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NEW GENERALIZATIONS OF EXTENDING MODULES VIA C-ESSENTIAL SUBMODULES

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Abstract In this paper, we present two new generalizations of extending modules by introducing a new generalization of essential submodules. A submodule N of a module W is defined as a C-essential submodule if every submodule of W that properly contains N is essential in W. A module W is called C-extending if every submodule of W is C-essential in a direct summand of W. Furthermore, W is called a CSIP-extending module if the intersection of every pair of direct summands is C-essential in a direct summand of W. Some related results are included about C-essential submodules, C-extending modules, and CSIP-extending modules.

1 Introduction

Throughout this paper, all rings are assumed to be associative with an identity element, and all modules are assumed to be unitary left modules unless specified otherwise. We use the notations $\subseteq$,$\subset$, $\leq$, and $<$ to show inclusion, proper inclusion, submodule, and proper submodule, respectively, and $N \leq^{\oplus} W$ means that N is a direct summand of W. The concepts of small and essential submodules are well established and serve as important tools in ring and module theory. Small submodules and essential submodules have been generalized in several ways [1, 5, 6, 7]. In 2025, M. Akbari Gelvardi and Y. Talebi [8] introduced a new generalization of small submodules. The authors named the C-small submodule for this generalization. A submodule N of a module W is called C-small if any proper submodule of N is small in W. We introduce the dual definition of C-small submodules, namely C-essential submodules. Also, we investigate some results about C-essential submodules and define a new generalization of extending modules, which we call a C-extending module. An extending module, also known as a CS-module, has the property that every submodule is essential in a direct summand. Equivalently, a module W is a CS-module, if every complement of W is a summand. Extending modules are a generalization of injective modules that remain of interest to many researchers. Many generalizations of these modules have been studied, and for a review of some of them, see the article [2]. Extending modules have been generalized in several ways. For instance, SIP-extending modules [3] and K-extending modules [10] are proper generalizations of extending modules. SIP-extending modules are a generalization of both extending and SIP-modules and K-extending modules generalize both extending and Rickart modules.

A submodule L of an A-module W is called $superfluous$ or $small$ in W (denoted by $L \ll W$), if for each submodule $N \leq W$, the equality $L + N = W$ implies $N = W$. Equivalently, L is small in W if for every proper submodule $N < W$, we have $L + N \neq W$. The sum of all small submodules in a module M, denoted by $Rad(M)$, is $radical$ of W. A module whose all nontrivial submodules are small, is called a $hollow$ module. Dually a submodule L of an A-module W is called $essential$ in W (denoted by $L \leq_e W$), if for each submodule $N \leq W$, the equality $L \cap N = 0$ implies $N = 0$. Equivalently, L is essential in W if for every proper submodule $N < W$, we have $L \cap N \neq 0$. A module whose all nontrivial submodules are essential, is called a $uniform$ module. An A-module W is called $local$ if it has a largest submodule, i.e., a submodule which contains all other submodules. Consider L and N are two submodules of W. We say L is a complement of N if L is maximal in $N \cap L = 0$. By Zorn's lemma, it is well known that every submodule has a complement. A submodule that has no proper essential extension in the module is called a closed submodule. A submodule $U \leq W$ is fully invariant if, for every endomorphism $\theta : W \to W$, the image $\theta(U)$ is contained within U. Any undefined algebraic terminology can be obtained in [2, 4, 9].

The second part of this article generalizes essential submodules by introducing C-essential submodules and examining their properties, accompanied by illustrative examples. The third part discusses C-extending modules, which, along with the forth part, serve as applications of C-essential submodules. Building on the concept of C-essential submodules, we generalize extending modules, analyse their properties, and present relevant examples. The forth part specifically focuses on CSIP-extending modules.

2 C-ESSENTIAL SUBMODULES

In this section, we start with our definition and examine some examples and preliminary results.

Definition 2.1. Let W be an R-module and $N \leq W$. N is called C-$essential$ in W, if any submodule K of W with $N \subset K$ (properly containing N) is essential in W. Equivalently, any submodule that properly contains N is essential in W.

If N is essential in W, then N is C-essential in W. To see this, consider an arbitrary submodule K such that $N \subset K$. Since N is an essential submodule, it has a nonzero intersection with all nonzero submodules, and since $N \subset K$, we conclude that K also has a nonzero intersection with all nonzero submodules. So K is essential in W. Inasmuch as K is an arbitrary submodule that contains N, N is C-essential. The converse may not be true. For instance consider $\mathbb{Z}_6$ as $\mathbb{Z}$-module. The submodules are $\{0, <\bar{1}>, <\bar{2}>, <\bar{3}>\}$. We know that $<\bar{1}> = \mathbb{Z}_6$ and is essential in $\mathbb{Z}_6$. The only submodule which properly containing $<\bar{2}>$ is $<\bar{1}>$ and it is essential. So $<\bar{2}>$ is C-essential. But $<\bar{2}>$ is not essential, because $<\bar{2}> \cap <\bar{3}> = 0$ and $<\bar{3}> \neq 0$. So the converse is not true in general.

