Palestine Journal of Mathematics

Vol 15(2)(2026) , 397–420 © Palestine Polytechnic University-PPU 2026

INTRINSIC THEORY OF GENERALIZED APPROXIMATION MATSUMOTO METRICS

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Communicated by: Zafar Ahsan

Dedicated to the memory of Professor Nabil L. Youssef

MSC 2020 Classifications: Primary 53C60; Secondary 53B40.

Keywords and phrases: Approximation Matsumoto metric, Geodesic spray, Barthel connection, Berwald connection.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper

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Abstract. In this paper, we present a coordinate-free investigation of the generalized approximation Matsumoto metric. Specifically, for a Finsler metric (M, L) and a one-form $\mathfrak{B}$, we examine various geometric structures associated with the metric

$$\widetilde{L}(x,y) = L(x,y) + \epsilon_1 \mathfrak{B}(x,y) + \epsilon_2 \frac{\mathfrak{B}^2(x,y)}{L(x,y)} + \epsilon_3 \frac{\mathfrak{B}^3(x,y)}{L^2(x,y)},$$

where ϵ_1, ϵ_2, and ϵ_3 are positive constants. Our analysis expresses these structures in terms of the corresponding objects of the base metric L. In particular, we derive the metric tensor, Cartan tensor, and other associated geometric quantities for $\widetilde{L}$, and we establish conditions under which the metric tensor of $\widetilde{L}$ is non-degenerate. To determine the geodesic spray, Barthel connection, and Berwald connection of $\widetilde{L}(x,y)$, we specialize the one-form $\mathfrak{B}$ to one induced by a concurrent π-vector field. Additionally, we compute the curvature of the Barthel connection for $\widetilde{L}$, and provide a concrete example to illustrate our results.

1 Introduction

In 1941, G. Randers [17] introduced a special class of Finsler spaces defined by the transformation $\widetilde{L} = L + B$, where B is a one-form on a differentiable manifold M and L is a Riemannian metric. The motivation behind this construction was to develop a unified field theory that incorporated both gravity and electromagnetism. Later, in 1974, M. Matsumoto [10] extended Randers spaces by allowing L to be a general Finsler metric rather than restricting it to the Riemannian case. Since then, Finsler metrics of Randers type have attracted considerable attention, leading to numerous local studies and a rich body of literature on their geometric properties.

The theory of special Finsler spaces forms a dynamic and expansive domain of research, offering a wide array of applications in disciplines such as Physics and Biology. Within this framework, the π-tensor fields—particularly those related to torsion and curvature under the Cartan connection—often adhere to specific structural properties. These conditions give rise to numerous subclasses of Finsler spaces, resulting in a broader spectrum of special spaces compared to traditional Riemannian geometry.

A considerable body of research has focused on local (coordinate-dependent) methods to study these spaces, including investigations into approximation Matsumoto metrics [1, 11, 12, 28]. Nonetheless, intrinsic (coordinate-independent) analyses remain relatively scarce. Significant progress in this area has been made by A. Tamim, L. Youssef, and others, who have contributed substantially to the development of intrinsic approaches to special Finsler spaces (refer to [3, 20, 22, 25, 30, 34]).

Numerous applications of Matsumoto-type metrics have been explored in the literature; for example, see [2, 9, 14, 15, 16, 18, 19]. Adopting the pullback formalism in Finsler geometry, recent studies have intrinsically explored several generalized Finsler metrics, such as the first and second approximation Matsumoto metrics [4, 24], the Shen square metric [23], and the generalized Randers metric [26]. In this paper, following the same formalism, we present a coordinate-free investigation of a generalized approximation Matsumoto metric—a broader class that encompasses the previously studied metrics.

By a generalized approximation Matsumoto metric, we refer to the deformation of a given Finsler metric L (not necessarily Riemannian) through a one-form $\mathfrak{B}$, resulting in:

$$\widetilde{L}(x,y) = L(x,y) + \epsilon_1\mathfrak{B}(x,y) + \epsilon_2\frac{\mathfrak{B}^2(x,y)}{L(x,y)} + \epsilon_3\frac{\mathfrak{B}^3(x,y)}{L^2(x,y)},$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are positive constants.

Under this construction, we proceed to compute intrinsic geometric objects associated with $\widetilde{L}$, such as the supporting form $\widetilde{\ell}$, angular metric tensor $\widetilde{\hbar}$, Finsler metric $\widetilde{g}$, and the Cartan torsion $\widetilde{\mathbf{T}}$. We further identify the condition under which $\widetilde{g}$ is non-degenerate (see Theorem 4.2):

$$\epsilon_2(2L^3p^2 - 3\mathfrak{B}^2L) + L^3 + \epsilon_3(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3) \neq 0.$$

Furthermore, we establish the relationship between the original and modified Barthel connections Γ and $\widetilde{\Gamma}$, respectively. The associated canonical sprays G and $\widetilde{G}$ are shown to be related via (see Theorem 5.4):

$$\begin{aligned}\widetilde{G} =& G - \frac{L^2\left(\mathfrak{B}^2\left(12\mathfrak{B}^3\epsilon_3^2 + 3L\epsilon_3(5\mathfrak{B}^2\epsilon_2 + L^2) + 4\mathfrak{B}L^2\epsilon_2^2\right) + L^2\epsilon_1(8\mathfrak{B}^3\epsilon_3 - L^3 + 3\mathfrak{B}^2L\epsilon_2)\right)}{\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L(\mathfrak{B}\epsilon_2 + L\epsilon_1)\right)\left(\epsilon_2(2L^3p^2 - 3\mathfrak{B}^2L) + L^3 + \epsilon_3(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3)\right)}\mathcal{C} \\ &+ \frac{2L^4(3\mathfrak{B}\epsilon_3 + L\epsilon_2)}{\epsilon_2(2L^3p^2 - 3\mathfrak{B}^2L) + L^3 + \epsilon_3(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3)}\gamma\overline{p},\end{aligned}$$

where $\mathcal{C}$ denotes the Liouville vector field.

To illustrate, consider the Finsler manifold $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq 0\}$ and L be a conic Finsler metric given by

$$L = \sqrt{\frac{x_1^2y_3^3}{y_2} + y_1^2},$$

where $(x_1, x_2, x_3; y_1, y_2, y_3) \in \mathcal{T}M \subset \mathbb{R}^3 \times \mathbb{R}^3$. Here, the corresponding π-form has components $\mathbf{B}_2 = \mathbf{B}_3 = 0$, $\mathbf{B}_1 = x_1$, leading to the one-form $\mathfrak{B}(x,y) = x_1y_1$. Consequently, the deformed metric becomes:

$$\begin{aligned}\widetilde{L}(x,y) &= L(x,y) + \epsilon_1\,\mathfrak{B}(x,y) + \epsilon_2\,\frac{\mathfrak{B}^2(x,y)}{L(x,y)} + \epsilon_3\,\frac{\mathfrak{B}^3(x,y)}{L^2(x,y)} \\ &= \sqrt{\frac{x_1^2y_3^3}{y_2} + y_1^2} + \epsilon_1\,x_1y_1 + \epsilon_2\,\frac{(x_1y_1)^2}{\sqrt{\frac{x_1^2y_3^3}{y_2} + y_1^2}} + \epsilon_3\,\frac{(x_1y_1)^3}{\frac{x_1^2y_3^3}{y_2} + y_1^2},\end{aligned}$$

where $(x_1, x_2, x_3; y_1, y_2, y_3) \in TM \subset \mathbb{R}^3 \times \mathbb{R}^3$. The metric $\widetilde{L}$ characterizes a specific form of a generalized approximation Matsumoto metric defined over M.

2 Notations and Preliminaries

Let M be a smooth manifold of dimension n, and consider its tangent bundle (TM, π, M), along with the differential bundle $(TTM, d\pi, TM)$. The vertical bundle of TM, denoted by $V(TM)$, is defined as the kernel of $d\pi$, that is, $V(TM) = \ker(d\pi)$. Additionally, let $\pi^{-1}(TM)$ represent the pullback bundle of the tangent bundle over TM. These structures yield the following short exact sequence of vector bundle morphisms (see [6]):

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} TTM \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where $\mathcal{T}M$ is the slit tangent bundle, γ is the natural injection and $\rho := (\pi_{TM}, d\pi)$.

The almost tangent structure (also known as the vertical endomorphism) on TM is given by $J = \gamma \circ \rho$. Let $C^\infty(TM)$ denote the algebra of smooth real-valued functions on TM, and let $\mathcal{P}$ represent the $C^\infty(TM)$-module consisting of smooth sections of the pullback bundle $\pi^{-1}(TM)$. The sections of this pullback bundle are referred to as π-vector fields and will be denoted using barred letters such as $\overline{X}$.

The Liouville vector field, also called the fundamental π-vector field, is defined by $\mathcal{C} := \gamma\,\overline{\eta}$, where $\overline{\eta}(u) = (u, u)$ for every $u \in \mathcal{T}M$. Here, $\mathcal{T}M := TM \setminus \{0\}$ denotes the slit tangent bundle.

We begin by briefly reviewing some fundamental concepts and key properties of the Klein–Grifone framework in Finsler geometry. For a more comprehensive treatment, the reader is referred to [6, 7, 8].

