - \phi_k(N)$ for some $k \geq 2$. Since N is a ϕ_k-n-absorbing primary submodule of M for every $k \geq 2$, therefore either $a_1 \cdots a_n \in \sqrt{(N:M)}$ or $\widehat{a_i} m \in N$ for some $1 \leq i \leq n$. Hence N is a ϕ_ω-n-absorbing primary submodule of M.

The converse is clear from part (3). □

Question: *If N is a ϕ-n-absorbing primary submodule of an R-module M, then under what conditions will N be an n-absorbing primary submodule of M?*

We obtain a few results that answer this question.

Theorem 2.6. *Let N be a ϕ-n-absorbing primary submodule of an R-module M such that $\phi(N)$ is an n-absorbing primary submodule of M. Then N is an n-absorbing primary submodule of M.*

Proof. Assume that N is a ϕ-n-absorbing primary submodule of M such that $\phi(N)$ is an n-absorbing primary submodule of M. Let $a_1 \cdots a_n m \in N$ for $a_1, \ldots, a_n \in R$ and $m \in M$. If $a_1 \cdots a_n m \in \phi(N)$, then either $a_1 \cdots a_n \in \sqrt{(\phi(N):M)}$ or $\widehat{a_i} m \in \phi(N)$ for some $1 \leq i \leq n$ as $\phi(N)$ is an n-absorbing primary submodule of M. Since $\phi(N) \subseteq N$, therefore either $a_1 \cdots a_n \in \sqrt{(N:M)}$ or $\widehat{a_i} m \in N$ for some $1 \leq i \leq n$ and we are done. If $a_1 \cdots a_n m \notin \phi(N)$, then $a_1 \cdots a_n m \in N - \phi(N)$ and N is a ϕ-n-absorbing primary submodule of M. Hence the result. □

Definition 2.7. Let M be an R-module, N be a proper submodule of M, $a_1, \ldots, a_n \in R$ and $m \in M$.

(1) If N is a weakly n-absorbing primary submodule of M with $a_1 \cdots a_n m = 0$, $a_1 \cdots a_n \notin \sqrt{(N:M)}$ and $\widehat{a_i} m \notin N$ for every $1 \leq i \leq n$, then $(a_1, \ldots, a_n, m)$ is called an $(n+1)$-tuple-zero of N.

(2) If N is a ϕ-n-absorbing submodule of M with $a_1 \cdots a_n m \in \phi(N)$, $a_1 \cdots a_n \notin (N:M)$ and $\widehat{a_i} m \notin N$ for every $1 \leq i \leq n$, then $(a_1, \ldots, a_n, m)$ is called a ϕ-$(n+1)$-tuple-zero of N.

(3) If N is a ϕ-n-absorbing primary submodule of M with $a_1 \cdots a_n m \in \phi(N)$, $a_1 \cdots a_n \notin \sqrt{(N:M)}$ and $\widehat{a_i} m \notin N$ for every $1 \leq i \leq n$, then $(a_1, \ldots, a_n, m)$ is called a ϕ-primary-$(n+1)$-tuple-zero of N.

Remark 2.8. If N is a ϕ-n-absorbing primary submodule of M that is not an n-absorbing primary submodule of M, then there exists a ϕ-primary-$(n+1)$-tuple-zero $(a_1, \ldots, a_n, m)$ of N for some $a_1, \ldots, a_n \in R$ and $m \in M$.

Proposition 2.9. *Let N be a ϕ-n-absorbing primary submodule of an R-module M. If $(a_1, \ldots, a_n, m)$ is a ϕ-primary-$(n+1)$-tuple-zero of N for $a_1, \ldots, a_n \in R$ and $m \in M$, then the following inclusions hold.*

(1) $a_1 \cdots a_n N \subseteq \phi(N)$.

(2) $a_{i_1} \cdots a_{i_{n-k}} (N:M)^k m \subseteq \phi(N)$ *for all* $1 \leq k \leq n-1$ *and* $i_1, i_2, \ldots, i_{n-k} \in \{1, 2, \ldots, n\}$.

(3) $(N:M)^n m \subseteq \phi(N)$.

(4) $a_{i_1} \cdots a_{i_{n-k}} (N:M)^k N \subseteq \phi(N)$ *for all* $1 \leq k \leq n-1$ *and* $i_1, i_2, \ldots, i_{n-k} \in \{1, 2, \ldots, n\}$.

Proof. Since $(a_1,\dots,a_n,m)$ is a ϕ-primary-$(n+1)$-tuple-zero of N, we have $a_1\cdots a_n m\in\phi(N)$, $a_1\cdots a_n\notin\sqrt{(N:M)}$ and $\widehat{a_i}m\notin N$ for every $1\le i\le n$.

(1) Assume that $a_1\cdots a_nN\nsubseteq\phi(N)$. Then there exists $n_0\in N$ such that $a_1\cdots a_nn_0\notin\phi(N)$. Therefore $a_1\cdots a_n(m+n_0)\notin\phi(N)$, which gives $a_1\cdots a_n(m+n_0)\in N-\phi(N)$. As N is a ϕ-n-absorbing primary submodule of M and $a_1\cdots a_n\notin\sqrt{(N:M)}$, we have $\widehat{a_i}(m+n_0)\in N$ for some $1\le i\le n$. Hence $\widehat{a_i}m\in N$ for some $1\le i\le n$, which is a contradiction as $(a_1,\dots,a_n,m)$ is a ϕ-primary-$(n+1)$-tuple-zero of N. Thus $a_1\cdots a_nN\subseteq\phi(N)$.

(2) We use induction on k. Let $k=1$. Assume that $a_{i_1}\cdots a_{i_{n-1}}(N:M)m\nsubseteq\phi(N)$. Then there exists $r\in(N:M)$ such that $a_{i_1}\cdots a_{i_{n-1}}rm\notin\phi(N)$. Therefore $a_{i_1}\cdots a_{i_{n-1}}(a_{i_n}+r)m\notin\phi(N)$. So $a_{i_1}\cdots a_{i_{n-1}}(a_{i_n}+r)m\in N-\phi(N)$. Since N is a ϕ-n-absorbing primary submodule of M, therefore either $a_{i_1}\cdots a_{i_{n-1}}(a_{i_n}+r)\in\sqrt{(N:M)}$ or $a_{i_1}\cdots a_{i_{n-1}}m\in N$ or $\widehat{a_{i_j}}(a_{i_n}+r)m\in N$ for some $1\le j\le n-1$ where $\widehat{a_{i_j}}$ denotes the element of R obtained by deleting a_{i_j} from the product $a_{i_1}\cdots a_{i_{n-1}}$. This implies that either $a_{i_1}\cdots a_{i_n}\in\sqrt{(N:M)}$ or $a_{i_1}\cdots a_{i_{n-1}}m\in N$ or $\widehat{a_{i_j}}a_{i_n}m\in N$ for some $1\le j\le n-1$, a contradiction. Hence the result holds for $k=1$. Suppose $k>1$ and assume that the result holds for all positive integers less than k. Now we prove the result for k. Assume that $a_{i_1}\cdots a_{i_{n-k}}(N:M)^km\nsubseteq\phi(N)$. Then there exist $r_1,r_2,\dots,r_k\in(N:M)$ such that $a_{i_1}\cdots a_{i_{n-k}}r_1r_2\cdots r_km\notin\phi(N)$. By induction hypothesis, we conclude that there exists $n_0\in\phi(N)$ such that $a_{i_1}\cdots a_{i_{n-k}}(a_{i_{n-(k-1)}}+r_1)(a_{i_{n-(k-2)}}+r_2)\cdots(a_{i_n}+r_k)m=n_0+a_{i_1}\cdots a_{i_{n-k}}r_1r_2\cdots r_km\notin\phi(N)$. So $a_{i_1}\cdots a_{i_{n-k}}(a_{i_{n-(k-1)}}+r_1)\cdots(a_{i_n}+r_k)m\in N-\phi(N)$. Since N is a ϕ-n-absorbing primary submodule of M, therefore either $a_{i_1}\cdots a_{i_{n-k}}(a_{i_{n-(k-1)}}+r_1)\cdots(a_{i_n}+r_k)\in\sqrt{(N:M)}$ or $\widehat{a_{i_j}}(a_{i_{n-(k-1)}}+r_1)(a_{i_{n-(k-2)}}+r_2)\cdots(a_{i_n}+r_k)m\in N$ for some $1\le j\le n-k$ where $\widehat{a_{i_j}}$ denotes the element of R obtained by deleting a_{i_j} from the product $a_{i_1}\cdots a_{i_{n-k}}$ or $a_{i_1}\cdots a_{i_{n-k}}\widehat{(a_{i_{n-(k-t)}}+r_t)}m\in N$ for some $1\le t\le k$ where $\widehat{(a_{i_{n-(k-t)}}+r_t)}$ denotes the element of R obtained by deleting $(a_{i_{n-(k-t)}}+r_t)$ from the product $(a_{i_{n-(k-1)}}+r_1)\cdots(a_{i_n}+r_k)$. This implies that either $a_{i_1}\cdots a_{i_n}\in\sqrt{(N:M)}$ or $\widehat{a_{i_j}}a_{i_{n-(k-1)}}\cdots a_{i_n}m\in N$ for some $1\le j\le n-k$ or $a_{i_1}\cdots a_{i_{n-k}}\widehat{a_{i_{n-(k-t)}}}m\in N$ for some $1\le t\le k$ where $\widehat{a_{i_{n-(k-t)}}}$ denotes the element of R obtained by deleting $a_{i_{n-(k-t)}}$ from the product $a_{i_{n-(k-1)}}\cdots a_{i_n}$, a contradiction. Hence $a_{i_1}\cdots a_{i_{n-k}}(N:M)^km\subseteq\phi(N)$.