Example 2.2. (1) Let W be uniform module. Every proper submodule of W is C-essential. Because every proper submodule of W is essential. For instance for a prime number p, the $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is uniform. So every submodule of $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is C-essential.

(2) Let p and q be distinct prime numbers. Consider $W = \mathbb{Z}_{pq^2}$ as $\mathbb{Z}$-module. The submodules of W are $\{0, <\bar{1}>, <\bar{p}>, <\bar{q}>, <\bar{q^2}>, <\bar{pq}>\}$. Now suppose $p = 3$ and $q = 2$ i.e., $W =$

$\mathbb{Z}_{12}$ as $\mathbb{Z}$-module. The submodules of W are $\{0, <\bar{1}>= \mathbb{Z}_{12}, <\bar{2}>, <\bar{3}>, <\bar{4}>, <\bar{6}>\}$. The submodule $<\bar{4}>$ is not essential but it is C-essential, because $<\bar{1}>$ and $<\bar{2}>$ properly containing $<\bar{4}>$ and both of them are essential submodules. Also by the same way, we can show that $<\bar{6}>$ is a C-essential submodule but is not an essential submodule in W.

Remark 2.3. Let W be an A-module. If $0 \leq_{ce} W$, then W is uniform module. Since every submodule of W contains a zero element, it must be essential in W.

Proposition 2.4. *Let W be an A-module and $N, K \leq_{ce} W$. Then $N + K \leq_{ce} W$.*

Proof. Let H be the submodule of W such that $N + K \subset H$. Since N is C-essential in W and $N \subset N + K \subset H$, so H is essential. Hence $N + K$ is C-essential in W. □

Proposition 2.5. *Let W be an A-module and $N, K \leq W$ such that $N \cap K \leq_{ce} W$. Then $N, K \leq_{ce} W$.*

Proof. Since $N \cap K \leq_{ce} W$ and $N \cap K \subset N$, N is essential and hence is C-essential. Similarly, since $N \cap K \leq_{ce} W$ and $N \cap K \subset K$, K is essential and hence is C-essential. □

Proposition 2.6. *Let W and N be two arbitrary A-modules and $f : W \to N$ be an A-module monomorphism. If $H \leq_{ce} N$ and $Imf \subset H$, then $f^{-1}(H) \leq_{ce} W$.*

Proof. Let G be a submodule of W such that $f^{-1}(H) \subset G$, so $f(f^{-1}(H)) \subset f(G)$ i.e $f(W) \cap H \subset f(G)$ and since $Imf \subset H$, we have $H \subset f(G)$. We must show that G is an essential submodule of W. So suppose that K is a submodule of W such that $G \cap K = 0$. So $f(G \cap K) = f(G) \cap f(K) = 0$. Since $H \subset f(G)$ and $H \leq_{ce} W$, we can conclude that $f(K) = 0$. Finally, inasmuch as f is a monomorphism, we conclude that $K \subset Ker(f) = 0$, so $K = 0$. Hence G is an essential submodule in W. Hence the result. □

Theorem 2.7. *Let W be an A-module and $L, K \leq W$. The following holds.*
(1) If $K \leq_{ce} W$, then for every submodule T with $K \subset T$, $T \leq_{ce} W$.
(2) If $K \leq_{ce} L$ and $L \leq_{ce} W$ and for any submodule G which containing K, $G \leq L$ or $L \leq G$, then $K \leq_{ce} W$.

Proof. (1) Suppose that G is a submodule of W which containing T. Since $K \leq_{ce} W$, so T is essential in W. Now consider another submodule T' which containing T. Since $K \subset T \subset T'$ and again by $K \leq_{ce} W$, we have T' is essential in W. So T is also C-essential in W.

(2) Consider a submodule G of W which containing K. By assumption, we have 2 cases. Case 1 is when $G \leq L$. In this case, since $K \leq_{ce} L$, G is essential in L and since $L \leq_{ce} M$, every submodule that contains L is essential in W. Now we can conclude that G is essential in M. Case 2 is when $L \leq G \leq W$. In this case, since $L \leq_{ce} W$, again G is essential in W. Hence, the proof is done. □

Let W be an A-module. There exists an A-module E that is an essential injective extension of W. The module E is called an $injective\ envelope$ of W. We use the symbol $E(W)$ to denote an injective envelope of W.