A nonlinear connection on M is defined as a vector 1-form Γ on TM that is smooth on the slit tangent bundle $\mathcal{T}M = TM \setminus \{0\}$ and continuous on the entire tangent bundle TM. It satisfies the conditions

$$J\Gamma = J, \quad \Gamma J = -J.$$

Associated with such a connection are the horizontal and vertical projectors, denoted respectively by

$$h = \frac{1}{2}(I + \Gamma), \quad v = \frac{1}{2}(I - \Gamma).$$

The torsion of Γ is defined as

$$t := \frac{1}{2}[J, \Gamma],$$

while its curvature is given by

$$\mathfrak{R} := -\frac{1}{2}[h, h].$$

Let D be a linear connection on the pullback bundle $\pi^{-1}(TM)$. The corresponding connection map K is defined as

$$K : TTM \to \pi^{-1}(TM), \quad X \mapsto D_X\overline{\eta},$$

where $\overline{\eta}$ is the fundamental π-vector field. The horizontal space at a point $u \in TM$ is then given by

$$H_u(TM) := \{X \in T_u(TM) \mid K(X) = 0\}.$$

The connection D is called regular if every tangent space $T_u(TM)$ can be decomposed as a direct sum of vertical and horizontal components:

$$T_u(TM) = V_u(TM) \oplus H_u(TM), \quad \forall u \in TM.$$

For a regular connection, the restrictions $\rho|_{H(TM)}$ and $K|_{V(TM)}$ act as vector bundle isomorphisms. The inverse of ρ restricted to $H(TM)$, denoted by

$$\beta := (\rho|_{H(TM)})^{-1}.$$

Let D be a regular connection on the pullback bundle $\pi^{-1}(TM)$, with horizontal map β, and let $\mathbf{T}$ and $\mathbf{K}$ denote its classical torsion and curvature tensor fields, respectively. Then the following hold:

(i) Covariant Derivatives: For any π-tensor field A of type $(0, p)$, the horizontal and vertical covariant derivatives, denoted by $\overset{h}{D}$ and $\overset{v}{D}$, are defined as:

$$(\overset{h}{D}\,A)(\overline{X}, \overline{X}_1, \ldots, \overline{X}_p) := (D_{\beta\overline{X}}A)(\overline{X}_1, \ldots, \overline{X}_p),$$

$$(\overset{v}{D}\,A)(\overline{X}, \overline{X}_1, \ldots, \overline{X}_p) := (D_{\gamma\overline{X}}A)(\overline{X}_1, \ldots, \overline{X}_p).$$

(ii) Torsion Tensors: The torsion tensor $\mathbf{T}$ gives rise to the following derived tensors:

- (h)h-torsion: $Q(\overline{X}, \overline{Y}) := \mathbf{T}(\beta\overline{X}, \beta\overline{Y})$,

- (h)hv-torsion: $T(\overline{X},\overline{Y}) := \mathbf{T}(\gamma\overline{X},\beta\overline{Y})$,
- (h)v-torsion: $V(\overline{X},\overline{Y}) := \mathbf{T}(\gamma\overline{X},\gamma\overline{Y})$.

(iii) Curvature Tensors: The curvature tensor $\mathbf{K}$ leads to the following tensors:

- Horizontal curvature: $R(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X},\beta\overline{Y})\overline{Z}$,
- Mixed curvature: $P(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X},\gamma\overline{Y})\overline{Z}$,
- Vertical curvature: $S(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z}$.

(iv) Contracted Torsion Tensors: By applying these curvature tensors to the fundamental vector field $\overline{\eta}$, we obtain:

$$\widehat{R}(\overline{X},\overline{Y}) := R(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X},\overline{Y}) := P(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X},\overline{Y}) := S(\overline{X},\overline{Y})\overline{\eta}.$$

A Finsler manifold is defined as follows:

Definition 2.1. Let M be an n-dimensional smooth manifold. A Finsler manifold is a pair (M,L), where

$$L : TM \to \mathbb{R}$$

is a function, known as the Finsler structure, satisfying the following conditions:

(a) $L(u) > 0$ for all $u \in \mathcal{T}M := TM \setminus \{0\}$, and $L(0) = 0$,

(b) L is of class C^∞ on $\mathcal{T}M$ and continuous on TM,

(c) L is positively homogeneous of degree one in the directional variable y, i.e., $\mathcal{L}_{\mathcal{C}}L = L$,

(d) The 2-form $\Omega := dd_J E$, with $E := \frac{1}{2}L^2$, is non-degenerate (i.e., has maximal rank).

The function E is referred as the energy function associated with L. The Finsler metric g, induced by L on the pullback bundle $\pi^{-1}(TM)$, is defined by:

$$g(\rho X, \rho Y) := \Omega(JX, Y), \quad \forall X, Y \in \mathfrak{X}(TM).$$

If the function L satisfies the above conditions only on a conic subset $U \subset \mathcal{T}M$ (i.e., $u \in U \Rightarrow \lambda u \in U$ for all $\lambda > 0$), then (M,L) is called a conic Finsler manifold.

A semispray is a vector field G on TM, which is smooth on the slit tangent bundle $\mathcal{T}M := TM \setminus \{0\}$ and of class C^1 on all of TM, satisfying the condition $JG = \mathcal{C}$, where $\mathcal{C}$ denotes the Liouville vector field. When a semispray G is 2-homogeneous with respect to the fiber coordinates—that is, $[\mathcal{C}, G] = G$—it is referred as a spray.

Proposition 2.2 ([8, 7]). *For a Finsler manifold (M,L), the following are associated:*

(a) *The canonical spray G, defined by the equation $i_G\, dd_J E = -dE$.*

(b) *The Barthel connection Γ, given by $\Gamma = [J, G]$.*

The existence and uniqueness of a particular linear connection on a Finsler manifold are established in the following theorem:

Theorem 2.3 ([31]). *Let (M,L) be a Finsler manifold, with g denoting the metric derived from L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ that satisfies:*

(i) *∇ preserves the metric: $\nabla g = 0$,*

(ii) *The horizontal-horizontal torsion vanishes: $Q = 0$,*

(iii) *The horizontal-vertical torsion tensor T satisfies the symmetry condition:*

$$g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y}).$$

This unique connection ∇ is known as the Cartan connection associated with the Finsler structure (M,L).

The next lemma provides a useful characterization related to the metric properties of both the Cartan and Berwald connections.

Lemma 2.4 ([31]). *Let (M, L) be a Finsler manifold, and let β be the horizontal lift corresponding to the Cartan connection ∇. Then:*

(a) $(D^{\circ}_{\gamma\overline{X}}g)(\overline{Y}, \overline{Z}) = 2\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})$, *and* $\nabla_{\gamma\overline{X}}g = 0$,

(b) $(D^{\circ}_{\beta\overline{X}}g)(\overline{Y}, \overline{Z}) = -2\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z})$, *and* $\nabla_{\beta\overline{X}}g = 0$,

where $\widehat{\mathbf{P}}$ *is the (v)hv-torsion of type* $(0, 3)$*, defined by* $\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$*, with* $\widehat{P}$ *being the corresponding (v)hv-torsion tensor of the Cartan connection.*

For a more comprehensive treatment of the pullback approach in global Finsler geometry, we refer the reader to [13, 20, 27, 35, 36].

Lemma 2.5. *Let (M, L) be a Finsler manifold. Then, the following identities hold:*

(a) $d_J L(\gamma\overline{X}) = 0$, *and* $D^{\circ}_{\gamma\overline{X}}L = dL(\gamma\overline{X}) = d_J L(\beta\overline{X}) = \ell(\overline{X})$,

(b) $d_h L(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}L = dL(\beta\overline{X}) = 0$,

(c) $(D^{\circ}_{\gamma\overline{X}}\ell)(\overline{Y}) = (\nabla_{\gamma\overline{X}}\ell)(\overline{Y}) = L^{-1}\hbar(\overline{X}, \overline{Y})$,

(d) $dd_J E(\gamma\overline{X}, \beta\overline{Y}) = g(\overline{X}, \overline{Y})$,

where g is the Finsler metric induced by L, and ℓ is the normalized supporting element, given by $\ell := L^{-1} i_{\overline{\eta}} g$.

3 Generalized approximation Matsumoto metric

In this section, we undertake an intrinsic study of the approximation Matsumoto metric. Our approach involves generalizing the classical construction by substituting the underlying Riemannian metric with a more general Finsler metric. The resulting formulation is referred as the generalized approximation Matsumoto metric. This study serves to broaden and unify several earlier developments presented in [4, 23, 24, 26].

Definition 3.1 ([33]). Let (M, L) be a Finsler manifold and let D° denote the Berwald connection on the pullback bundle $\pi^{-1}(TM)$. A π-vector field $\overline{Y} \in \mathcal{X}(\pi(M))$ is said to be independent of the directional argument y if and only if $D^{\circ}_{\gamma\overline{X}}\overline{Y} = 0$ for every $\overline{X} \in \mathcal{X}(\pi(M))$. Similarly, a scalar or vector π-form ω is independent of y if it satisfies $D^{\circ}_{\gamma\overline{X}}\,\omega = 0$ for all $\overline{X} \in \mathcal{X}(\pi(M))$.