(3) Assume that $(N:M)^nm\nsubseteq\phi(N)$. Then there exists $r_1,\dots,r_n\in(N:M)$ such that $r_1\cdots r_nm\notin\phi(N)$. By hypothesis and part (2), there exists $n_0\in\phi(N)$ such that $(a_1+r_1)\cdots(a_n+r_n)m=n_0+r_1\cdots r_nm\notin\phi(N)$. So $(a_1+r_1)\cdots(a_n+r_n)m\in N-\phi(N)$ and N is a ϕ-n-absorbing primary submodule of M. This implies that either $(a_1+r_1)\cdots(a_n+r_n)\in\sqrt{(N:M)}$ or $\widehat{(a_i+r_i)}m\in N$ for some $1\le i\le n$. Therefore either $a_1\cdots a_n\in\sqrt{(N:M)}$ or $\widehat{a_i}m\in N$ for some $1\le i\le n$, a contradiction. Hence $(N:M)^nm\subseteq\phi(N)$.

(4) We use induction on k. Let $k=1$. Assume that $a_{i_1}\cdots a_{i_{n-1}}(N:M)N\nsubseteq\phi(N)$. Then there exists $r\in(N:M)$ and $n_0\in N$ such that $a_{i_1}\cdots a_{i_{n-1}}rn_0\notin\phi(N)$. Therefore $a_{i_1}\cdots a_{i_{n-1}}(a_{i_n}+r)(m+n_0)\notin\phi(N)$. So $a_{i_1}\cdots a_{i_{n-1}}(a_{i_n}+r)(m+n_0)\in N-\phi(N)$. Since N is a ϕ-n-absorbing primary submodule of M, therefore either $a_{i_1}\cdots a_{i_{n-1}}(a_{i_n}+r)\in\sqrt{(N:M)}$ or $a_{i_1}\cdots a_{i_{n-1}}(m+n_0)\in N$ or $\widehat{a_{i_j}}(a_{i_n}+r)(m+n_0)\in N$ for some $1\le j\le n-1$ where $\widehat{a_{i_j}}$ denotes the element of R obtained by deleting a_{i_j} from the product $a_{i_1}\cdots a_{i_{n-1}}$. This implies that either $a_{i_1}\cdots a_{i_n}\in\sqrt{(N:M)}$ or $a_{i_1}\cdots a_{i_{n-1}}m\in N$ or $\widehat{a_{i_j}}a_{i_n}m\in N$ for some $1\le j\le n-1$, a contradiction. Hence the result holds for $k=1$. Suppose $k>1$ and assume that the result holds for all positive integers less than k. Now we prove the result for k. Assume that $a_{i_1}\cdots a_{i_{n-k}}(N:M)^kN\nsubseteq\phi(N)$. Then there exist $r_1,r_2,\dots,r_k\in(N:M)$ and $n_0\in N$ such that $a_{i_1}\cdots a_{i_{n-k}}r_1r_2\cdots r_kn_0\notin\phi(N)$. By part (1), part (2) and induction hypothesis, we conclude that there exists $n'\in\phi(N)$ such that $a_{i_1}\cdots a_{i_{n-k}}(a_{i_{n-(k-1)}}+r_1)(a_{i_{n-(k-2)}}+r_2)\cdots(a_{i_n}+r_k)(m+n_0)=n'+a_{i_1}\cdots a_{i_{n-k}}r_1r_2\cdots r_kn_0\notin\phi(N)$. So $a_{i_1}\cdots a_{i_{n-k}}(a_{i_{n-(k-1)}}+r_1)\cdots(a_{i_n}+r_k)(m+n_0)\in N-\phi(N)$. Since N is a ϕ-n-absorbing primary submodule of M, therefore either $a_{i_1}\cdots a_{i_{n-k}}(a_{i_{n-(k-1)}}+r_1)\cdots(a_{i_n}+r_k)\in\sqrt{(N:M)}$ or $\widehat{a_{i_j}}(a_{i_{n-(k-1)}}+r_1)\cdots(a_{i_n}+r_k)(m+n_0)\in N$ for some $1\le j\le n-k$ where $\widehat{a_{i_j}}$ denotes the element of R obtained by deleting a_{i_j} from the product $a_{i_1}\cdots a_{i_{n-k}}$ or $a_{i_1}\cdots a_{i_{n-k}}\widehat{(a_{i_{n-(k-t)}}+r_t)}(m+n_0)\in N$ for some $1\le t\le k$ where $\widehat{(a_{i_{n-(k-t)}}+r_t)}$ denotes the element of R obtained by deleting $(a_{i_{n-(k-t)}}+r_t)$ from the product $(a_{i_{n-(k-1)}}+r_1)\cdots(a_{i_n}+r_k)$. This implies that either $a_{i_1}\cdots a_{i_n}\in\sqrt{(N:M)}$

or $\widehat{a_{i_j}}a_{i_{n-(k-1)}}\cdots a_{i_n}m \in N$ for some $1 \le j \le n-k$ or $a_{i_1}\cdots a_{i_{n-k}}\widehat{a_{i_{n-(k-t)}}}m \in N$ for some $1 \le t \le k$ where $\widehat{a_{i_{n-(k-t)}}}$ denotes the element of R obtained by deleting $a_{i_{n-(k-t)}}$ from the product $a_{i_{n-(k-1)}}\cdots a_{i_n}$, a contradiction. Hence $a_{i_1}\cdots a_{i_{n-k}}(N:M)^kN \subseteq \phi(N)$. □

Theorem 2.10. *Let N be a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary. Then $(N:M)^nN \subseteq \phi(N)$.*

Proof. Since N is a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary, therefore there exists a ϕ-primary-$(n+1)$-tuple-zero $(a_1,\dots,a_n,m)$ of N for some $a_1,\dots,a_n \in R$ and $m \in M$. Assume that $(N:M)^nN \nsubseteq \phi(N)$. Then there exists $r_1,\dots,r_n \in (N:M)$ and $n_0 \in N$ such that $r_1\cdots r_nn_0 \notin \phi(N)$. By Proposition 2.9, $(a_1+r_1)\cdots(a_n+r_n)(m+n_0) \notin \phi(N)$. So $(a_1+r_1)\cdots(a_n+r_n)(m+n_0) \in N-\phi(N)$ and N is a ϕ-n-absorbing primary submodule of M. Therefore either $(a_1+r_1)\cdots(a_n+r_n) \in \sqrt{(N:M)}$ or $(\widehat{a_i+r_i})(m+n_0) \in N$ for some $1 \le i \le n$. This implies that either $a_1\cdots a_n \in \sqrt{(N:M)}$ or $\widehat{a_i}m \in N$ for some $1 \le i \le n$, a contradiction. Hence $(N:M)^nN \subseteq \phi(N)$. □

Corollary 2.11. *If N is a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary, then $\sqrt{(N:M)} = \sqrt{(\phi(N):M)}$.*