Corollary 2.8. *Let W and N be R-modules. If $N \leq_{ce} E(W)$, then $E(W) = E(N)$.*

Proof. We know that $W \leq_e E(W)$, so $W \leq_{ce} E(W)$. Since $N \leq_{ce} E(W)$ and $N \subset E(N) \leq E(W)$, we have, $E(N) \leq_e E(W)$. So the inclusion $E(N) \to E(W)$ is an injective envelope of $E(N)$, i.e., $E(E(N)) = E(W)$. Hence $E(N) = E(W)$. □

Proposition 2.9. *Every maximal submodule of an A-module is a C-essential submodule.*

Proof. Let W be an A-module and X be a maximal submodule of W. Since X is maximal, the only submodule that contains X is W, and it is well known that W is essential in itself. So X is C-essential. □

Lemma 2.10. *A submodule N of an A-module W is C-essential in W if and only if for each $0 \neq x \in W$ and $N \subset K \leq W$ there exists an $r \in R$ such that $0 \neq rx \in K$.*

Proof. Suppose that N is C-essential in W. Consider $0 \neq x \in W$ and $N \subset K \leq M$. So $K \leq_e W$. Therefore, $Rx \cap K \neq 0$, i.e., there exists an $a \in A$ such that $0 \neq rx \in K$. Conversely, suppose that for each $0 \neq x \in W$ and $N \subset K \leq M$ there exists an $a \in A$ such that $0 \neq ax \in K$. So K has a nonzero intersection with all nonzero submodules of W, i.e., K is essential in W. Since K is an arbitrary submodule which containing N, N is C-essential in W. □

Theorem 2.11. *Let $K_1 \leq W_1 \leq W$ and $K_2 \leq W_2 \leq W$ and $W = W_1 \oplus W_2$. Then $K_1 \oplus K_2 \leq_{ce} W_1 \oplus W_2$ if and only if $K_1 \leq_{ce} W_1$ and $K_2 \leq_{ce} W_2$.*

Proof. First suppose that $K_1 \leq_{ce} W_1$ and $K_2 \leq_{ce} W_2$. We want to show that $K_1 \oplus K_2 \leq_{ce} W_1 \oplus W_2$. Let $0 \neq z_i \in W_i$. Since $K_1 \leq_{ce} W_1$, so there exists an element $0 \neq a \in A$, such that $0 \neq az_1 \in K_1'$ for all K_1' with $K_1 \leq K_1'$. Similarly Since $K_2 \leq_{ce} W_2$, so there exists an element $0 \neq a \in A$, such that $0 \neq az_2 \in K_2'$ for all K_2' with $K_2 \leq K_2'$. So $0 \neq az_1 + az_2 \in K_1' \oplus K_2'$. If $az_2 \notin K_2'$, then by Lemma 2.10, there exists $a' \in A$, with $0 \neq a'az_2 \in K_2'$ and we have $0 \neq a'az_1 + a'az_2 \in K_1' \oplus K_2'$. So $K_1 + K_2$ is C-essential in $W_1 \oplus W_2$. Conversely by contrary, suppose that K_1 is not C-essential in W_1, it means that there exists a submodule K_1', with $K_1 \subset K_1'$ which is not essential in W_1. It means that $K_1' \cap L_1 = 0$ for some $0 \neq L_1 \leq W_1$. Similarly we have $K_2' \leq W_2$ which is containing K_2. Now we have $(K_1' \oplus K_2') \cap L_1 = 0$. So there exist $k_1 \in K_1, k_2 \in K_2$ and $l_1 \in L_1$ such that $k_1 + k_2 = l_1$. Then $k_2 = l_1 - k_1 \in W_1 \cap W_2$, which is contradiction. □

Corollary 2.12. *Let $\{K_\alpha\}$ and $\{N_\alpha\}$ be two independent family of submodules of a module W. If $K_\alpha \leq_{ce} N_\alpha$ for each α, then $\oplus K_\alpha \leq_{ce} \oplus N_\alpha$.*

Proof. First, consider the case when the index set consists of exactly two elements, say $\{1, 2\}$. By assumption, $K_1 \leq_{ce} W_1$ and $K_2 \leq_{ce} W_2$. By Theorem 2.11, $K_1 \oplus K_2 \leq_{ce} N_1 \oplus N_2$ i.e., the theorem is true for index sets with two elements. Now consider the index set which contains n elements, and the result is true for $n-1$ elements. Then $K_1 \oplus K_2 \oplus \cdots \oplus K_{n-1} \leq_{ce} N_1 \oplus N_2 \oplus \cdots \oplus N_{n-1}$. Using above result, we see that $(K_1 \oplus K_2 \oplus \cdots \oplus K_{n-1}) \oplus K_n \leq_{ce} (N_1 \oplus N_2 \oplus \cdots \oplus N_{n-1}) \oplus N_n$. Therefore, the statement is true for all finite index sets. Now we want to show that the statement is also true for infinite index sets. For this consider distinct indices $\alpha(0), \alpha(1), \ldots, \alpha(n)$. Now any nonzero submodule $W \leq \oplus N_\alpha$ contains a nonzero element, which must belong to $N_{\alpha(1)} \oplus N_{\alpha(2)} \oplus \cdots \oplus N_{\alpha(n)}$ for some $\alpha(i)$. So $W \cap \oplus N_{\alpha(i)} \neq 0$ for $1 \leq i \leq n$. Hence $M \cap \oplus N_{\alpha(i)} \cap \oplus K_{\alpha(i)} \neq 0; 1 \leq i \leq n$ and since $\oplus K_{\alpha_i} \leq_{ce} \oplus N_{\alpha(i)}; 1 \leq i \leq n$, so $M \cap \oplus K_\alpha \neq 0$. Hence, the proof is done. □