Definition 3.2. Let (M, L) be a Finsler manifold. Consider a deformation of the Finsler structure L given by:

$$\widetilde{L}(x, y) = L(x, y) + \epsilon_1 \mathfrak{B}(x, y) + \epsilon_2 \frac{\mathfrak{B}^2(x, y)}{L(x, y)} + \epsilon_3 \frac{\mathfrak{B}^3(x, y)}{L^2(x, y)}, \tag{3.1}$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are positive constants, and $\mathfrak{B}(x, y) := \mathbf{B}(\overline{\eta})$ is a scalar π-form that does not depend on the directional variable y.

If $\widetilde{L}$ satisfies the properties of a Finsler structure on M, it is called a generalized approximation Matsumoto metric.

In particular, by choosing specific values for $\epsilon_1, \epsilon_2, \epsilon_3$, $\widetilde{L}$ reduces to various well-known metrics:

(i) A Generalized Randers Metric (GRM) when $\epsilon_1 = 1$, $\epsilon_2 = 0$, $\epsilon_3 = 0$, [26].

(ii) A Generalized Shen Square Metric (GSSM) when $\epsilon_1 = 2$, $\epsilon_2 = 1$, $\epsilon_3 = 0$, [23].

(iii) A Generalized First Approximation Matsumoto Metric (GFAMM) when $\epsilon_1 = 1$, $\epsilon_2 = 1$, $\epsilon_3 = 0$, [4].

(iv) A Generalized Second Approximation Matsumoto Metric (GSAMM) when all $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, [24].

To study the geometric objects associated with $\widetilde{L}$, we need the following lemmas.

Lemma 3.3. *Under the change $L \mapsto \widetilde{L}$, the vertical counterpart for Berwald connection $D^{\circ}_{\gamma\overline{X}}\overline{Y}$ is invariant. i.e. $\widetilde{D}^{\circ}_{\gamma\overline{X}}\overline{Y} = D^{\circ}_{\gamma\overline{X}}\overline{Y}$.*

Proof. Under the change $L \mapsto \widetilde{L}$, the difference between the horizontal maps $\widetilde{\beta}$ and β is a vertical vector field, means that $\widetilde{\beta} = \beta + \gamma\overline{\mu}$, for some π-vector field $\overline{\mu}$. Hence, the proof follows from [31] by the property

$$D^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \beta\overline{Y}],$$

together with the facts that $\rho \circ \gamma$ vanishes identically and the vertical distribution is integrable. In more details

$$\widetilde{D}^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \widetilde{\beta}\overline{Y}] = \rho[\gamma\overline{X}, \beta\overline{Y}] + \rho[\gamma\overline{X}, \gamma\overline{\mu}] = \rho[\gamma\overline{X}, \beta\overline{Y}] = D^{\circ}_{\gamma\overline{X}}\overline{Y}.$$

Hence, the result follows. □

Lemma 3.4. *Let (M, L) be a Finsler manifold equipping a scalar π-form $\mathbf{B}$ which is independent of the directional argument y, and $\overline{p}$ its the associated π-vector field given by $i_{\overline{p}}\, g := \mathbf{B}$. Then, the one form $\mathfrak{B}(x, y) := \mathbf{B}(\overline{\eta})$ has the following properties*

(a) $d_J\mathfrak{B}(\gamma\overline{X}) = 0, D^{\circ}_{\gamma\overline{X}}\mathfrak{B} = d\mathfrak{B}(\gamma\overline{X}) = d_J\mathfrak{B}(\beta\overline{X}) = \mathbf{B}(\overline{X})$.

(b) $d_h\,\mathfrak{B}(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}\,\mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) = L\,\ell(D^{\circ}_{\beta\overline{X}}\,\overline{p})$, $d\mathfrak{B}(G) = L\,\ell(D^{\circ}_{G}\,\overline{p})$.

(c) $D^{\circ}_{\gamma\overline{X}}\,\overline{p} = -2T(\overline{X}, \overline{p})$.

Proof. The proof follows from the facts that $\rho\circ\gamma$ and $K\circ\beta$ vanish identically, $\rho\circ\beta = id_{\mathfrak{X}(\pi(M))}$, $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$, $i_{\overline{\eta}}\widehat{\mathbf{P}} = 0$, $D^{\circ}_{\gamma\overline{X}}\mathbf{B} = 0$ and taking into account Definition 3.1 and Lemma 2.4. In more details, we have the following.
(a) For $d_J\mathfrak{B}(\gamma\overline{X})$, we have

$$d_J\mathfrak{B}(\gamma\overline{X}) = (J \circ \gamma\overline{X}) \cdot \mathfrak{B} = \gamma\,(\rho \circ \gamma)\overline{X} \cdot \mathfrak{B} = 0.$$

Moreover, $d_J\mathfrak{B}(\gamma\overline{X})$ can be obtained as follows:

$$\begin{aligned} d_J\mathfrak{B}(\beta\overline{X}) &= J\,(\beta\overline{X} \cdot \mathfrak{B}) = \gamma\,(\rho \circ \beta)\overline{X} \cdot \mathfrak{B} = \gamma\overline{X} \cdot \mathfrak{B} \\ &= (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{\eta}) + \mathbf{B}(D^{\circ}_{\gamma\overline{X}}\overline{\eta}) \\ &= \mathbf{B}(\overline{X}). \end{aligned}$$

(b) One can see that

$$\begin{aligned} d_h\mathfrak{B}(\beta\overline{X}) &= (\beta \circ \rho \circ \beta\overline{X}) \cdot \mathfrak{B} = \beta\overline{X} \cdot \mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) \\ &= \beta\overline{X} \cdot g(\overline{p}, \overline{\eta}) = (D^{\circ}_{\beta\overline{X}}\, g)(\overline{p}, \overline{\eta}) + g(D^{\circ}_{\beta\overline{X}}\,\overline{p}, \overline{\eta}) + g(\overline{p}, D^{\circ}_{\beta\overline{X}}\overline{\eta}) \\ &= -2\widehat{\mathbf{P}}(\overline{X}, \overline{p}, \overline{\eta}) + g(D^{\circ}_{\beta\overline{X}}\,\overline{p}, \overline{\eta}) + 0 \\ &= L\,\ell(D^{\circ}_{\beta\overline{X}}\,\overline{p}). \end{aligned}$$

That is, we get that $d\mathfrak{B}(G) = L\,\ell(D^{\circ}_{G}\,\overline{p})$.
(c) We have

$$\begin{aligned} 0 &= (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y}) \\ &= D^{\circ}_{\gamma\overline{X}}\mathbf{B}(\overline{Y}) - \mathbf{B}(D^{\circ}_{\gamma\overline{X}}\overline{Y}) \\ &= D^{\circ}_{\gamma\overline{X}}\, g(\overline{p}, \overline{Y}) - g(\overline{p}, D^{\circ}_{\gamma\overline{X}}\overline{Y}) \\ &= (D^{\circ}_{\gamma\overline{X}}\, g)(\overline{p}, \overline{Y}) + g(D^{\circ}_{\gamma\overline{X}}\,\overline{p}, \overline{Y}) \\ &= 2\mathbf{T}(\overline{p}, \overline{X}, \overline{Y}) + g(D^{\circ}_{\gamma\overline{X}}\,\overline{p}, \overline{Y}). \end{aligned}$$

Thus, we conclude that $D^{\circ}_{\gamma\overline{X}}\,\overline{p} = -2T(\overline{X}, \overline{p})$. □

Adopting to the normalized supporting element ℓ and the angular metric tensor $\hbar$, we have the following proposition.

Proposition 3.5. *Under the generalized approximation Matsumoto metric* (3.1)*, we have*

(i) The supporting form $\widetilde{\ell}$ *and* ℓ *are related by*

$$\widetilde{\ell}(\overline{X}) = \frac{1}{L^3}\left(-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2L\epsilon_2\right)\ell(\overline{X}) + \frac{1}{L^2}\left(\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right) + \epsilon_1L^2\right)\mathbf{B}(\overline{X}). \tag{3.2}$$

(ii) The angular metric tensors $\widetilde{\hbar}$ *and* $\hbar$ *are related by*

$$\begin{aligned}\widetilde{\hbar}(\overline{X},\overline{Y}) \;=\;& \frac{1}{L^6}\left(-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2L\epsilon_2\right)\hbar(\overline{X},\overline{Y})\\ &+ \left(3\mathfrak{B}\epsilon_3 + L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2L\epsilon_2\right)\Big(\frac{2}{L^4}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+\frac{2\mathfrak{B}^2}{L^6}\ell(\overline{X})\,\ell(\overline{Y}) - \frac{2\mathfrak{B}^2}{L^6}\left\{\mathbf{B}(\overline{X})\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\ell(\overline{X})\right\}\Big).\end{aligned} \tag{3.3}$$

Proof. Under the generalized approximation Matsumoto metric (3.1), taking Lemma 3.4 into account, we have the following:

(i) Due to the facts that $\rho\circ\gamma = 0$ and that $\rho\circ\beta = \rho\circ\widetilde{\beta} = id_{\mathfrak{X}(\pi(M))}$, it follows that

$$\begin{aligned}\widetilde{\ell}(\overline{X}) &= d_J\widetilde{L}(\widetilde{\beta}\overline{X}) = d_J\widetilde{L}(\beta\overline{X})\\ &= \frac{\partial\widetilde{L}}{\partial L}\,d_JL(\beta\overline{X}) + \frac{\partial\widetilde{L}}{\partial\mathfrak{B}}\,d_J\mathfrak{B}(\beta\overline{X})\\ &= \frac{1}{L^3}\left(-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2L\epsilon_2\right)\,\ell(\overline{X}) + \frac{1}{L^2}\left(\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right) + \epsilon_1L^2\right)\,\mathbf{B}(\overline{X}).\end{aligned}$$