Proof. Since N is a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary, by Theorem 2.10, we have $(N:M)^nN \subseteq \phi(N)$. Then $(N:M)^{n+1} = (N:M)^n(N:M) = ((N:M)^nN:M) \subseteq (\phi(N):M)$. Therefore $\sqrt{(N:M)} \subseteq \sqrt{(\phi(N):M)}$. As $\phi(N) \subseteq N$, $\sqrt{(\phi(N):M)} \subseteq \sqrt{(N:M)}$. Thus $\sqrt{(N:M)} = \sqrt{(\phi(N):M)}$. □

Theorem 2.12. *Let N be a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary, where $\phi \le \phi_{n+2}$. Then $(N:M)^nN = (N:M)^{n+1}N$.*

Proof. Clearly, $(N:M)^{n+1}N \subseteq (N:M)^nN$. Since N is a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary, by Theorem 2.10, we have $(N:M)^nN \subseteq \phi(N)$. As $\phi \le \phi_{n+2}$, therefore $\phi(N) \subseteq (N:M)^{n+1}N$. Thus $(N:M)^nN = (N:M)^{n+1}N$. □

Corollary 2.13. *If N is a ϕ-n-absorbing primary submodule of an R-module M with $\phi \le \phi_{n+2}$, then N is a ϕ_ω-n-absorbing primary submodule of M.*

Proof. If N is an n-absorbing primary submodule of M, then N is a ϕ_ω-n-absorbing primary submodule of M by Theorem 2.5(3). So assume that N is not an n-absorbing primary submodule of M. Then by Theorem 2.12, $(N:M)^nN = (N:M)^{n+1}N$. Therefore $\phi_\omega(N) = \bigcap_{i=1}^{\infty}(N:M)^iN = (N:M)^nN$. By hypothesis, $\phi \le \phi_{n+2}$ and by Remark 2.3, $\phi_{n+2} \le \phi_{n+1}$. So $\phi \le \phi_{n+1}$ and N is a ϕ-n-absorbing primary submodule of M. Therefore by Lemma 2.4, N is a ϕ_{n+1}-n-absorbing primary submodule of M and $\phi_{n+1}(N) = (N:M)^nN = \phi_\omega(N)$. Hence N is a ϕ_ω-n-absorbing primary submodule of M. □

Let M be a multiplication R-module. Then $N = (N:M)M$ for every submodule N of M. Let N be a ϕ-n-absorbing primary submodule of M. Then for any $k \ge 2$, $\phi_k(N) = (N:M)^{k-1}N = (N:M)^{k-1}(N:M)M = (N:M)^kM = N^k$. If we consider M to be a multiplication R-module, then Theorem 2.10 and Theorem 2.12 give the following results.

Corollary 2.14. *If N is a ϕ-n-absorbing primary submodule of a multiplication R-module M that is not n-absorbing primary, then $N^{n+1} \subseteq \phi(N)$.*

Proof. Since N is a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary, by Theorem 2.10, we have $(N:M)^nN \subseteq \phi(N)$, that is, $\phi_{n+1}(N) \subseteq \phi(N)$. Since M is a multiplication R-module, therefore $\phi_{n+1}(N) = N^{n+1}$. Hence $N^{n+1} \subseteq \phi(N)$. □

Corollary 2.15. *Let N be a ϕ-n-absorbing primary submodule of a multiplication R-module M that is not n-absorbing primary, where $\phi \le \phi_{n+2}$. Then $N^{n+1} = N^{n+2}$.*

Proof. Since N is a ϕ-n-absorbing primary submodule of an R-module M that is not n-absorbing primary, by Theorem 2.12, $(N:M)^nN = (N:M)^{n+1}N$, that is, $\phi_{n+1}(N) = \phi_{n+2}(N)$. Since M is a multiplication R-module, this implies that $N^{n+1} = N^{n+2}$. □

We prove several results concerning characterizations of ϕ-n-absorbing primary submodule.

Theorem 2.16. *Every ϕ-n-absorbing primary submodule of an R-module is a ϕ-m-absorbing primary submodule for $m > n$.*

Proof. It is sufficient to prove that every ϕ-n-absorbing primary submodule of an R-module is a ϕ-$(n+1)$-absorbing primary submodule. Assume that N is a ϕ-n-absorbing primary submodule of an R-module M. Let $a_1 \cdots a_n a_{n+1} m \in N - \phi(N)$ for $a_1, \ldots, a_n, a_{n+1} \in R$ and $m \in M$. Denote $a_n a_{n+1}$ by $a_{n'}$. Then $a_1 \cdots a_{n'} m \in N - \phi(N)$ and N is a ϕ-n-absorbing primary submodule of M. Therefore either $a_1 \cdots a_{n'} \in \sqrt{(N : M)}$ or $\widehat{a_i} m \in N$ for some $i \in \{1, 2, \ldots, n-1, n'\}$. If $i \neq n'$, then we are done. If $i = n'$, then we have $a_1 \cdots a_{n-1} m \in N$, which implies that $a_1 \cdots a_{n-1} a_n m \in N$ or $a_1 \cdots a_{n-1} a_{n+1} m \in N$. Hence N is a ϕ-$(n+1)$-absorbing primary submodule of M. □

Theorem 2.17. *[19, Theorem 2.5] Let N be an n-absorbing primary submodule of a cyclic multiplication R-module M. Then $\sqrt{(N : M)}$ is an n-absorbing ideal of R.*

Theorem 2.18. *If N is a ϕ-n-absorbing primary submodule of a cyclic multiplication R-module M such that $\sqrt{(\phi(N) : M)}$ is an n-absorbing ideal of R, then $\sqrt{(N : M)}$ is also an n-absorbing ideal of R.*

Proof. If N is an n-absorbing primary submodule of M, then by Theorem 2.17, $\sqrt{(N : M)}$ is an n-absorbing ideal of R. So assume that N is not an n-absorbing primary submodule of M. Then by Corollary 2.11, $\sqrt{(N : M)} = \sqrt{(\phi(N) : M)}$. By our hypothesis, $\sqrt{(\phi(N) : M)}$ is an n-absorbing ideal of R. Therefore $\sqrt{(N : M)}$ is an n-absorbing ideal of R. □

Theorem 2.19. *Let $f : M \to M'$ be an epimorphism of R-modules M and M', $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ and $\phi' : \mathcal{S}(M') \to \mathcal{S}(M') \cup \{\emptyset\}$ be functions. Then the following statements hold.*

(1) If N' is a ϕ'-n-absorbing primary submodule of M' and $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$, then $f^{-1}(N')$ is a ϕ-n-absorbing primary submodule of M.

(2) If N is a ϕ-n-absorbing primary submodule of M containing $Ker(f)$ and $\phi'(f(N)) = f(\phi(N))$, then $f(N)$ is a ϕ'-n-absorbing primary submodule of M'.

Proof. (1) Since f is an epimorphism, therefore $f^{-1}(N')$ is a proper submodule of M. Let $a_1 \cdots a_n m \in f^{-1}(N') - \phi(f^{-1}(N'))$ for $a_1, \ldots, a_n \in R$ and $m \in M$. As $a_1 \cdots a_n m \in f^{-1}(N')$, we have $a_1 \cdots a_n f(m) \in N'$. Since $a_1 \cdots a_n m \notin \phi(f^{-1}(N')) = f^{-1}(\phi'(N')$, therefore $a_1 \cdots a_n f(m) \notin \phi'(N')$. So $a_1 \cdots a_n f(m) \in N' - \phi'(N')$ and N' is a ϕ'-n-absorbing primary submodule of M'. Therefore either $a_1 \cdots a_n \in \sqrt{(N' : M')}$ or $\widehat{a_i} f(m) \in N'$ for some $1 \leq i \leq n$. This implies that either $a_1 \cdots a_n \in \sqrt{(f^{-1}(N') : M)}$ or $\widehat{a_i} m \in f^{-1}(N')$ for some $1 \leq i \leq n$. Hence $f^{-1}(N')$ is a ϕ-n-absorbing primary submodule of M.