Theorem 2.13. *Let W be an A-module and $L \leq N \leq W$. If $L \leq_e N$ and $N/L \leq_{ce} W/L$, then $N \leq_{ce} W$.*

Proof. Consider an arbitrary submodule G of W which containing N. We must show that G is essential in W. Since $N/L \subset G/L$ and $N/L \leq_{ce} W/L$, we conclude that G/L is essential in W/L. It means that $G/L \cap (H+L)/L \neq 0$ i.e $G \cap (H+L) \neq L$ for all arbitrary submodule $H \leq W$. By modularity, we have $L + (G \cap H) \neq L$ and hence $G \cap H \neq 0$. Since H was an arbitrary submodule, G is essential in W. □

Definition 2.14. Let W and N be two A-modules. An A-monomorphism $f : W \to N$ is said to be C-essential, in case $Imf \leq_{ce} N$.

Theorem 2.15. *Let W be an A-module and N be a submodule of W. Then the following statements are equivalent:*
(1) $N \leq_{ce} W$.
(2) The inclusion map $i_N : N \to W$ is a C-essential monomorphism.
(3) For every module K which properly containing N and for any A-module L and for each $h \in Hom(W, L)$, $(Kerh \cap K = 0)$ implies $Kerh = 0$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $N \leq_{ce} W$ and $i_N : N \to W$ is an inclusion map. We must show that $Imi_N \leq_{ce} W$. Since i_N is inclusion map, so $Imi_N = N$ and $N \leq_{ce} W$. So it is straightforward.

(2) $\Longrightarrow$ (3): Consider K is an R-module which containing N and $h \in Hom(M, L)$ such that $Kerh \cap K = 0$. Since i_N is C-essential, so $N \leq_{ce} W$. Hence K is essential in W and so $Kerh = 0$.

(3) $\Longrightarrow$ (1): Consider a submodule K of W such that $N \subset K$ and U is a submodule of W such that $K \cap U = 0$. We must show that $U = 0$. Let $L = W/U$ and $h : M \to W/U$ be a natural epimorphism. Then $(Kerh = U) \cap K = 0$, so by (2) $Kerh = U = 0$. Hence, the proof is done. □

Corollary 2.16. *Let W and L be A-modules, $f : L \to W$ be an A-monomorphism and there exists a submodule $K \leq W$ with $Imf \subset K$. If f is C-essential, then for all homomorphisms h, if hf is monic, then h is monic.*

Proof. Suppose that the monomorphism $f : L \to W$ is C-essential and for all homomorphisms h, hf is monic. Let K be a submodule which properly containing $N = Imf$, then $Imf \leq_{ce} W$ and $K \leq_e M$. Consider an inclusion map $i_k : K \to W$. Since $Imf \subset Img = K$, by [4, Theorem 3.6] we conclude that There exists a monomorphism $\phi : L \to K$ such that $f\phi = i_K$. Since hf is monic, we conclude that hi_K is monic. It means that $Ker(hi_K) = Kerh \cap K = 0$. Now by Theorem 2.15, we have $Kerh = 0$, i.e., h is monic. □

3 C-EXTENDING MODULES

Extending modules can be considered a generalization of injective, semisimple, and uniform modules. An extending module has every submodule as an essential submodule of a direct summand. Thus, the concept of extending modules is broadened to include various related module types, representing a significant area of study in module theory. In this section, we use the concept of C-essential submodules to introduce a new generalization of extending modules. Some examples and related results are investigated and presented in this section.

Definition 3.1. Let W be an A-module. W is called C-$extending$ module, if every submodule of W is C-essential in a direct summand of W. More generally, W is called $uniform$-C-$extending$, if every uniform submodule is C-essential in a direct summand of W.

Lemma 3.2. *Any direct summand of a (uniform-) C-extending module is also a (uniform-) C-extending module.*

Proof. Let W be a C-extending module and N be a direct summand of W. Consider a submodule K of N. Since K is also a submodule of W, we conclude that K is C-essential in a direct summand of W. Inasmuch as $N \leq^{\oplus} W$ and $K \leq N$, so K is C-essential in N. For the parenthetical version, just consider a uniform submodule. □

Example 3.3. (1) It is clear that every extending module is a C-extending module. So semisimple modules, injective modules, and uniform modules are C-extending modules. Also, quasi-continuous and continuous modules has C-extending property, since they have the extending property.