(ii) Applying the previous item, Lemma 3.4, Lemma 2.5, together with Lemma 3.3, one can show that

$$\begin{aligned}\widetilde{\hbar}(\overline{X},\overline{Y}) &= \widetilde{L}(\widetilde{D}^{\circ}_{\gamma\overline{X}}\widetilde{\ell})(\overline{Y}) = \widetilde{L}(D^{\circ}_{\gamma\overline{X}}\widetilde{\ell})(\overline{Y})\\ &= \widetilde{L}\,D^{\circ}_{\gamma\overline{X}}\left\{\left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2L\epsilon_2}{L^3}\right)\ell(\overline{Y}) + \left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)}{L^2} + \epsilon_1\right)\mathbf{B}(\overline{Y})\right\}\\ &= \left(L + \epsilon_1\mathfrak{B} + \epsilon_2\frac{\mathfrak{B}^2}{L} + \epsilon_3\frac{\mathfrak{B}^3}{L^2}\right)\left(D^{\circ}_{\gamma\overline{X}}\left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2L\epsilon_2}{L^3}\right)\right)\ell(\overline{Y})\\ &+ \left(L + \epsilon_1\mathfrak{B} + \epsilon_2\frac{\mathfrak{B}^2}{L} + \epsilon_3\frac{\mathfrak{B}^3}{L^2}\right)\left(D^{\circ}_{\gamma\overline{X}}\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)}{L^2} + \epsilon_1\right)\right)\mathbf{B}(\overline{Y})\\ &+ \left(L + \epsilon_1\mathfrak{B} + \epsilon_2\frac{\mathfrak{B}^2}{L} + \epsilon_3\frac{\mathfrak{B}^3}{L^2}\right)\left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2L\epsilon_2}{L^3}\right)\,(D^{\circ}_{\gamma\overline{X}}\ell)(\overline{Y})\\ &+ \left(L + \epsilon_1\mathfrak{B} + \epsilon_2\frac{\mathfrak{B}^2}{L} + \epsilon_3\frac{\mathfrak{B}^3}{L^2}\right)\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)}{L^2} + \epsilon_1\right)\,(D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y}).\end{aligned}$$

Hence, the result follows. □

4 The metric and Cartan tensors

In this section, we derive several geometric quantities related to the deformed metric $\widetilde{L}(x,y)$, expressed in terms of those corresponding to the original metric L. The next proposition establishes the relation between the fundamental tensor g and its counterpart $\widetilde{g}$.

Proposition 4.1. *The Finsler metric $\widetilde{g}$ associated with the special generalized approximation Matsumoto metric* (3.1) *is given by the following relation:*

$$\begin{aligned}
\widetilde{g}(\overline{X},\overline{Y}) \;=\;& \frac{1}{L^6}\left(-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)g(\overline{X},\overline{Y})\\
&+\frac{1}{L^4}\Big(\left(\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)+\epsilon_1L^2\right)^2+2\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)\Big)\\
&\times\mathbf{B}(\overline{X})\mathbf{B}(\overline{Y})+\frac{1}{L^6}\Big(\mathfrak{B}^3\left(12\mathfrak{B}^3\epsilon_3^2+3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2+L^2\right)+4\mathfrak{B}L^2\epsilon_2^2\right)\\
&+\mathfrak{B}L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3-L^3+3\mathfrak{B}^2L\epsilon_2\right)\Big)\ell(\overline{X})\,\ell(\overline{Y})+\frac{1}{L^5}\Big(L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2\right)\\
&-\mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3+L^3\right)+4\mathfrak{B}L^2\epsilon_2^2+15\mathfrak{B}^2L\epsilon_2\epsilon_3\right)\Big)\left\{\mathbf{B}(\overline{X})\ell(\overline{Y})+\mathbf{B}(\overline{Y})\ell(\overline{X})\right\}.
\end{aligned}$$

Consequently, the Cartan torsion $\widetilde{\mathbf{T}}$ of the special generalized approximation Matsumoto metric has the form

$$\begin{aligned}
2\widetilde{\mathbf{T}}(\overline{X},\quad&\overline{Y},\overline{Z})=\frac{2}{L^6}\left(-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)\mathbf{T}(\overline{X},\overline{Y},\overline{Z})\\
&+\frac{1}{L^7}\left(\mathfrak{B}^3\left(12\mathfrak{B}^3\epsilon_3^2+3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2+L^2\right)+4\mathfrak{B}L^2\epsilon_2^2\right)+\mathfrak{B}L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3-L^3+3\mathfrak{B}^2L\epsilon_2\right)\right)\\
&\times\left\{\hbar(\overline{X},\overline{Z})\,\ell(\overline{Y})+\hbar(\overline{Y},\overline{Z})\,\ell(\overline{X})\right\}+\frac{1}{L^6}\Big(L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2\right)\\
&-\mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3+L^3\right)+4\mathfrak{B}L^2\epsilon_2^2+15\mathfrak{B}^2L\epsilon_2\epsilon_3\right)\Big)\left\{\mathbf{B}(\overline{X})\,\hbar(\overline{Y},\overline{Z})+\mathbf{B}(\overline{Y})\,\hbar(\overline{X},\overline{Z})\right\}\\
&+D^{\circ}_{\gamma\overline{Z}}\left(\frac{1}{L^6}\left(-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)\right)\;g(\overline{X},\overline{Y})\\
&+D^{\circ}_{\gamma\overline{Z}}\Big(\frac{1}{L^4}(\mathfrak{B}(3\mathfrak{B}\epsilon_3+2L\epsilon_2)+\epsilon_1L^2)^2+2(3\mathfrak{B}\epsilon_3+L\epsilon_2)(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1 \qquad (4.1)\\
&+\mathfrak{B}^2L\epsilon_2))\Big)\mathbf{B}(\overline{X})\mathbf{B}(\overline{Y})+D^{\circ}_{\gamma\overline{Z}}\Big(\frac{1}{L^6}\mathfrak{B}^3(12\mathfrak{B}^3\epsilon_3^2+3L\epsilon_3(5\mathfrak{B}^2\epsilon_2+L^2)+4\mathfrak{B}L^2\epsilon_2^2)\\
&+\mathfrak{B}L^2\epsilon_1(8\mathfrak{B}^3\epsilon_3-L^3+3\mathfrak{B}^2L\epsilon_2))\Big)\ell(\overline{X})\,\ell(\overline{Y})+D^{\circ}_{\gamma\overline{Z}}\Big(\frac{1}{L^5}(L^2\epsilon_1(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2)\\
&-\mathfrak{B}^2(3\epsilon_3(4\mathfrak{B}^3\epsilon_3+L^3)+4\mathfrak{B}L^2\epsilon_2^2+15\mathfrak{B}^2L\epsilon_2\epsilon_3))\Big)\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

where $D^{\circ}_{\gamma\overline{X}}f(L,\mathfrak{B})=d_Jf(\beta\overline{X})=\frac{\partial f}{\partial L}\,\ell(\overline{X})+\frac{\partial f}{\partial\mathfrak{B}}\,\mathbf{B}(\overline{X})$.

Proof. In view of the special generalized approximation Matsumoto metric *(3.1)*, using Proposition 3.5 together with the fact that $\widetilde{\hbar}:=\widetilde{g}-\widetilde{\ell}\otimes\widetilde{\ell}$, then we have

$$\begin{aligned}
\widetilde{g}(\overline{X},\overline{Y}) \;=\;& \widetilde{\hbar}(\overline{X},\overline{Y})+\widetilde{\ell}(\overline{X})\,\widetilde{\ell}(\overline{Y})\\
=\;& \frac{1}{L^6}\left(-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)\hbar(\overline{X},\overline{Y})\\
&+\frac{2}{L^4}\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\
&+\frac{2}{L^6}\mathfrak{B}^2\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)\ell(\overline{X})\,\ell(\overline{Y})\\
&-\frac{2\mathfrak{B}^2}{L^6}\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)\left\{\mathbf{B}(\overline{X})\ell(\overline{Y})+\mathbf{B}(\overline{Y})\ell(\overline{X})\right\}\\
&+\left(\frac{1}{L^3}\left(-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2\right)\ell(\overline{X})+\frac{1}{L^2}\left(\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)+\epsilon_1L^2\right)\mathbf{B}(\overline{X})\right)\\
&\times\left(\frac{1}{L^3}\left(-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2\right)\,\ell(\overline{Y})+\frac{1}{L^2}\left(\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)+\epsilon_1L^2\right)\,\mathbf{B}(\overline{Y})\right)
\end{aligned}$$

Consequently, using the expression of the metric $\widetilde{g}$, taking into account the fact that $(D^{\circ}_{\gamma\overline{Z}}g)(\overline{X},\overline{Y}) = 2\mathbf{T}(\overline{X},\overline{Y},\overline{Z})$ (Lemma 2.4), it follows the expression of the Cartan torsion $\widetilde{\mathbf{T}}$ of the special generalized approximation Matsumoto metric. This completes the proof. $\square$