(2) Let $a_1 \cdots a_n m' \in f(N) - \phi'(f(N))$ for $a_1, \ldots, a_n \in R$ and $m' \in M'$. Then $a_1 \cdots a_n m' = f(n_0)$ for some $n_0 \in N$. As $m' \in M'$ and f is an epimorphism, there exists $m \in M$ such that $f(m) = m'$. Therefore $a_1 \cdots a_n f(m) = f(n_0)$, that is, $f(a_1 \cdots a_n m - n_0) = 0$. This implies that $a_1 \cdots a_n m - n_0 \in Ker(f) \subseteq N$ and hence $a_1 \cdots a_n m \in N$. Since $a_1 \cdots a_n m' \notin \phi'(f(N))$, we have $f(a_1 \cdots a_n m) \notin f(\phi(N))$ and so $a_1 \cdots a_n m \notin \phi(N)$. Therefore $a_1 \cdots a_n m \in N - \phi(N)$ and N is a ϕ-n-absorbing primary submodule of M. So either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or $\widehat{a_i} m \in N$ for some $1 \leq i \leq n$. This implies that either $a_1 \cdots a_n \in \sqrt{(f(N) : M')}$ or $\widehat{a_i} m' \in f(N)$ for some $1 \leq i \leq n$. Hence $f(N)$ is a ϕ'-n-absorbing primary submodule of M'. □

Theorem 2.20. *Let N be a proper submodule of an R-module M. Then the following conditions are equivalent.*

(1) N is a ϕ-n-absorbing primary submodule of M.

(2) For $a_1, \ldots, a_{n-1} \in R$ and $m \in M$ with $a_1 \cdots a_{n-1} m \notin N$, $(N : a_1 \cdots a_{n-1} m) \subseteq \bigcup_{i=1}^{n-1} (N : \widehat{a_i} m) \cup (\sqrt{(N : M)} : a_1 \cdots a_{n-1}) \cup (\phi(N) : a_1 \cdots a_{n-1} m)$, where $\widehat{a_i}$ denotes the element of R obtained by deleting a_i from the product $a_1 \cdots a_{n-1}$.

Proof. (1) $\Rightarrow$ (2) Let $r \in (N : a_1 \cdots a_{n-1}m)$, that is, $ra_1 \cdots a_{n-1}m \in N$. If $ra_1 \cdots a_{n-1}m \in \phi(N)$, then $r \in (\phi(N) : a_1 \cdots a_{n-1}m)$. If $ra_1 \cdots a_{n-1}m \notin \phi(N)$, then $ra_1 \cdots a_{n-1}m \in N - \phi(N)$. Since N is a ϕ-n-absorbing primary submodule of M and $a_1 \cdots a_{n-1}m \notin N$, therefore either $ra_1 \cdots a_{n-1} \in \sqrt{(N : M)}$ or $r\widehat{a_i}m \in N$ for some $1 \leq i \leq n-1$ where $\widehat{a_i}$ denotes the element of R obtained by deleting a_i from the product $a_1 \cdots a_{n-1}$. This implies that $r \in (\sqrt{(N : M)} : a_1 \cdots a_{n-1})$ or $r \in (N : \widehat{a_i}m)$ for some $1 \leq i \leq n-1$. Thus $(N : a_1 \cdots a_{n-1}m) \subseteq \bigcup_{i=1}^{n-1} (N : \widehat{a_i}m) \cup (\sqrt{(N : M)} : a_1 \cdots a_{n-1}) \cup (\phi(N) : a_1 \cdots a_{n-1}m)$.

(2) $\Rightarrow$ (1) Let $a_1 \cdots a_n m \in N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and $m \in M$. Suppose that $a_1 \cdots a_{n-1}m \notin N$. Then by assumption, $(N : a_1 \cdots a_{n-1}m) \subseteq \bigcup_{i=1}^{n-1} (N : \widehat{a_i}m) \cup (\sqrt{(N : M)} : a_1 \cdots a_{n-1}) \cup (\phi(N) : a_1 \cdots a_{n-1}m)$. As $a_1 \cdots a_n m \in N - \phi(N)$, we have $a_n \in (N : a_1 \cdots a_{n-1}m)$ but $a_n \notin (\phi(N) : a_1 \cdots a_{n-1}m)$. Therefore either $a_n \in (N : \widehat{a_i}m)$ for some $1 \leq i \leq n-1$ or $a_n \in (\sqrt{(N : M)} : a_1 \cdots a_{n-1})$, that is, either $\widehat{a_i}a_n m \in N$ for some $1 \leq i \leq n-1$ or $a_1 \cdots a_n \in \sqrt{(N : M)}$. Hence N is a ϕ-n-absorbing primary submodule of M. □

Theorem 2.21. *N is a ϕ-n-absorbing primary submodule of an R-module M if and only if for $a_1, \ldots, a_n \in R$ such that $a_1 \cdots a_n \notin \sqrt{(N : M)}$, $(N : a_1 \cdots a_n) = \bigcup_{i=1}^{n} (N : \widehat{a_i}) \cup (\phi(N) : a_1 \cdots a_n)$.*

Proof. Assume that N is a ϕ-n-absorbing primary submodule of M and $a_1 \cdots a_n \notin \sqrt{(N : M)}$ for $a_1, \ldots, a_n \in R$. Let $m \in (N : a_1 \cdots a_n)$, that is, $a_1 \cdots a_n m \in N$. If $a_1 \cdots a_n m \in \phi(N)$, then $m \in (\phi(N) : a_1 \cdots a_n)$. If $a_1 \cdots a_n m \notin \phi(N)$, then $a_1 \cdots a_n m \in N - \phi(N)$. Since N is a ϕ-n-absorbing primary submodule of M and $a_1 \cdots a_n \notin \sqrt{(N : M)}$, therefore $\widehat{a_i}m \in N$ for some $1 \leq i \leq n$, that is, $m \in (N : \widehat{a_i})$ for some $1 \leq i \leq n$. Hence $(N : a_1 \cdots a_n) \subseteq \bigcup_{i=1}^{n} (N : \widehat{a_i}) \cup (\phi(N) : a_1 \cdots a_n)$. We know that $(N : \widehat{a_i}) \subseteq (N : a_1 \cdots a_n)$ for every $1 \leq i \leq n$ and $(\phi(N) : a_1 \cdots a_n) \subseteq (N : a_1 \cdots a_n)$. Therefore $(N : a_1 \cdots a_n) = \bigcup_{i=1}^{n} (N : \widehat{a_i}) \cup (\phi(N) : a_1 \cdots a_n)$.

Conversely, assume that for $a_1, \ldots, a_n \in R$ if $a_1 \cdots a_n \notin \sqrt{(N : M)}$, then $(N : a_1 \cdots a_n) = \bigcup_{i=1}^{n} (N : \widehat{a_i}) \cup (\phi(N) : a_1 \cdots a_n)$. Let $a_1 \cdots a_n m \in N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and $m \in M$. Then $m \in (N : a_1 \cdots a_n)$ and $m \notin (\phi(N) : a_1 \cdots a_n)$. If $a_1 \cdots a_n \in \sqrt{(N : M)}$, then we are done. If $a_1 \cdots a_n \notin \sqrt{(N : M)}$, then by assumption, $(N : a_1 \cdots a_n) = \bigcup_{i=1}^{n} (N : \widehat{a_i}) \cup (\phi(N) : a_1 \cdots a_n)$. Therefore $m \in (N : \widehat{a_i})$ for some $1 \leq i \leq n$, that is, $\widehat{a_i}m \in N$ for some $1 \leq i \leq n$. Hence N is a ϕ-n-absorbing primary submodule of M. □

As defined in [23], a commutative ring R is called a um-ring provided R has the property that an R-module which is equal to a finite union of submodules must be equal to one of them.

Theorem 2.22. *Let R be a um-ring, M be an R-module and N be a submodule of M. Then the following conditions are equivalent.*

(1) N is a ϕ-n-absorbing primary submodule of M.

(2) If $a_1 \cdots a_n \notin \sqrt{(N : M)}$ for $a_1, \ldots, a_n \in R$, then either $(N : a_1 \cdots a_n) = (N : \widehat{a_i})$ for some $1 \leq i \leq n$ or $(N : a_1 \cdots a_n) = (\phi(N) : a_1 \cdots a_n)$.

(3) If $a_1 \cdots a_n K \subseteq N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and a submodule K of M, then either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or $\widehat{a_i}K \subseteq N$ for some $1 \leq i \leq n$.

Proof. (1) $\Rightarrow$ (2) Let N be a ϕ-n-absorbing primary submodule of M such that $a_1 \cdots a_n \notin \sqrt{(N : M)}$ for $a_1, \ldots, a_n \in R$. Then by Theorem 2.21, we have $(N : a_1 \cdots a_n) = \bigcup_{i=1}^{n} (N :$

$\widehat{a_i}) \cup (\phi(N) : a_1 \cdots a_n)$. Since R is a um-ring, therefore either $(N : a_1 \cdots a_n) = (N : \widehat{a_i})$ for some $1 \leq i \leq n$ or $(N : a_1 \cdots a_n) = (\phi(N) : a_1 \cdots a_n)$.