(2) Take a prime number p. We know that the $\mathbb{Z}$-module, $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not extending module [9]. So for instance the $\mathbb{Z}$-module, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not extending. It is clear that every submodule of $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is C-essential in a direct summand of $\mathbb{Z}_2 \oplus \mathbb{Z}_8$. So $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is C-extending but not extending module.

Corollary 3.4. *Let W be a C-extending A-module and $B \leq^{\oplus} W$. Then W/B is a C-extending module.*

Proof. Since $B \leq^{\oplus} W$, we have the exact sequence $0 \to B \to W \to W/B \to 0$, which is split. So we have $W = B \oplus W/B$. It means that W/B is a direct summand of W. Now, by Lemma 3.2, W/B is a C-extending module. □

Definition 3.5. A submodule N of a module W is called $C-closed$ if N has no C-essential proper extension in W. Every C-closed submodule is closed. The following example shows that the converse is not true in general.

Example 3.6. Suppose that $W = \mathbb{Z}_{18}$ as $\mathbb{Z}$-module. The set of all submodules of W is $\{0, <\bar{2}>, <\bar{3}>, <\bar{6}>, <\bar{9}>, \mathbb{Z}_{18}\}$. Consider the submodule $<\bar{9}>$. Since $<\bar{9}> \subset <\bar{3}>$ and $<\bar{9}> \cap <\bar{6}> = 0, <\bar{6}> \subset <\bar{3}>$, so we conclude that $<\bar{9}>$ is not essential in $<\bar{3}>$ and since the only submodule which containing $<\bar{9}>$ is $<\bar{3}>$, so it is clear that $<\bar{9}>$ has no proper essential extension in W. It means that $<\bar{9}>$ is a closed submodule but is not C-closed. Because $<\bar{9}>$ is C-essential in $<\bar{3}>$. So we find an example which is closed but not C-closed.

A module W is extending if and only if every closed submodule of W is a direct summand. The following proposition gives the C-extending version.

Proposition 3.7. *A module W is C-extending if every C-closed submodule of W is a direct summand of W. Conversely, is true if every closed submodule of W is maximal, then W is C-extending.*

Proof. First, suppose that W is a C-extending module. Consider an arbitrary submodule N of W such that N is a C-closed submodule. Since W is C-extending, there exists a direct summand L of W such that N is C-essential in L. Now, since N is C-closed, N has no proper C-essential extension, and we conclude that N is a direct summand of W. Conversely Suppose that every C-closed submodule of W is a direct summand of W. Let N be a submodule of M. By Zorn's lemma, there exists a closed submodule L, such that $N \leq_{ce} L$. Since L is maximal, so L is C-closed and by assumption, it is a direct summand. Hence W is C-extending. □

Lemma 3.8. *Let W_1 and W_2 be two A-modules. Then W_1 is W_2-injective if and only if for each submodule $N \leq W$ which $N \cap W_1 = 0$, there exists a submodule W' of W such that $W = W_1 \oplus W'$ and $N \subset W'$.*

Proof. See [9, Lemma 7.5]. □

Theorem 3.9. *Let $W = W_1 \oplus W_2$ which W_i's are C-extending modules. Then W is C-extending if and only if for all C-closed submodules $K \leq W$, K is a direct summand of W such that $K \cap W_1 = 0$ or $K \cap W_2 = 0$.*

Proof. It is clear that if W is C-extending, then every C-closed submodule K of W is a direct summand of W such that either $K \cap W_1 = 0$ or $K \cap W_2 = 0$. Conversely suppose that for all C-closed submodule $K \leq W$, K is a direct summand of W such that $K \cap W_1 = 0$ or $K \cap W_2 = 0$. We want to prove that W is C-extending. Let $L \leq W$ be a C-closed submodule of W. There exists a complement H of L such that $L \cap W_2 \leq_e H$ and consequently $L \cap W_2 \leq_{ce} H$. It is clear that $H \cap W_1 = 0$. By assumption, there exists a submodule H' of W such that $W = H \oplus H'$, and we can write $L = H \oplus (L \cap H')$. So we can say $L \cap H'$ is a closed submodule of W. Hence $(L \cap H') \cap W_2 = 0$. By assumption, $L \cap H'$ is a direct summand of W and consequently is a direct summand of H'. So L is a direct summand of W. It means that W is a C-extending module. □

The following example shows that the direct sum of two C-extending modules is not necessarily C-extending.

Example 3.10. Suppose that $A = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and α_{ij} be the matrix in A which is 1 in (i,j)-position and elsewhere is 0. Then $A_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} = \alpha_{11} A_A \oplus \alpha_{22} A_A$ is a direct sum of C-extending modules(Sine both of them are uniform modules, so they are extending and hence C-extending). But A_A is not C-extending. Because the submodule $\begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 2\mathbb{Z} \end{pmatrix}$ is not C-essential in a direct summand.