Theorem 4.2. *The metric tensor* $\widetilde{g}$ *of* $\widetilde{L}$ *is non-degenerate if and only if*

$$\epsilon_2\left(2L^3p^2 - 3\mathfrak{B}^2L\right) + L^3 + \epsilon_3\left(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3\right) \neq 0. \tag{4.2}$$

That is, the generalized approximation Matsumoto metric is a Finsler structure (or, conic Finsler structure) if and only if the condition (4.2) *is satisfied.*

Proof. Let $\widetilde{g}$ denote the Finsler metric corresponding to the generalized approximation Matsumoto metric, as defined in equation (3.1). To verify that $\widetilde{g}$ is non-degenerate, assume that $\widetilde{g}(\overline{X},\overline{Y}) = 0$ for every $\overline{X} \in \pi^{-1}(TM)$. Applying Proposition 4.1, we derive

$$\begin{aligned} 0 \;=\; & \frac{1}{L^6}\left(-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2L\epsilon_2\right) g(\overline{X},\overline{Y}) \\ & + \frac{1}{L^4}\Big(\left(\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right) + \epsilon_1L^2\right)^2 + 2\left(3\mathfrak{B}\epsilon_3 + L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2L\epsilon_2\right)\Big) \\ & \times\mathbf{B}(\overline{X})\mathbf{B}(\overline{Y}) + \frac{1}{L^6}\Big(\mathfrak{B}^3\left(12\mathfrak{B}^3\epsilon_3^2 + 3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2 + L^2\right) + 4\mathfrak{B}L^2\epsilon_2^2\right) \\ & + \mathfrak{B}L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3 - L^3 + 3\mathfrak{B}^2L\epsilon_2\right)\Big)\ell(\overline{X})\,\ell(\overline{Y}) + \frac{1}{L^5}\Big(L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3 + L^3 - 3\mathfrak{B}^2L\epsilon_2\right) \\ & - \mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3 + L^3\right) + 4\mathfrak{B}L^2\epsilon_2^2 + 15\mathfrak{B}^2L\epsilon_2\epsilon_3\right)\Big)\left\{\mathbf{B}(\overline{X})\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\ell(\overline{X})\right\}. \end{aligned}$$

From which, by substituting $\overline{X} = \overline{p}$, noting that $\ell(\overline{p}) = \frac{\mathfrak{B}}{L}$ and $\mathbf{B}(\overline{p}) = g(\overline{p},\overline{p}) =: p^2$, one can show that

$$\xi_1\,\ell(\overline{Y}) + \xi_2\,\mathbf{B}(\overline{Y}) = 0, \tag{4.3}$$

where

$$\begin{aligned} \xi_1 \;:=\; & \frac{1}{L^7}\Big(\mathfrak{B}^2\left(-12\mathfrak{B}^3\epsilon_3^2 - 3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2 + L^2\right) - 4\mathfrak{B}L^2\epsilon_2^2\right)(Lp - \mathfrak{B})(\mathfrak{B} + Lp) \\ & + \left(L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3 + L^3 - 3\mathfrak{B}^2L\epsilon_2\right)\right)(Lp - \mathfrak{B})(\mathfrak{B} + Lp)\Big), \\ \xi_2 \;:=\; & \frac{1}{L^6}\Big(L^6p^2\epsilon_1^2 + L^6 + 2\epsilon_2\left(L^6p^2 + \mathfrak{B}^3L\epsilon_3\left(10L^2p^2 - 9\mathfrak{B}^2\right)\right) + \epsilon_2^2\left(6\mathfrak{B}^2L^4p^2 - 5\mathfrak{B}^4L^2\right) \\ & + 2\mathfrak{B}L^2\epsilon_1\left(\epsilon_2\left(3L^3p^2 - 2\mathfrak{B}^2L\right) + L^3 + \mathfrak{B}\epsilon_3\left(6L^2p^2 - 5\mathfrak{B}^2\right)\right) \\ & + \mathfrak{B}\epsilon_3\left(6L^5p^2 - 4\mathfrak{B}^2L^3 + \mathfrak{B}^3\epsilon_3\left(15L^2p^2 - 14\mathfrak{B}^2\right)\right)\Big). \end{aligned}$$

Similarly, by substituting $\overline{X} = \overline{\eta}$, taking into account the facts that $\ell(\overline{\eta}) = L$ and $\mathbf{B}(\overline{\eta}) = \mathfrak{B}$, we obtain

$$\xi_3\,\ell(\overline{Y}) + \xi_4\,\mathbf{B}(\overline{Y}) = 0, \tag{4.4}$$

with

$$\begin{aligned} \xi_3 \;:=\; & \frac{\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L\left(\mathfrak{B}\epsilon_2 + L\epsilon_1\right)\right)\left(L^3 - \mathfrak{B}^2\left(2\mathfrak{B}\epsilon_3 + L\epsilon_2\right)\right)}{L^5}, \\ \xi_4 \;:=\; & \frac{\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L\left(\mathfrak{B}\epsilon_2 + L\epsilon_1\right)\right)\left(L^2\epsilon_1 + \mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)\right)}{L^4}. \end{aligned}$$

Now, the system of the algebraic equations (4.3) and (4.4) has non-trivial solution for $\ell(\overline{Y})$ or $\mathbf{B}(\overline{Y})$ if and only if

$$\frac{\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2L\epsilon_2\right)^3}{L^{11}}\left(\epsilon_2\left(2L^3p^2 - 3\mathfrak{B}^2L\right) + L^3 + \epsilon_3\left(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3\right)\right) = 0.$$

Hence, as $\widetilde{L} \neq 0$ over $\mathcal{T}M$, then we conclude that

$$\epsilon_2 \left(2L^3p^2 - 3\mathfrak{B}^2L\right) + L^3 + \epsilon_3 \left(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3\right) = 0.$$

Therefore, $\ell(\overline{Y}) = \mathbf{B}(\overline{Y}) = 0$ if and only if the Finsler structure L and the π-form $\mathfrak{B}$ satisfy the condition

$$\epsilon_2 \left(2L^3p^2 - 3\mathfrak{B}^2L\right) + L^3 + \epsilon_3 \left(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3\right) \neq 0.$$

Combining this result with the initial relation in the proof, the assumption that $\widetilde{L} \neq 0$, and the known non-degeneracy of the original Finsler metric g, we conclude that $\overline{Y}$ must be zero. Therefore, the generalized approximation Matsumoto metric tensor $\widetilde{g}$ is non-degenerate if and only if the condition given in (4.2) holds. This concludes the proof. □

Form now on, we consider that the generalized approximation Matsumoto metric $\widetilde{L}$ satisfies the condition (4.2).

5 Geodesic spray and Berwald connection

To simplify the resulting expressions and facilitate the computation of the geodesic spray for the generalized approximation Matsumoto metric, we consider a specific case of the 1-form. This leads us to the following definition.

Definition 5.1. [33] Let (M, L) be a Finsler manifold. A π-vector field $\overline{p} \in \mathfrak{X}(\pi(M))$ is said to be concurrent if it satisfies the following conditions:

$$\nabla_{\beta\overline{X}}\,\overline{p} = -\overline{X} = D^{\circ}_{\beta\overline{X}}\,\overline{p}, \quad \nabla_{\gamma\overline{X}}\,\overline{p} = 0 = D^{\circ}_{\gamma\overline{X}}\,\overline{p}. \tag{5.1}$$

Furthermore, if $\mathbf{B}$ denotes the π-form corresponding to $\overline{p}$ through the metric duality given by g, that is, $\mathbf{B} = i_{\overline{p}}\,g$, then $\mathbf{B}$ satisfies the following properties:

$$(\nabla_{\beta\overline{X}}\mathbf{B})(\overline{Y}) = -g(\overline{X}, \overline{Y}) = (D^{\circ}_{\beta\overline{X}}\mathbf{B})(\overline{Y}), \quad (\nabla_{\gamma\overline{X}}\mathbf{B})(\overline{Y}) = 0 = (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y}).$$

If the π-vector field $\overline{p}(x, y)$ associated with the given scalar π-form $\mathbf{B}$, means that $\mathbf{B}$ is a concurrent vector field over (M, L), then $\widetilde{L}(x, y)$ will be called a special generalized approximation Matsumoto metric.

In [33], Nabil et al. investigated an intrinsic study of concurrent π-vector fields in Finsler geometry. Moreover, they characterized the concurrent π-vector fields. That is, we have the following.

Lemma 5.2. *Let (M, L) be a Finsler manifold equipping a scalar π-form $\mathbf{B}$ which is independent of the directional argument y, and $\overline{p}$ its the associated π-vector field is concurrent π-vector field. Then $d\mathfrak{B}(G) = -L^2$.*

Proof. The proof follows by applying Lemma 3.4 (b) and taking Definition 5.1 into account. □

Theorem 5.3. *[33] A concurrent π-vector field $\overline{p}$ and its associated π-form **B** are independent of the directional argument y.*

Now, we find the relationship between the canonical (geodesic) spray $\widetilde{G}$ corresponding to the special generalized approximation Matsumoto metric $\widetilde{L}$, in terms of the geodesic spray G of L. Precisely, we have the following theorem.