(2) ⇒ (3) Let $a_1 \cdots a_n K \subseteq N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and a submodule K of M. Then $K \subseteq (N : a_1 \cdots a_n)$ and $K \nsubseteq (\phi(N) : a_1 \cdots a_n)$. If $a_1 \cdots a_n \in \sqrt{(N : M)}$, then we are done. If $a_1 \cdots a_n \notin \sqrt{(N : M)}$, then by assumption, either $(N : a_1 \cdots a_n) = (N : \widehat{a_i})$ for some $1 \leq i \leq n$ or $(N : a_1 \cdots a_n) = (\phi(N) : a_1 \cdots a_n)$. Therefore $K \subseteq (N : \widehat{a_i})$ for some $1 \leq i \leq n$, that is, $\widehat{a_i}K \subseteq N$ for some $1 \leq i \leq n$.

(3) ⇒ (1) Let $a_1 \cdots a_n m \in N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and $m \in M$. Then $a_1 \cdots a_n \langle m \rangle \subseteq N - \phi(N)$ where $\langle m \rangle$ denotes the submodule of M generated by m. By assumption, either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or $\widehat{a_i}\langle m \rangle \subseteq N$ for some $1 \leq i \leq n$, that is, either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or $\widehat{a_i}m \in N$ for some $1 \leq i \leq n$. Hence N is a ϕ-n-absorbing primary submodule of M. □

Theorem 2.23. *Let M be an R-module, $k \geq 2$ be an integer, $m \in M$ such that $Rm \neq M$ and $(0 : m) \subseteq \sqrt{(Rm : M)}$. Then Rm is a ϕ-n-absorbing primary submodule of M for some ϕ with $\phi \leq \phi_k$ if and only if Rm is an n-absorbing primary submodule of M.*

Proof. Assume that Rm is a ϕ-n-absorbing primary submodule of M for some $\phi \leq \phi_k$. Let $a_1, \ldots, a_n \in R$ and $m' \in M$ such that $a_1 \cdots a_n m' \in Rm$. If $a_1 \cdots a_n m' \notin (Rm : M)^{k-1}Rm$, then $a_1 \cdots a_n m' \in Rm - \phi(Rm)$ as $\phi \leq \phi_k$. Since Rm is a ϕ-n-absorbing primary submodule of M, therefore either $a_1 \cdots a_n \in \sqrt{(Rm : M)}$ or $\widehat{a_i}m' \in Rm$ for some $1 \leq i \leq n$ and we are done. So assume that $a_1 \cdots a_n m' \in (Rm : M)^{k-1}Rm$. We have $a_1 \cdots a_n(m' + m) \in Rm$. If $a_1 \cdots a_n(m' + m) \notin (Rm : M)^{k-1}Rm$, then $a_1 \cdots a_n(m' + m) \in Rm - \phi(Rm)$ and Rm is a ϕ-n-absorbing primary submodule of M. Therefore either $a_1 \cdots a_n \in \sqrt{(Rm : M)}$ or $\widehat{a_i}(m' + m) \in Rm$ for some $1 \leq i \leq n$, that is, either $a_1 \cdots a_n \in \sqrt{(Rm : M)}$ or $\widehat{a_i}m' \in Rm$ for some $1 \leq i \leq n$ and we are done. So assume that $a_1 \cdots a_n(m' + m) \in (Rm : M)^{k-1}Rm$. Therefore $a_1 \cdots a_n m \in (Rm : M)^{k-1}Rm$. This implies that there exists $r \in (Rm : M)$ such that $a_1 \cdots a_n m = rm$. Therefore $a_1 \cdots a_n - r \in (0 : m) \subseteq \sqrt{(Rm : M)}$. Thus $a_1 \cdots a_n \in \sqrt{(Rm : M)}$, which shows that Rm is an n-absorbing primary submodule of M.

Conversely, if Rm is an n-absorbing primary submodule of M, then Rm is a ϕ-n-absorbing primary submodule of M for any ϕ. □

Let R be an integral domain. An R-module M is said to be torsion-free if $rm = 0$ implies $r = 0$ or $m = 0$, where $r \in R$ and $m \in M$.

Corollary 2.24. *Let R be an integral domain, M be a torsion-free R-module, $m \in M$ with $Rm \neq M$ and $k \geq 2$ be an integer. Then Rm is a ϕ-n-absorbing primary submodule of M for some ϕ with $\phi \leq \phi_k$ if and only if Rm is an n-absorbing primary submodule of M.*

Theorem 2.25. *Let M be a multiplication R-module, r be an element of R such that $rM \neq M$ and $(0 : r) \subseteq rM$. Then rM is an almost n-absorbing primary submodule of M if and only if rM is an n-absorbing primary submodule of M.*

Proof. Assume that rM is an almost n-absorbing primary submodule of M. Let $a_1 \cdots a_n m \in rM$ for $a_1, \ldots, a_n \in R$ and $m \in M$. If $a_1 \cdots a_n m \notin r^2M$, then either $a_1 \cdots a_n \in \sqrt{(rM : M)}$ or $\widehat{a_i}m \in rM$ for some $1 \leq i \leq n$ as rM is an almost n-absorbing primary submodule of M and we are done. So assume that $a_1 \cdots a_n m \in r^2M$. Note that $(a_1 + r)a_2 \cdots a_n m \in rM$. If $(a_1 + r)a_2 \cdots a_n m \notin r^2M$, then either $(a_1 + r)a_2 \cdots a_n \in \sqrt{(rM : M)}$ or $\widehat{a_1}m \in rM$ or $(a_1 + r)a_2 \cdots \widehat{\widehat{a_i}} \cdots a_n m \in rM$ for some $2 \leq i \leq n$ where $a_2 \cdots \widehat{\widehat{a_i}} \cdots a_n$ denotes the element of R obtained by deleting a_i from the product $a_2 \cdots a_n$. Therefore either $a_1 \cdots a_n \in \sqrt{(rM : M)}$ or $\widehat{a_i}m \in rM$ for some $1 \leq i \leq n$ and we are done. So assume that $(a_1 + r)a_2 \cdots a_n m \in r^2M$. Then $ra_2 \cdots a_n m \in r^2M$. This implies that there exists $m_0 \in M$ such that $ra_2 \cdots a_n m = r^2m_0$. Therefore $a_2 \cdots a_n m - rm_0 \in (0 : r) \subseteq rM$. Hence $a_2 \cdots a_n m \in rM$, that is, $\widehat{a_1}m \in rM$, which shows that rM is an n-absorbing primary submodule of M.

The converse follows by Theorem 2.5(3). □

3 ϕ-n-absorbing primary submodule of some well-known modules

Let N be a submodule of an R-module M and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. Define the function $\phi_N : \mathcal{S}(M/N) \to \mathcal{S}(M/N) \cup \{\emptyset\}$ by $\phi_N(K/N) = (\phi(K) + N)/N$ if $\phi(K) \neq \emptyset$ and $\phi_N(K/N) = \emptyset$ if $\phi(K) = \emptyset$, where K is a submodule of M such that $N \subseteq K$.

Theorem 3.1. *Let N and K be submodules of an R-module M with $N \subseteq K$. If K is a ϕ-n-absorbing primary submodule of M, then K/N is a ϕ_N-n-absorbing primary submodule of M/N.*

Proof. Let $a_1 \cdots a_n(m + N) \in K/N - \phi_N(K/N)$ for $a_1, \dots, a_n \in R$ and $m \in M$, that is, $a_1 \cdots a_n(m + N) \in K/N - (\phi(K) + N)/N$. Then $a_1 \cdots a_n m \in K - \phi(K)$ and K is a ϕ-n-absorbing primary submodule of M. Therefore either $a_1 \cdots a_n \in \sqrt{(K : M)}$ or $\widehat{a_i} m \in K$ for some $1 \leq i \leq n$. Hence either $a_1 \cdots a_n \in \sqrt{(K/N : M/N)}$ or $\widehat{a_i}(m + N) \in K/N$ for some $1 \leq i \leq n$. Thus K/N is a ϕ_N-n-absorbing primary submodule of M/N. □

Corollary 3.2. *Let N and K be submodules of an R-module M with $N \subseteq K$. If K is a ϕ-n-absorbing primary submodule of M with $\phi(K) \subseteq N$, then K/N is a weakly n-absorbing primary submodule of M/N.*