Theorem 3.11. *Let $W = W_1 \oplus W_2 \oplus W_3 \oplus \cdots \oplus W_n$ be the finite direct sum of relatively injective modules. Then W is C-extending if and only if W_i is C-extending for all $i = 1, 2, 3, \ldots, n$.*

Proof. The necessity is obtained from Lemma 3.2. Suppose that $W = W_1 \oplus W_2 \oplus W_3 \oplus \cdots \oplus W_n$ is the finite direct sum of relatively injective modules and $W_1, W_2, \ldots, W_n$ are C-extending modules. We want to show that W is C-extending. By induction, it is sufficient to show that the result is true for $n = 2$. Consider $W = W_1 \oplus W_2$ and take an arbitrary C-closed submodule K of W which $K \cap W_1 = 0$. By Lemma 3.8, there exists a submodule W' of W such that $K \subset W'$ and $W = W_1 \oplus W'$. It is clear that $W' \cong W_2$. So W' is C-extending. Hence K is a direct summand of W' and consequently is a direct summand of W. Similarly, every C-closed submodule H of W which $H \cap W_2 = 0$ is a direct summand of W. Hence By Theorem 3.9, W is C-extending. □

The set of all elements of an A-module W, which annihilate some essential left ideal of a ring A, is a submodule which is named the $singular\ submodule$ and denoted by $Z(W)$. The second singular submodule, denoted by $Z_2(W)$ is a submodule of W, which containing $Z(W)$ such that $Z_2(W)/Z(W)$ is the singular submodule of the factor module $W/Z(W)$.

Theorem 3.12. *Let W be an A-module. W is a C-extending module if and only if there exists a submodule W' of W, such that $W = Z_2(W) \oplus W'$, $Z_2(W)$ is W'-injective, and both of W' and $Z_2(M)$ are C-extending.*

Proof. First consider that W is C-extending module. Since $W/Z_2(W)$ is nonsingular, so $Z_2(W)$ is closed. So by assumption there exists a submodule $W' \leq W$, such that $W = Z_2(W) \oplus W'$. We know that every summand of C-extending module is C-extending (By Lemma 3.2). So both of W' and $Z_2(W)$ are C-extending modules. Now we must show that $Z_2(W)$ is W'-injective. Let G be an arbitrary submodule of W such that $G \cap Z_2(W) = 0$. Since W is C-extending, so every submodule is C-essential in a direct summand, hence there exist submodules U and V such that $W = U \oplus V$ and G is C-essential in U. Since $G \cap Z_2(W) = 0$ and G is essential in U, so we conclude that $U \cap Z_2(W) = 0$. Consequently $Z_2(W) \subset V$. Therefore $W = U \oplus V = Z_2(W) \oplus (V \cap W') \oplus U$ and $G \subset (V \cap W') \oplus U$. Now by Lemma 3.8 we conclude that $Z_2(W)$ is W'-injective. Conversely suppose that $W = Z_2(W) \oplus W'$, $Z_2(W)$ is W'-injective and both of W' and $Z_2(M)$ is C-extending. We want to show that W is C-extending. Since $Z(W) \subset Z_2(W)$, we conclude that W' is nonsingular and hence $Hom(Z_2(W), W') = 0$. Therefore W' is $Z_2(W)$-injective and by Theorem 3.9, W is C-extending. This complete the proof. □

Definition 3.13. A module W is $C-uniform\ module$, if every submodule of W is C-essential in W. It is clear that every uniform module is C-uniform. The next proposition is obtained immediately.

Proposition 3.14. *Let W be an indecomposable module. W is C-extending if and only if W is C-uniform.*

Proof. If W is indecomposable and C-extending, then every submodule of W is C-extending in W. It means that W is C-uniform module. Conversely suppose that W is indecomposable and C-uniform. It means that every submodule of W is C-extending in W and since W is a summand of itself, so W is C-extending. □

Theorem 3.15. *Let W be a C-extending module, which has finite uniform dimension. Then every submodule of W has ACC on C-closed submodules.*

Proof. Let n be the number of orthogonal idempotents in $End(W)$. Take an arbitrary submodule $N \leq M$. Let $N_1 \subset N_2 \subset N_3 \ldots$ be an ascending chain of C-closed submodules of N. Since W is C-extending, so the submodule N_{n+1} is C-essential in a direct summand of $C_1 \leq^{\oplus} M$. So there exists a submodule D_1 such that $W = C_1 \oplus D_1$. If $D_1 = 0$, then N_{n+1} is C-essential in W and it means that $N_{n+2}, N_{n+3}, \ldots$ are essential submodules in W, a contradiction. By Lemma 3.2 we know that every direct summand of a C-extending module is C-extending. So C_1 is C-extending and we conclude that there exists a direct summand $C_2 \leq^{\oplus} W$ such that N_n is C-essential in C_2. So there exists a submodule D_2 such that $W = C_2 \oplus D_2$. Similarly $D_2 \neq 0$. By repeat this process, we find that $W = D_1 \oplus D_2 \oplus D_3 \oplus \cdots \oplus D_n \oplus C_{n+1}$, which is contradiction with the number of orthogonal idempotents in $End(W)$. □

A module W is called C_{11}-module or has C_{11} property, if every submodule has a complement in W which is a direct summand of W. Recall that a direct summand of a C_{11}-module is not C_{11}-module.