Theorem 5.4. *The canonical spray $\widetilde{G}$ associated with the special generalized approximation Matsumoto metric (3.1), is given by*

$$\begin{aligned}\widetilde{G} \;=\; & G - \frac{L^2\left(\mathfrak{B}^2\left(12\mathfrak{B}^3\epsilon_3^2 + 3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2 + L^2\right) + 4\mathfrak{B}L^2\epsilon_2^2\right) + L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3 - L^3 + 3\mathfrak{B}^2L\epsilon_2\right)\right)}{\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L\left(\mathfrak{B}\epsilon_2 + L\epsilon_1\right)\right)\left(\epsilon_2\left(2L^3p^2 - 3\mathfrak{B}^2L\right) + L^3 + \epsilon_3\left(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3\right)\right)}\,\mathcal{C} \\ & + \frac{2L^4\left(3\mathfrak{B}\epsilon_3 + L\epsilon_2\right)}{\epsilon_2\left(2L^3p^2 - 3\mathfrak{B}^2L\right) + L^3 + \epsilon_3\left(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3\right)}\,\gamma\overline{p},\end{aligned}$$

where, $\mathcal{C}$ is the Liouville vector field defined by $\mathcal{C} := \gamma\,\overline{\eta}$ and $p^2 := \mathbf{B}(\overline{p}) = g(\overline{p}, \overline{p})$.

Proof. Due to the special generalized approximation Matsumoto metric *(3.1)*, taking into account the expression of the exterior π-form $\widetilde{\Omega} := \frac{1}{2} dd_J \widetilde{L}^2$, the fact that the difference between two sprays is a vertical vector field (i.e. $\widetilde{G} = G + \gamma\overline{\mu}$, for some π-vector field $\overline{\mu}$) and using Proposition 2.2, one can show that

$$\begin{aligned} -d\widetilde{E}(X) &= i_{\widetilde{G}}\,\widetilde{\Omega}(X) = i_{G+\gamma\overline{\mu}}\,(\tfrac{1}{2}\,dd_J\widetilde{L}^2)(X) \\ &= \tfrac{1}{2} i_G\, dd_J\widetilde{L}^2(X) + \tfrac{1}{2} i_{\gamma\overline{\mu}}\, dd_J\widetilde{L}^2(X). \end{aligned} \tag{5.2}$$

Therefore, after some computation together the fact thats $\beta\overline{\eta} = G$ and $X = hX + vX = \beta\rho X + \gamma K X$, together with Lemma 3.4, we have

$$\begin{aligned} d\widetilde{E}(X) &= \frac{1}{2} d\widetilde{L}^2(X) = \widetilde{L}\, d\widetilde{L}(X) \\ &= \left(L + \epsilon_1\,\mathfrak{B} + \epsilon_2\,\frac{\mathfrak{B}^2}{L} + \epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2}{L^3}\right) dL(X) \\ &\quad + \left(L + \epsilon_1\,\mathfrak{B} + \epsilon_2\,\frac{\mathfrak{B}^2}{L} + \epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2)}{L^2} + \epsilon_1\right) d\mathfrak{B}(X). \\ \frac{1}{2}\, i_G\, dd_J\widetilde{L}^2(X) &= \frac{1}{2}\{dd_J\widetilde{L}^2(\beta\overline{\eta}, X)\} \\ &= \frac{1}{2}\left\{G\cdot d_J\widetilde{L}^2(X) - X\cdot d_J\widetilde{L}^2(G) - d_J\widetilde{L}^2[G,X]\right\} \\ &= \frac{1}{2}\left\{G\cdot(2\widetilde{L}\widetilde{\ell}(\rho X)) - X\cdot(2\widetilde{L}\widetilde{\ell}(\overline{\eta})) - 2\widetilde{L}\ell(\rho[G,X])\right\} \\ &= ((G\cdot\widetilde{L})\,\widetilde{\ell}(\rho X) + \widetilde{L}\,G\cdot\widetilde{\ell}(\rho X)) - (X\cdot\widetilde{L}^2) - \widetilde{L}\,\ell(\rho[G,X]). \end{aligned}$$

From which taking into account Lemmas 3.4, and the following facts

$$\begin{aligned} G\cdot\widetilde{L} &= d\widetilde{L}(G) \\ &= \left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2}{L^3}\right) dL(G) + \left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2)}{L^2} + \epsilon_1\right) d\mathfrak{B}(G) \\ &= -\left(\mathfrak{B}\,(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2) + \epsilon_1\, L^2\right), \\ X\cdot\widetilde{L} &= d\widetilde{L}(X) \\ &= \left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2}{L^3}\right) dL(X) + \left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2)}{L^2} + \epsilon_1\right) d\mathfrak{B}(X), \\ \widetilde{\ell}(\overline{X}) &= \left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2}{L^3}\right) \ell(\overline{X}) + \left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2)}{L^2} + \epsilon_1\right) \mathbf{B}(\overline{X}), \\ \rho[G,X] &= \rho[G, hX + vX] = D^\circ_G \rho X - KX, \\ (D^\circ_G\,\mathbf{B})(\overline{X}) &= -g(\overline{X}, \overline{\eta}) = -L\,\ell(\overline{X}), \\ (D^\circ_G\,\ell)(\overline{X}) &= (\nabla_G\,\ell)(\overline{X}) = 0, \\ d\mathfrak{B}(X) &= \mathbf{B}(KX) - L\ell(\rho X), \\ dL(X) &= dL(\gamma KX) = \ell(KX). \end{aligned}$$

The above relation reduces to

$$\begin{aligned} \frac{1}{2}\, i_G\, dd_J\widetilde{L}^2(X) \quad &= -\left(\mathfrak{B}\,(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2) + \epsilon_1\, L^2\right)\left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2}{L^3}\right) \ell(\rho X) \\ &\quad - \left(\mathfrak{B}\,(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2) + \epsilon_1\, L^2\right)\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2)}{L^2} + \epsilon_1\right) \mathbf{B}(\rho X) \\ &\quad + \left(L + \epsilon_1\,\mathfrak{B} + \epsilon_2\,\frac{\mathfrak{B}^2}{L} + \epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right) G\cdot\left(\left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2}{L^3}\right) \ell(\rho X)\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,G\cdot\left(\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)\,\mathbf{B}(\rho X)\right)\\
&-2\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2}{L^3}\right)\,dL(X)\\
&-2\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)\,d\mathfrak{B}(X)\\
&-\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2}{L^3}\right)\,\ell(\rho[G,X])\\
&-\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)\,\mathbf{B}(\rho[G,X])
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\frac{1}{2}\,i_G\,dd_J\widetilde{L}^2(X)=-\left(\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)+\epsilon_1\,L^2\right)\,\left(\frac{-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2}{L^3}\right)\,\ell(\rho X)\\
&-\left(\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)+\epsilon_1\,L^2\right)\,\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)\,\mathbf{B}(\rho X)\\
&+\left(L^3+\epsilon_1\,\mathfrak{B}\,L^2+\epsilon_2\,\mathfrak{B}^2L+\epsilon_3\,\mathfrak{B}^3\right)\,\left(\frac{2\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+L\epsilon_2)}{L^3}\right)\ell(\rho X)\\
&-\left(L^3+\epsilon_1\,\mathfrak{B}\,L^2+\epsilon_2\,\mathfrak{B}^2L+\epsilon_3\,\mathfrak{B}^3\right)\,\left(\frac{6\mathfrak{B}\epsilon_3+2L\epsilon_2}{L^2}\right)\mathbf{B}(\rho X)\\
&+\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2}{L^3}\right)G\cdot(\ell(\rho X))\\
&+\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)\,G\cdot(\mathbf{B}(\rho X))\\
&-2\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2}{L^3}\right)\,dL(X)\\
&-2\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)\,d\mathfrak{B}(X)\\
&-\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2}{L^3}\right)\,\ell(D^\circ_G\rho X-KX)\\
&-\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)\,\mathbf{B}(D^\circ_G\rho X-KX))\\
=\quad&-\frac{L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2\right)}{L^3}\,\ell(\rho X)\\
&+\frac{\mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3+L^3\right)+4\mathfrak{B}L^2\epsilon_2^2+15\mathfrak{B}^2L\epsilon_2\epsilon_3\right)}{L^3}\,\ell(\rho X)\\
&-L^2\,\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)^2\,\mathbf{B}(\rho X)\\
&-\frac{2\,(3\mathfrak{B}\epsilon_3+L\epsilon_2)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)}{L^2}\,\mathbf{B}(\rho X)\\
&-\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{-2\mathfrak{B}^3\epsilon_3+L^3-\mathfrak{B}^2L\epsilon_2}{L^3}\right)\,dL(X)\\
&-\left(L+\epsilon_1\,\mathfrak{B}+\epsilon_2\,\frac{\mathfrak{B}^2}{L}+\epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\,\left(\frac{\mathfrak{B}\,(3\mathfrak{B}\epsilon_3+2L\epsilon_2)}{L^2}+\epsilon_1\right)\,d\mathfrak{B}(X)).
\end{aligned}
$$