Proof. Let $N \neq a_1 \cdots a_n(m + N) \in K/N$ for $a_1, \dots, a_n \in R$ and $m \in M$. As $\phi(K) \subseteq N$, we have $\phi_N(K/N) = N$. Therefore $a_1 \cdots a_n(m + N) \in K/N - \phi_N(K/N)$. Since K is a ϕ-n-absorbing primary submodule of M, by Theorem 3.1, K/N is a ϕ_N-n-absorbing primary submodule of M/N. This implies that either $a_1 \cdots a_n \in \sqrt{(K/N : M/N)}$ or $\widehat{a_i}(m+N) \in K/N$ for some $1 \leq i \leq n$. Hence K/N is a weakly n-absorbing primary submodule of M/N. □

Theorem 3.3. *Let N and K be submodules of an R-module M with $N \subseteq K$. If $N \subseteq \phi(K)$ and K/N is a ϕ_N-n-absorbing primary submodule of M/N, then K is a ϕ-n-absorbing primary submodule of M.*

Proof. Let $a_1 \cdots a_n m \in K - \phi(K)$ for $a_1, \dots, a_n \in R$ and $m \in M$, that is, $a_1 \cdots a_n(m + N) \in K/N - \phi(K)/N$. As $N \subseteq \phi(K)$, we have $a_1 \cdots a_n(m + N) \in K/N - \phi_N(K/N)$. Since K/N is a ϕ_N-n-absorbing primary submodule of M/N, therefore either $a_1 \cdots a_n \in \sqrt{(K/N : M/N)}$ or $\widehat{a_i}(m + N) \in K/N$ for some $1 \leq i \leq n$. This implies that either $a_1 \cdots a_n \in \sqrt{(K : M)}$ or $\widehat{a_i} m \in K$ for some $1 \leq i \leq n$. Hence K is a ϕ-n-absorbing primary submodule of M. □

Theorem 3.4. *Let N and K be submodules of an R-module M with $N \subseteq K$. If N is a ϕ-n-absorbing primary submodule of M, $\phi(N) \subseteq \phi(K)$ and K/N is a weakly n-absorbing primary submodule of M/N, then K is a ϕ-n-absorbing primary submodule of M.*

Proof. Let $a_1 \cdots a_n m \in K - \phi(K)$ for $a_1, \dots, a_n \in R$ and $m \in M$. If $a_1 \cdots a_n m \in N$, then $a_1 \cdots a_n m \in N - \phi(N)$ as $\phi(N) \subseteq \phi(K)$. Since N is a ϕ-n-absorbing primary submodule of M, therefore either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or $\widehat{a_i} m \in N$ for some $1 \leq i \leq n$. As $N \subseteq K$, we have $\sqrt{(N : M)} \subseteq \sqrt{(K : M)}$. This implies that either $a_1 \cdots a_n \in \sqrt{(K : M)}$ or $\widehat{a_i} m \in K$ for some $1 \leq i \leq n$ and we are done. If $a_1 \cdots a_n m \notin N$, then $N \neq a_1 \cdots a_n(m+N) \in K/N$ and K/N is a weakly n-absorbing primary submodule of M/N. Therefore either $a_1 \cdots a_n \in \sqrt{(K/N : M/N)}$ or $\widehat{a_i}(m + N) \in K/N$ for some $1 \leq i \leq n$, that is, either $a_1 \cdots a_n \in \sqrt{(K : M)}$ or $\widehat{a_i} m \in K$ for some $1 \leq i \leq n$. Thus K is a ϕ-n-absorbing primary submodule of M. □

Corollary 3.5. *K is a ϕ-n-absorbing primary submodule of an R-module M if and only if $K/\phi(K)$ is a weakly n-absorbing primary submodule of $M/\phi(K)$.*

Proof. Assume that K is a ϕ-n-absorbing primary submodule of M. In Corollary 3.2, set $N = \phi(K)$. Then $K/\phi(K)$ is a weakly n-absorbing primary submodule of $M/\phi(K)$.

Conversely, assume that $K/\phi(K)$ is a weakly n-absorbing primary submodule of $M/\phi(K)$. Let $a_1 \cdots a_n(m + \phi(K)) \in K/\phi(K) - \phi_{\phi(K)}(K/\phi(K))$ for $a_1, \dots, a_n \in R$ and $m \in M$. But $\phi_{\phi(K)}(K/\phi(K)) = \phi(K)$. Therefore $\phi(K) \neq a_1 \cdots a_n(m + \phi(K)) \in K/\phi(K)$ and $K/\phi(K)$ is a weakly n-absorbing primary submodule of $M/\phi(K)$. This implies that either $a_1 \cdots a_n \in \sqrt{(K/\phi(K) : M/\phi(K))}$ or $\widehat{a_i}(m + \phi(K)) \in K/\phi(K)$ for some $1 \leq i \leq n$. Hence $K/\phi(K)$ is a $\phi_{\phi(K)}$-n-absorbing primary submodule of $M/\phi(K)$. Setting $N = \phi(K)$ in Theorem 3.3 implies that K is a ϕ-n-absorbing primary submodule of M. □

Theorem 3.6. *Let N be a ϕ-n-absorbing primary submodule of an R-module M. Then for $a_1, \ldots, a_n \in R$ and $m \in M$, $(a_1, \ldots, a_n, m)$ is a ϕ-primary-$(n+1)$-tuple-zero of N if and only if $(a_1, \ldots, a_n, m + \phi(N))$ is an $(n+1)$-tuple-zero of $N/\phi(N)$.*

Proof. Suppose that $(a_1, \ldots, a_n, m)$ is a ϕ-primary-$(n+1)$-tuple-zero of N. So $a_1 \cdots a_n m \in \phi(N)$, $a_1 \cdots a_n \notin \sqrt{(N : M)}$ and $\widehat{a_i} m \notin N$ for every $1 \leq i \leq n$. This implies that $a_1 \cdots a_n(m + \phi(N)) = \phi(N)$, $a_1 \cdots a_n \notin \sqrt{(N/\phi(N) : M/\phi(N))}$ and $\widehat{a_i}(m + \phi(N)) \notin N/\phi(N)$ for every $1 \leq i \leq n$. Since by Corollary 3.5, $N/\phi(N)$ is a weakly-n-absorbing primary submodule of $M/\phi(N)$, therefore $(a_1, \ldots, a_n, m + \phi(N))$ is an $(n+1)$-tuple-zero of $N/\phi(N)$.

The converse part can be easily proved by the same argument. □

Let S be a multiplicatively closed subset of R and M be an R-module. We know that every submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. Define $\phi_S : \mathcal{S}(S^{-1}M) \to \mathcal{S}(S^{-1}M) \cup \{\emptyset\}$ by $\phi_S(S^{-1}N) = S^{-1}\phi(N)$ if $\phi(N) \neq \emptyset$ and $\phi_S(S^{-1}N) = \emptyset$ if $\phi(N) = \emptyset$ for every submodule N of M. Note that $\phi_S(S^{-1}N) \subseteq S^{-1}N$.

We denote the set of all zero divisors of an R-module M by $Zd(M)$, which is given by $Zd(M) = \{r \in R : rm = 0 \text{ for some } 0 \neq m \in M\}$.

Theorem 3.7. *Let S be a multiplicatively closed subset of R and N be a proper submodule of an R-module M. If N is a ϕ-n-absorbing primary submodule of M with $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is a ϕ_S-n-absorbing primary submodule of $S^{-1}M$.*

Proof. Assume that N is a ϕ-n-absorbing primary submodule of M. Let $\frac{a_1}{s_1} \cdots \frac{a_n}{s_n} \frac{m}{s} \in S^{-1}N - \phi_S(S^{-1}N)$ for $a_1, \ldots, a_n \in R$, $s_1, \ldots, s_n, s \in S$ and $m \in M$. Then there exists $s' \in S$ such that $s'a_1 \cdots a_n m \in N$. Also, $s'a_1 \cdots a_n m \notin \phi(N)$ for if $s'a_1 \cdots a_n m \in \phi(N)$, then $\frac{a_1}{s_1} \cdots \frac{a_n}{s_n} \frac{m}{s} = \frac{s'a_1 \cdots a_n m}{s's_1 \cdots s_n s} \in S^{-1}\phi(N) = \phi_S(S^{-1}N)$, a contradiction. Hence $a_1 \cdots a_n(s'm) \in N - \phi(N)$ and N is a ϕ-n-absorbing primary submodule of M. Therefore either $a_1 \cdots a_n \in \sqrt{(N : M)}$ or $\widehat{a_i} s'm \in N$ for some $1 \leq i \leq n$. This implies that either $\frac{a_1}{s_1} \cdots \frac{a_n}{s_n} \in S^{-1}\sqrt{(N : M)} = \sqrt{S^{-1}(N : M)} \subseteq \sqrt{(S^{-1}N : S^{-1}M)}$ or $\frac{\widehat{a_i}}{\widehat{s_i}} \frac{m}{s} = \frac{\widehat{a_i} s'm}{\widehat{s_i} s's} \in S^{-1}N$ for some $1 \leq i \leq n$ where $\widehat{s_i}$ denotes the element of S obtained by deleting s_i from the product $s_1 \cdots s_n$. Thus $S^{-1}N$ is a ϕ_S-n-absorbing primary submodule of $S^{-1}M$. □