Theorem 3.16. *Let W be a C_{11}-module and U be a submodule of W. If for every direct summand V of W, $U \cap V$ is C-essential in U, then U is a C_{11}-module.*

Proof. Take a submodule A of U. Since $A \leq U \leq W$ and W is C_{11}-module, so there exists a complement N_2 of A such that $W = N_1 \oplus N_2$ for some $N_1 \leq M$. We know that $A \oplus N_2 \leq_e M$. Therefore $(N_2 \cap U) \cap A = U \cap (N_2 \cap A) = 0$. So we have $(N_2 \cap U) \oplus A = U \cap (N_2 \cap A) \leq_e U$. Since $(N_2 \cap U) \leq^{\oplus} U$, so by assumption, there exists a direct summand V of U such that $N_2 \cap U \leq_{ce} V$. Since $(N_2 \cap U) \cap A = 0$ and $(N_2 \cap U) \leq_{ce} V$, so $V \cap A = 0$. Also we have $(N_2 \cap U) \oplus A \leq_e U$. So $V \oplus A \leq_e U$. Hence V is a complement in U. It means that an arbitrary submodule A of U has a complement V in U such that is a direct summand of U. Hence U is C_{11}-module. □

Definition 3.17. We call that an A-module W is C-extending, if for every index set I and every direct sum $\oplus_{i \in I} X_i$ of submodules X_i of W, there exists a family $\{W_i | i \in I\}$, such that $\oplus_{i \in I}$ is a direct summand of W and for each $i \in I$, we have X_i is C-essential in W_i. If the index set has the cardinality n, we say that W is $n - C - extending$. If the index set if finite, we sat that W has $finite\ C - extending$ property.

Let $Y = \{G_\beta | \beta \in I\}$ of submodules of a module W. Then Y is called $local\ summand$ of W, if $\sum_{\beta \in I} G_\beta$ is direct and for every finite subset J of I, $\sum_{\beta \in J} G_\beta$ is a summand of W. If the $\sum_{\beta \in I} G_\beta$ is summand, we say that this $local\ summand\ is\ summand$[9].

Lemma 3.18. *Let W be an arbitrary A-module. Then*
(1) W is 1-C-extending if and only if M is C-extending.
(2) W is finite C-extending if and only if W is 2-C-extending if and only if W has C-extending property and C_3.
(3) W has C-extending property if and only if W has finite C-extending property and every local summand of W is summand.

Proof. (1) First, suppose that W has the C-extending property. We want to show that W is 1-C-extending. Since W is C-extending, so for every submodule X_i of W, there exists a direct summand W_i of W such that $X_i \leq_{ce} W_i$. It means that W is 1-C-extending. Conversely, if W is 1-C-extending, then for each submodule X_i of W, there exists a direct summand W_i such that $X_i \leq_{ce} W_i$, and this is exactly the definition of C-extending modules.

(2) If W is finite C-extending, then it is obvious that it is also 2-C-extending. Take 2 submodules, X_1 and X_2, by assumption, there exist 2 direct summand W_1 and W_2 such that $X_1 \leq_{ce} W_1$

and $X_2 \leq_{ce} W_2$. So by Theorem 2.2, $X_1 \oplus X_2 \leq_{ce} W_1 \oplus W_2$. Since W has C_3 property, It is clear that $W_1 \oplus W_2$ is also direct summand. Conversely is straightforward.

(3) It is clear that if W has a C-extending property, then it has a finite C-extending property. Consider a local summand $\sum_{\beta \in I} G_\beta$. Since W is C-extending and for every finite subset J of I, the sum $\sum_{\beta \in J} G_\beta$ is a summand, we conclude that the local summand $\sum_{\beta \in I} G_\beta$ is a summand. □

4 CSIP-EXTENDING MODULES

Extending modules are generalized by SIP-extending modules, which encompass both SIP modules and extending modules. A module W has summand intersection property (SIP) if the intersection of every pair of direct summands of W is a direct summand of W. If this property holds for any family of direct summands of W, we say W has strong summand intersection property (SSIP). A module W is referred to as an SIP-extending module if, for any two direct summands of W, their intersection is essential within some direct summand of W. Also, W is called an SSIP-extending module if the intersection of any collection of its direct summands is essential in a direct summand of W[9]. In this section, we introduce CSIP-extending modules.

Definition 4.1. Let W be an A-module. W is called a CSIP-extending module if the intersection of every pair of direct summands of W is C-essential in a direct summand of W.