On the other hand, using Proposition 4.1, we have

$$\begin{aligned}
&\frac{1}{2}\, i_{\gamma\overline{\mu}}\, dd_J \widetilde{L}^2(X) = \widetilde{g}(\overline{\mu}, \rho X) \\
= \quad &\frac{\left(-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2 L\epsilon_2\right)}{L^6}\, g(\overline{\mu}, \rho X) \\
&+\left(\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)}{L^2} + \epsilon_1\right)^2 + \frac{2\left(3\mathfrak{B}\epsilon_3 + L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2 L\epsilon_2\right)}{L^4}\right) \\
&\times \mathbf{B}(\overline{\mu})\mathbf{B}(\rho X) \\
&+\left(\frac{\mathfrak{B}^3\left(12\mathfrak{B}^3\epsilon_3^2 + 3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2 + L^2\right) + 4\mathfrak{B}L^2\epsilon_2^2\right) + \mathfrak{B}L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3 - L^3 + 3\mathfrak{B}^2 L\epsilon_2\right)}{L^6}\right) \\
&\times \ell(\overline{\mu})\, \ell(\rho X) \\
&+\left(\frac{L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3 + L^3 - 3\mathfrak{B}^2 L\epsilon_2\right) - \mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3 + L^3\right) + 4\mathfrak{B}L^2\epsilon_2^2 + 15\mathfrak{B}^2 L\epsilon_2\epsilon_3\right)}{L^5}\right) \\
&\times \left\{\mathbf{B}(\overline{\mu})\, \ell(\rho X) + \mathbf{B}(\rho X)\, \ell(\overline{\mu})\right\}.
\end{aligned}$$

Plugging the last two relations into Equation (5.2), after some calculation, it follows that

$$\begin{aligned}
&-\left(L + \epsilon_1\,\mathfrak{B} + \epsilon_2\,\frac{\mathfrak{B}^2}{L} + \epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2}{L^3}\right) dL(X) \\
&-\left(L + \epsilon_1\,\mathfrak{B} + \epsilon_2\,\frac{\mathfrak{B}^2}{L} + \epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)}{L^2} + \epsilon_1\right) d\mathfrak{B}(X)) = \\
&-\frac{L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3 + L^3 - 3\mathfrak{B}^2 L\epsilon_2\right)}{L^3}\,\ell(\rho X) + \frac{\mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3 + L^3\right) + 4\mathfrak{B}L^2\epsilon_2^2 + 15\mathfrak{B}^2 L\epsilon_2\epsilon_3\right)}{L^3}\,\ell(\rho X) \\
&-L^2\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)}{L^2} + \epsilon_1\right)^2 \mathbf{B}(\rho X) - \frac{2\left(3\mathfrak{B}\epsilon_3 + L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2 L\epsilon_2\right)}{L^2}\,\mathbf{B}(\rho X) \\
&-\left(L + \epsilon_1\,\mathfrak{B} + \epsilon_2\frac{\mathfrak{B}^2}{L} + \epsilon_3\frac{\mathfrak{B}^3}{L^2}\right)\left(\frac{-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2}{L^3}\right) dL(X) \\
&-\left(L + \epsilon_1\,\mathfrak{B} + \epsilon_2\,\frac{\mathfrak{B}^2}{L} + \epsilon_3\,\frac{\mathfrak{B}^3}{L^2}\right)\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)}{L^2} + \epsilon_1\right) d\mathfrak{B}(X)) \\
&+\frac{\left(-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2 L\epsilon_2\right)}{L^6}\, g(\overline{\mu}, \rho X) \\
&+\left(\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3 + 2L\epsilon_2\right)}{L^2} + \epsilon_1\right)^2 + \frac{2\left(3\mathfrak{B}\epsilon_3 + L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2 L\epsilon_2\right)}{L^4}\right) \\
&\times \mathbf{B}(\overline{\mu})\mathbf{B}(\rho X) \\
&+\left(\frac{\mathfrak{B}^3\left(12\mathfrak{B}^3\epsilon_3^2 + 3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2 + L^2\right) + 4\mathfrak{B}L^2\epsilon_2^2\right) + \mathfrak{B}L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3 - L^3 + 3\mathfrak{B}^2 L\epsilon_2\right)}{L^6}\right) \\
&\times \ell(\overline{\mu})\, \ell(\rho X) \\
&+\left(\frac{L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3 + L^3 - 3\mathfrak{B}^2 L\epsilon_2\right) - \mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3 + L^3\right) + 4\mathfrak{B}L^2\epsilon_2^2 + 15\mathfrak{B}^2 L\epsilon_2\epsilon_3\right)}{L^5}\right) \\
&\times \left\{\mathbf{B}(\overline{\mu})\, \ell(\rho X) + \mathbf{B}(\rho X)\, \ell(\overline{\mu})\right\}.
\end{aligned}$$

In view of the non-degenerate property of the Finsler metric g, one can show that

$$\frac{\left(-2\mathfrak{B}^3\epsilon_3 + L^3 - \mathfrak{B}^2 L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L^2\epsilon_1 + \mathfrak{B}^2 L\epsilon_2\right)}{L^6}\,\overline{\mu}$$

$$
\begin{aligned}
= \ & \{\frac{L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2\right)-\mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3+L^3\right)+4\mathfrak{B}L^2\epsilon_2^2+15\mathfrak{B}^2L\epsilon_2\epsilon_3\right)}{L^4}\\
& -\left(\frac{L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2\right)-\mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3+L^3\right)+4\mathfrak{B}L^2\epsilon_2^2+15\mathfrak{B}^2L\epsilon_2\epsilon_3\right)}{L^6}\right)\mathbf{B}(\overline{\mu})\\
& -\left(\frac{\mathfrak{B}^3\left(12\mathfrak{B}^3\epsilon_3^2+3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2+L^2\right)+4\mathfrak{B}L^2\epsilon_2^2\right)+\mathfrak{B}L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3-L^3+3\mathfrak{B}^2L\epsilon_2\right)}{L^7}\right)\ell(\overline{\mu})\}\overline{\eta}\\
& +\{L^2\left(\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)}{L^2}+\epsilon_1\right)^2+\frac{2\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)}{L^4}\right) \qquad (5.3)\\
& -\left(\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)}{L^2}+\epsilon_1\right)^2+\frac{2\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)}{L^4}\right)\mathbf{B}(\overline{\mu})\\
& -\left(\frac{L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2\right)-\mathfrak{B}^2\left(3\epsilon_3\left(4\mathfrak{B}^3\epsilon_3+L^3\right)+4\mathfrak{B}L^2\epsilon_2^2+15\mathfrak{B}^2L\epsilon_2\epsilon_3\right)}{L^5}\right)\ell(\overline{\mu})\}\overline{p}.
\end{aligned}
$$

where $\ell(\overline{\mu})$ and $\mathbf{B}(\overline{\mu})$ are geometric quantities determined by the following two equations

$$
\begin{aligned}
A_1\,\ell(\overline{\mu})+B_1\,\mathbf{B}(\overline{\mu}) &= C_1.\\
A_2\,\ell(\overline{\mu})+B_2\,\mathbf{B}(\overline{\mu}) &= C_2,
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
A_1 &:= \frac{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(L^3-\mathfrak{B}^2\left(2\mathfrak{B}\epsilon_3+L\epsilon_2\right)\right)}{L^6}.\\
B_1 &:= \frac{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(L^2\epsilon_1+\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)\right)}{L^5}.\\
C_1 &:= \frac{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(L^2\epsilon_1+\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)\right)}{L^3}.\\
A_2 &:= \frac{\left(L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2\right)\right)\left(L^2p^2-\mathfrak{B}^2\right)}{L^7}\\
&\quad +\frac{\left(\mathfrak{B}^2\left(-12\mathfrak{B}^3\epsilon_3^2-3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2+L^2\right)-4\mathfrak{B}L^2\epsilon_2^2\right)\right)\left(L^2p^2-\mathfrak{B}^2\right)}{L^7}.\\
B_2 &:= \frac{-14\mathfrak{B}^6\epsilon_3^2+L^6p^2\epsilon_1^2+L^6+6\mathfrak{B}L^5p^2\epsilon_3-4\mathfrak{B}^3L^3\epsilon_3+15\mathfrak{B}^4L^2p^2\epsilon_3^2}{L^6}\\
&\quad +\frac{2\epsilon_2\left(L^6p^2+\epsilon_3\left(10\mathfrak{B}^3L^3p^2-9\mathfrak{B}^5L\right)\right)+\epsilon_2^2\left(6\mathfrak{B}^2L^4p^2-5\mathfrak{B}^4L^2\right)}{L^6}\\
&\quad +\frac{2\mathfrak{B}L^2\epsilon_1\left(\epsilon_2\left(3L^3p^2-2\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-5\mathfrak{B}^3\right)\right)}{L^6}.\\
C_2 &:= \frac{\mathfrak{B}\left(L^2\epsilon_1\left(-8\mathfrak{B}^3\epsilon_3+L^3-3\mathfrak{B}^2L\epsilon_2\right)-\mathfrak{B}^2\left(12\mathfrak{B}^3\epsilon_3^2+3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2+L^2\right)+4\mathfrak{B}L^2\epsilon_2^2\right)\right)}{L^4}\\
&\quad +L^2p^2\left(\left(\frac{\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)}{L^2}+\epsilon_1\right)^2+\frac{2\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L^2\epsilon_1+\mathfrak{B}^2L\epsilon_2\right)}{L^4}\right).\\
p^2 &:= \mathbf{B}(\overline{p}).
\end{aligned}
$$