Lemma 3.8. *[16, Lemma 1] Let R be a ring, S a multiplicatively closed subset of R, M an R-module and N a submodule of M. If $S \cap Zd(M/N) = \emptyset$, then $S \cap Zd(R/(N : M)) = \emptyset$.*

Theorem 3.9. *Let S be a multiplicatively closed subset of R and N be a proper submodule of a finitely generated R-module M. If $S^{-1}N$ is a ϕ_S-n-absorbing primary submodule of $S^{-1}M$ with $S \cap Zd(N/\phi(N)) = S \cap Zd(M/N) = \emptyset$, then N is a ϕ-n-absorbing primary submodule of M.*

Proof. Assume that $S^{-1}N$ is a ϕ_S-n-absorbing primary submodule of $S^{-1}M$ such that $S \cap Zd(N/\phi(N)) = S \cap Zd(M/N) = \emptyset$. Let $a_1 \cdots a_n m \in N - \phi(N)$ for $a_1, \ldots, a_n \in R$ and $m \in M$. Then $\frac{a_1 \cdots a_n m}{1} \in S^{-1}N$. If $\frac{a_1 \cdots a_n m}{1} \in \phi_S(S^{-1}N) = S^{-1}\phi(N)$, then there exists $s_0 \in S$ such that $s_0 a_1 \cdots a_n m \in \phi(N)$, a contradiction as $S \cap Zd(N/\phi(N)) = \emptyset$. So $\frac{a_1 \cdots a_n m}{1} \in S^{-1}N - \phi_S(S^{-1}N)$ and $S^{-1}N$ is a ϕ_S-n-absorbing primary submodule of $S^{-1}M$. Therefore either $\frac{a_1 \cdots a_n}{1} \in \sqrt{(S^{-1}N : S^{-1}M)}$ or $\frac{\widehat{a_i} m}{1} \in S^{-1}N$ for some $1 \leq i \leq n$. If $\frac{a_1 \cdots a_n}{1} \in \sqrt{(S^{-1}N : S^{-1}M)}$, then $\frac{a_1 \cdots a_n}{1} \in \sqrt{S^{-1}(N : M)}$ as $S^{-1}(N : M) = (S^{-1}N : S^{-1}M)$ since M is a finitely generated R-module. Therefore $\frac{a_1 \cdots a_n}{1} \in S^{-1}\sqrt{(N : M)}$. So there exists $s \in S$ such that $(sa_1 \cdots a_n)^k \in (N : M)$ for some positive integer k, that is, $s^k(a_1 \cdots a_n)^k \in (N : M)$. If $(a_1 \cdots a_n)^k \notin (N : M)$, then $s^k \in S \cap Zd(R/(N : M))$, which is a contradiction because by Lemma 3.8, $S \cap Zd(R/(N : M)) = \emptyset$ as $S \cap Zd(M/N) = \emptyset$. Therefore $(a_1 \cdots a_n)^k \in (N : M)$ which implies that $a_1 \cdots a_n \in \sqrt{(N : M)}$. If $\frac{\widehat{a_i} m}{1} \in S^{-1}N$ for some $1 \leq i \leq n$, then there exists $s' \in S$ such that $s'\widehat{a_i} m \in N$ for some $1 \leq i \leq n$. If $\widehat{a_i} m \notin N$ for every $1 \leq i \leq n$, then $s' \in S \cap Zd(M/N)$, which is a contradiction. Therefore $\widehat{a_i} m \in N$ for some $1 \leq i \leq n$. Hence N is a ϕ-n-absorbing primary submodule of M. □

Theorem 3.10. *Let M_1 be an R_1-module, M_2 be an R_2-module, $M = M_1 \times M_2$, $\psi_i : \mathcal{S}(M_i) \to \mathcal{S}(M_i) \cup \{\emptyset\}$ be functions for $i = 1, 2$, $\phi = \psi_1 \times \psi_2$ and $N = N_1 \times M_2$ be a proper submodule of M for a proper submodule N_1 of M_1. Then the following statements hold.*

(1) If N is a ϕ-n-absorbing primary submodule of M, then N_1 is a ψ_1-n-absorbing primary submodule of M_1.

(2) If N_1 is a ψ_1-n-absorbing primary submodule of M_1 and $\psi_2(M_2) = M_2$, then N is a ϕ-n-absorbing primary submodule of M.

(3) If N_1 is an n-absorbing primary submodule of M_1, then N is a ϕ-n-absorbing primary submodule of M.

(4) If N is a ϕ-n-absorbing primary submodule of M and $\psi_2(M_2) \neq M_2$, then N_1 is an n-absorbing primary submodule of M_1.

Proof. (1) Assume that N is a ϕ-n-absorbing primary submodule of M. Let $a_1 \cdots a_n m_1 \in N_1 - \psi_1(N_1)$ for $a_1, \ldots, a_n \in R_1$ and $m_1 \in M_1$. Then $(a_1, 1) \cdots (a_n, 1)(m_1, m_2) \in (N_1 \times M_2) - (\psi_1(N_1) \times \psi_2(M_2))$ for $m_2 \in M_2$, that is, $(a_1, 1) \cdots (a_n, 1)(m_1, m_2) \in N - \phi(N)$. Since N is a ϕ-n-absorbing primary submodule of M, therefore either $(a_1, 1) \cdots (a_n, 1) \in \sqrt{(N : M)}$ or $\widehat{(a_i, 1)}(m_1, m_2) \in N$ for some $1 \leq i \leq n$ where $\widehat{(a_i, 1)}$ denotes the element of $R_1 \times R_2$ obtained by deleting $(a_i, 1)$ from the product $(a_1, 1) \cdots (a_n, 1)$. This implies that either $a_1 \cdots a_n \in \sqrt{(N_1 : M_1)}$ or $\widehat{a}_i m_1 \in N_1$ for some $1 \leq i \leq n$. Hence N_1 is a ψ_1-n-absorbing primary submodule of M_1.

(2) Assume that N_1 is a ψ_1-n-absorbing primary submodule of M_1 and $\psi_2(M_2) = M_2$. Let $(s_1, t_1) \cdots (s_n, t_n)(m_1, m_2) \in N - \phi(N)$ for $(s_1, t_1), \ldots, (s_n, t_n) \in R_1 \times R_2$ and $(m_1, m_2) \in M$. Then $(s_1, t_1) \cdots (s_n, t_n)(m_1, m_2) \in (N_1 \times M_2) - \phi(N_1 \times M_2)$. As $\psi_2(M_2) = M_2$, we have $s_1 \cdots s_n m_1 \in N_1 - \psi_1(N_1)$. Since N_1 is a ψ_1-n-absorbing primary submodule of M_1, therefore either $s_1 \cdots s_n \in \sqrt{(N_1 : M_1)}$ or $\widehat{s}_i m_1 \in N_1$ for some $1 \leq i \leq n$. This implies that either $(s_1, t_1) \cdots (s_n, t_n) \in \sqrt{(N : M)}$ or $\widehat{(s_i, t_i)}(m_1, m_2) \in N$ for some $1 \leq i \leq n$ where $\widehat{(s_i, t_i)}$ denotes the element of $R_1 \times R_2$ obtained by deleting (s_i, t_i) from the product $(s_1, t_1) \cdots (s_n, t_n)$. Thus N is a ϕ-n-absorbing primary submodule of M.

(3) Assume that N_1 is an n-absorbing primary submodule of M_1. Let $(s_1, t_1), \ldots, (s_n, t_n) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ such that $(s_1, t_1) \cdots (s_n, t_n)(m_1, m_2) \in N = N_1 \times M_2$. Then $s_1 \cdots s_n m_1 \in N_1$. As N_1 is an n-absorbing primary submodule of M_1, we have either $s_1 \cdots s_n \in \sqrt{(N_1 : M_1)}$ or $\widehat{s}_i m_1 \in N_1$ for some $1 \leq i \leq n$. This implies that either $(s_1, t_1) \cdots (s_n, t_n) \in \sqrt{(N : M)}$ or $\widehat{(s_i, t_i)}(m_1, m_2) \in N$ for some $1 \leq i \leq n$. Thus N is an n-absorbing primary submodule of M. Hence N is a ϕ-n-absorbing primary submodule of M for any ϕ.