Example 4.2. Every C-extending module is a CSIP-extending module. Let W be a C-extending module. Consider two direct summands $N, K \leq^{\oplus} W$. Since the intersection of N and K is a submodule of W, by definition, $K \cap N$ is C-essential in a direct summand of W. Hence, W is a CSIP-extending module.

Theorem 4.3. *Let W be an A-module. W is a CSIP-extending module if and only if for every pair of summands G and D with $\pi : W \to G$, the projection map, the kernel of the restriction map $\pi|_D$ is C-essential in a direct summand of W.*

Proof. Suppose that W is a CSIP-extending module. Let G and D be two arbitrary summands of W with the projection $\pi : W \to G$. Let $S' = Ker\pi$. We have $W = G \oplus G'$. Now, by assumption, $Ker\pi|_D = D \cap G'$ is C-essential in a direct summand of W. Conversely, suppose that for every pair of summands G and D with $\pi : W \to G$ the projection map, the kernel of the restriction map $\pi|_D$ is C-essential in a direct summand of W. Consider two direct summands G and D of W. Let G' be the complement of G in W and $\phi : W \to G'$ be the projection map. Now, by hypothesis, we have $Ker\phi|_D = G \cap D$ is C-essential in a direct summand of W. Hence, the proof is done. □

Theorem 4.4. *Let $W = \oplus W_i$, where W_i's are fully invariant CSIP-extending submodules of W. If each W_i is CSIP-extending, then W is CSIP-extending.*

Proof. For any direct summand $G \leq^{\oplus} W$, we can write $G = \oplus(G \cap W_i)$, because each W_i is fully invariant. Now take two direct summands G and D of W. We must show that the intersection of these direct summands is C-essential in a direct summand of W. So we have $G \cap D = [\oplus(G \cap W_i)] \cap [\oplus(D \cap W_i)] = \oplus[(G \cap W_i) \cap (D \cap W_i)]$. Since each W_i is CSIP-extending module, so there exists direct summand $L_i \leq^{\oplus} W$ such that $[(G \cap W_i) \cap (D \cap W_i)] \leq_{ce} L_i$. So W is a CSIP-extending module. □

Definition 4.5. Let W be an A-module. We define the C-closure of a submodule N of W as the maximal C-essential extension in W. It means that the maximal extension in which N is C-essential in it.

Lemma 4.6. *Let W be an A-module and $G \leq W$ which has a unique C-closure S in W. Then S is the sum of all submodules L of W containing N such that G is C-essential in L*

Proof. Let H be the sum of submodules L of W such that G is a C-essential submodule of L. Since G is C-essential in its C-closure S, we conclude that $S \subset H$. Conversely, let L be any submodule of W such that G is a C-essential submodule of L. Let L' be any C-closure of L in W. Clearly, L' is a C-closure of G in W and so $L' = S$. Thus, $L \subset S$. It follows that $H \subset S$ and hence $H = S$. □

Lemma 4.7. *Let W be a CSIP-extending module, and let N be a direct summand of W. Suppose that N is the unique C-closure in W of any of its C-essential submodules. Then N is also a CSIP-extending module.*

Proof. Suppose that $W = N \oplus N'$ for some submodule N' of W and N is the unique C-closure in W of any C-essential submodules. Let K and L be two direct summands of N. We must show that $K \cap L$ is C-essential in a direct summand of W. Since K and L are also direct summands of W and W is a CSIP-extending module, we conclude that there exists a direct summand $D_1 \leq^{\oplus} W$ such that $K \cap L \leq_{ce} D_1$. So K and L are essential in D_1. Since $N \cap N' = 0$ and K and L are essential in W, so $D_1 \cap N' = 0$. Also $N \cap T_1$ is essential in D_1, because $K \cap L \subset N \cap D_1$ and $K \cap L \leq_{ce} D_1$. Consider $\phi : W \to D_1$ be a canonical projection alone D_1'. Then suppose that $g : N \to T_1$ is a restriction of ϕ to N. Take $K = g^{-1}(N \cap T_1)$. Now K is an essential submodule of N, because $N \cap D_1 \leq_e D_1$. We claim that $K = (N \cap D_1) \oplus (N \oplus D_1')$. For this, consider $x \in K$. Since $K \subset N \leq W$ and $W = D \oplus D'$, so there exists two elements $y \in T_1$ and $z \in T_1'$ such that $x = y + z$. Now since $g(x) = y \in N$, hence $z \in N$ and we conclude that $x \in (N \cap D_1) \oplus (N' \cap D_1')$. The converse is clear. let $N \cap D_1' = Z$, then K is essential submodule in $D_1 \oplus Z$ because $K \leq_e N \cap D_1$ and $N \cap D_1 \leq_e D_1$. Since $K \leq_e D_1 \oplus Z$, so $K \leq_{ce} D_1 \oplus Z$. Since N is the unique C-closure of K, we must have $D_1 \oplus Z \subset N$. It follows that D_1 is a direct summand of N and $K \cap L$ is C-essential in D_1. □

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