(4) Let N be a ϕ-n-absorbing primary submodule of M. Then by part (1), N_1 is a ψ_1-n-absorbing primary submodule of M_1. Assume that N_1 is not an n-absorbing primary submodule of M_1. Then there exists a ψ_1-primary-$(n+1)$-tuple-zero $(a_1, \ldots, a_n, m_1)$ of N_1 for $a_1, \ldots, a_n \in R_1$ and $m_1 \in M_1$. Therefore $a_1 \cdots a_n m_1 \in \psi_1(N_1)$, $a_1 \cdots a_n \notin \sqrt{(N_1 : M_1)}$ and $\widehat{a}_i m_1 \notin N_1$ for every $1 \leq i \leq n$. As $\psi_2(M_2) \neq M_2$, there exists $m_2 \in M_2$ such that $m_2 \notin \psi_2(M_2)$. This implies that $(a_1, 1) \cdots (a_n, 1)(m_1, m_2) \in (N_1 \times M_2) - (\psi_1(N_1) \times \psi_2(M_2))$, that is, $(a_1, 1) \cdots (a_n, 1)(m_1, m_2) \in N - \phi(N)$. Since N is a ϕ-n-absorbing primary submodule of M, therefore either $(a_1, 1) \cdots (a_n, 1) \in \sqrt{(N : M)}$ or $\widehat{(a_i, 1)}(m_1, m_2) \in N$ for some $1 \leq i \leq n$. This implies that either $a_1 \cdots a_n \in \sqrt{(N_1 : M_1)}$ or $\widehat{a}_i m_1 \in N_1$ for some $1 \leq i \leq n$, which is a contradiction. Hence N_1 is an n-absorbing primary submodule of M_1. □

Theorem 3.11. *[20, Theorem 2.19] Let M be an R-module. If N is a weakly n-absorbing primary submodule of M that is not n-absorbing primary, then $(N : M)^n N = \{0\}$.*

Theorem 3.12. *Let M_1 be an R_1-module, M_2 be an R_2-module and $R = R_1 \times R_2$. If N_1 is a weakly n-absorbing primary submodule of M_1, then $N = N_1 \times M_2$ is a ϕ-n-absorbing primary submodule of $M = M_1 \times M_2$ as an R-module for some ϕ with $\phi_\omega \leq \phi \leq \phi_1$.*

Proof. If N_1 is an n-absorbing primary submodule of M_1, then by Theorem 3.10(3), N is a ϕ-n-absorbing primary submodule of M for any ϕ. Assume that N_1 is not an n-absorbing primary submodule of M_1. Then, by Theorem 3.11, $(N_1 : M_1)^n N_1 = \{0\}$. Therefore $(N : M)^n N =$

$\{0\} \times M_2$. So $\phi_\omega(N) = \{0\} \times M_2$ and $N - \phi_\omega(N) = (N_1 - \{0\}) \times M_2$. Let $(s_1, t_1), \ldots, (s_n, t_n) \in R_1 \times R_2$ and $(m_1, m_2) \in M$ such that $(s_1, t_1) \cdots (s_n, t_n)(m_1, m_2) \in N - \phi_\omega(N)$. This gives $s_1 \cdots s_n m_1 \in N_1 - \{0\}$ and N_1 is a weakly n-absorbing primary submodule of M_1. Therefore either $s_1 \cdots s_n \in \sqrt{(N_1 : M_1)}$ or $\widehat{s_i} m_1 \in N_1$ for some $1 \leq i \leq n$. This implies that either $(s_1, t_1) \cdots (s_n, t_n) \in \sqrt{(N : M)}$ or $\widehat{(s_i, t_i)}(m_1, m_2) \in N$ for some $1 \leq i \leq n$. Hence N is a ϕ_ω-n-absorbing primary submodule of M. Thus, by Lemma 2.4, N is a ϕ-n-absorbing primary submodule of M for any ϕ with $\phi_\omega \leq \phi \leq \phi_1$. □

References

[1] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math, **29(4)**, 831–840, (2003).

[2] M. S. Bhatwadekar and P. K. Sharma, *Unique factorization and birth of almost primes*, Commun. Algebra, **33(1)**, 43–49, (2005).

[3] D. D. Anderson and M. Batanieh, *Generalizations of prime ideals*, Comm. Algebra, **36(2)**, 686–696, (2008).

[4] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Aust. Math. Soc., **75(3)**, 417–429, (2007).

[5] A. Y. Darani, *Generalizations of primary ideals in commutative rings*, Novi Sad J. Math., **42(1)**, 27–35, (2012).

[6] A. Badawi, U. Tekir and E. Yetkin, *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc., **51(4)**, 1163–1173, (2014).

[7] A. Badawi, U. Tekir and E. Yetkin, *On weakly 2-absorbing primary ideals of commutative rings*, J. Korean Math. Soc., **52(1)**, 97–111, (2015).

[8] A. Badawi, U. Tekir, E. A. Ugurlu, G. Ulucak and E. Y. Celikel, *Generalizations of 2-absorbing primary ideals of commutative rings*, Turk. J. Math., **40(3)**, 703–717, (2016).

[9] D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Commun. Algebra, **39(5)**, 1646–1672, (2011).

[10] A. E. Becker, *Results on n-absorbing ideals of commutative rings*, M. S. thesis, University of Wisconsin-Milwaukee, Milwaukee, U. S. A., (2015).

[11] M. Ebrahimpour and R. Nekooei, *On generalizations of prime ideals*, Comm. Algebra, **40(4)**, 1268–1279, (2012).

[12] H. Mostafanasab and A. Y. Darani, *On ϕ-n-absorbing primary ideals of commutative rings*, J. Korean Math. Soc., **53(3)**, 549–582, (2016).

[13] N. Zamani, *ϕ-prime submodules*, Glasgow Math. J., **52(2)**, 253–259, (2010).

[14] A. Y. Darani and F. Soheilnia, *2-Absorbing and Weakly 2-Absorbing Submodules*, Thai Journal of Mathematics, **9(3)**, 577–584, (2011).

[15] M. K. Dubey and P. Aggarwal, *On 2-absorbing primary submodules of modules over commutative ring with unity*, Asian-European Journal of Mathematics, **8(4)**, (2015).

[16] R. Moradi and M. Ebrahimpour, *On ϕ-2-absorbing primary submodules*, Acta Math Vietnam, **42**, 27–35, (2017).

[17] H. Mostafanasab, U. Tekir, E. Y. Celikel, E. A. Ugurlu, G. Ulucak and A. Y. Darani, *Generalizations of 2-absorbing and 2-absorbing primary submodules*, Hacet. J. Math. Stat., **48(4)**, 1001–1016, (2019).

[18] A. Y. Darani and F. Soheilnia, *On n-Absorbing Submodules*, Math. Commun., **17**, 547–557, (2012).

[19] N. I. Shaikh and R. P. Deore, *On n-absorbing primary submodules*, Palestine Journal of Mathematics, **10(2)**, 845–851, (2021).

[20] P. Yiarayong and M. Siripitukdet, *On n-absorbing primary submodules*, Southeast Asian Bulletin of Mathematics, **45**, 273–286, (2021).

[21] N. I. Shaikh and R. P. Deore, *On almost n-absorbing primary submodules*, Palestine Journal of Mathematics, **11(4)**, 267–274, (2022).

[22] M. Ebrahimpour and R. Nekooei, *On generalization of prime submodules*, Bull. Iranian Math. Soc., **39(5)**, 919–939, (2013).

[23] P. Quartararo and H. S. Butts, *Finite unions of ideals and modules*, Proc. Amer. Math. Soc., **52**, 91–96, (1975).

Author information

Nilofer I. Shaikh, Department of Mathematics, University of Mumbai, Mumbai-400098, Maharashtra, India.
E-mail: `shaikhnilofer23@gmail.com`

Rajendra P. Deore, Department of Mathematics, University of Mumbai, Mumbai-400098, Maharashtra, India.
E-mail: `rpdeore@gmail.com`

Received: 2025-07-24
Accepted: 2025-09-20