Making use of the condition (4.2), the system (5.4) has the following solution

$$
\begin{aligned}
\ell(\overline{\mu}) &= \frac{L^3\left(L^3-\mathfrak{B}^2\left(2\mathfrak{B}\epsilon_3+L\epsilon_2\right)\right)\left(L^2\epsilon_1+\mathfrak{B}\left(3\mathfrak{B}\epsilon_3+2L\epsilon_2\right)\right)}{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(\epsilon_2\left(2L^3p^2-3\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-8\mathfrak{B}^3\right)\right)},\\
\mathbf{B}(\overline{\mu}) &= \frac{L^4\left(\mathfrak{B}L\epsilon_1\left(\epsilon_2\left(2L^2p^2-3\mathfrak{B}^2\right)+L^2\right)+2\epsilon_2\left(L^4p^2+\epsilon_2\left(\mathfrak{B}^2L^2p^2-2\mathfrak{B}^4\right)\right)\right)}{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(\epsilon_2\left(2L^3p^2-3\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-8\mathfrak{B}^3\right)\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathfrak{B}L^3\epsilon_3\left(6L^4p^2+\epsilon_1\left(6\mathfrak{B}L^3p^2-8\mathfrak{B}^3L\right)-3\mathfrak{B}^2L^2+\epsilon_2\left(8\mathfrak{B}^2L^2p^2-15\mathfrak{B}^4\right)\right)}{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(\epsilon_2\left(2L^3p^2-3\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-8\mathfrak{B}^3\right)\right)}\\
&+\frac{6\mathfrak{B}^4L^2\epsilon_3^2\left(L^2p^2-2\mathfrak{B}^2\right)}{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(\epsilon_2\left(2L^3p^2-3\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-8\mathfrak{B}^3\right)\right)}.
\end{aligned}
$$

Consequently, in view of Equation (5.3) taking into account the assumption $\widetilde{G}=G+\gamma\overline{\mu}$, it follows that the canonical sprays G and $\widetilde{G}$, are related by

$$
\begin{aligned}
\widetilde{G} \;=\; & G-\frac{L^2\left(\mathfrak{B}^2\left(12\mathfrak{B}^3\epsilon_3^2+3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2+L^2\right)+4\mathfrak{B}L^2\epsilon_2^2\right)+L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3-L^3+3\mathfrak{B}^2L\epsilon_2\right)\right)}{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(\epsilon_2\left(2L^3p^2-3\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-8\mathfrak{B}^3\right)\right)}\,\mathcal{C}\\
&+\frac{2L^4\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)}{\epsilon_2\left(2L^3p^2-3\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-8\mathfrak{B}^3\right)}\,\gamma\overline{p}.
\end{aligned}
$$

Hence, the proof is completed. □

Theorem 5.5. *The Barthel connection $\widetilde{\Gamma}$ associated with the special generalized approximation Matsumoto metric* (3.1) *is given by*

$$\widetilde{\Gamma}=\Gamma-\lambda_1\,J-d_J\lambda_1\otimes\gamma\overline{\eta}+d_J\lambda_2\otimes\gamma\overline{p},$$

where

$$
\begin{aligned}
\lambda_1 &:= \frac{L^2\left(\mathfrak{B}^2\left(12\mathfrak{B}^3\epsilon_3^2+3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2+L^2\right)+4\mathfrak{B}L^2\epsilon_2^2\right)+L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3-L^3+3\mathfrak{B}^2L\epsilon_2\right)\right)}{\left(\mathfrak{B}^3\epsilon_3+L^3+\mathfrak{B}L\left(\mathfrak{B}\epsilon_2+L\epsilon_1\right)\right)\left(\epsilon_2\left(2L^3p^2-3\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-8\mathfrak{B}^3\right)\right)},\\
\lambda_2 &:= \frac{2L^4\left(3\mathfrak{B}\epsilon_3+L\epsilon_2\right)}{\epsilon_2\left(2L^3p^2-3\mathfrak{B}^2L\right)+L^3+\epsilon_3\left(6\mathfrak{B}L^2p^2-8\mathfrak{B}^3\right)}.
\end{aligned}
$$

Consequently, the horizontal map $\widetilde{\beta}$ associated with the special generalized approximation Matsumoto metric has the form

$$\widetilde{\beta}\overline{X}=\beta\overline{X}-\frac{1}{2}\left\{\lambda_1\,\gamma\overline{X}+d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta}-d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\right\}.$$

Proof. From Theorem 5.4 and the formula [5]:

$$[fX,J]=f[X,J]+df\wedge i_XJ-d_Jf\otimes X,$$

and using the given assumption for λ_1 and λ_2, one can show that

$$
\begin{aligned}
\widetilde{\Gamma} &= [J,\widetilde{G}]=[J,G-\lambda_1\,\gamma\overline{\eta}+\lambda_2\gamma\overline{p}]=[J,G]+[\lambda_1\,\gamma\overline{\eta}-\lambda_2\gamma\overline{p},J]\\
&= [J,G]+\lambda_1[\gamma\overline{\eta},J]+d\lambda_1\wedge i_{\gamma\overline{\eta}}\,J-d_J\lambda_1\otimes\gamma\overline{\eta}\\
&\quad -\lambda_2[\gamma\overline{p},J]-d\lambda_2\wedge i_{\gamma\overline{p}}\,J+d_J\lambda_2\otimes\gamma\overline{p}.
\end{aligned}
$$

On the other hand, we obtain

$$
\begin{aligned}
d_Jp^2 &= 0\\
i_{\gamma\overline{\eta}}\,J &= 0=i_{\gamma\overline{p}}\,J, \qquad\qquad (\text{as } J\circ\gamma=0),
\end{aligned}
$$

whereas

$$
\begin{aligned}
[\gamma\overline{p},J]X &= [\gamma\overline{p},JX]-J[\gamma\overline{p},X]\\
&= \gamma\{\nabla_{\gamma\overline{p}}\,\rho X-\nabla_{JX}\,\overline{p}\}-\gamma\{\nabla_{\gamma\overline{p}}\,\rho X-T(\overline{p},\rho X)\}=0.
\end{aligned}
$$

$$[\gamma\overline{\eta},J]X=-JX.$$

Therefore,

$$\widetilde{\Gamma}=\Gamma-\lambda_1\,J-d_J\lambda_1\otimes\gamma\overline{\eta}+d_J\lambda_2\otimes\gamma\overline{p}.$$

Consequently, using the fact that $\Gamma = 2\beta\circ\rho - I$, the horizontal map $\widetilde{\beta}$ associated with the special generalized Matsumoto metric has the form

$$\widetilde{\beta}\overline{X} = \beta\overline{X} - \frac{1}{2}\left\{\lambda_1\,\gamma\overline{X} + d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta} - d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\right\}.$$

This completes the proof. □

Theorem 5.6. *The Barthel curvature tensor* $\widetilde{\Re}$ *associated with the special generalized second approximation Matsumoto metric* (3.1) *is determined by*

$$\widetilde{\Re} = \Re - [h, \mathbb{L}] - N_{\mathbb{L}},$$

where $N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L}, \mathbb{L}]$ *is the Nijenhuis torsion of a vector 1-form* $\mathbb{L}$ *defined by*

$$\mathbb{L} := -\frac{1}{2}\left\{\lambda_1\,J + d_J\lambda_1\otimes\gamma\overline{\eta} - d_J\lambda_2\otimes\gamma\overline{p}\right\}. \tag{5.5}$$

Proof. In view of Theorem 5.5, we conclude that the horizontal projection $\widetilde{h}$ and vertical projection $\widetilde{v}$ associated with the special generalized approximation Matsumoto metric has the form

$$\widetilde{h} = h + \mathbb{L}, \quad \widetilde{v} = v - \mathbb{L},$$

where $\mathbb{L}$ is defined by (5.5). Now, the proof follows from the fact that $\widetilde{\Re} = -\frac{1}{2}[\widetilde{h}, \widetilde{h}]$, and taking into account the proprties of the Frölicher-Nijenhuis bracket. □

The Berwald vertical counterpart is given by Lemma 3.3 and the Berwald horizontal counterpart is given by the following result.

Proposition 5.7. *For the special generalized approximation Matsumoto metric* (3.1)*, the Berwald horizontal counterpart is given by*

$$\begin{aligned}
\widetilde{D^\circ}_{\beta\overline{X}}\overline{Y} &= D^\circ{}_{\beta\overline{X}}\overline{Y} - \frac{1}{2}\{\lambda_1\,D^\circ_{\gamma\overline{X}}\overline{Y} + d_J\lambda_1(\beta\overline{X})\,D^\circ_{\gamma\overline{\eta}}\overline{Y}\\
&\quad - d_J\lambda_1(\beta\overline{X})\,\overline{Y} - d_J\lambda_1(\beta\overline{Y})\,\overline{X} - d_J\lambda_2(\beta\overline{X})\,D^\circ_{\gamma\overline{p}}\overline{Y}\}\\
&\quad + \frac{1}{2}\left\{dd_J\lambda_1(\gamma\overline{Y}, \beta\overline{X})\,\overline{\eta} - dd_J\lambda_2(\gamma\overline{Y}, \beta\overline{X}))\,\overline{p}\right\}.
\end{aligned}$$

Proof. The proof follows from the fact that $v := \gamma\circ K$, $h := \beta\circ\rho$, $\gamma D^\circ{}_{hX}\overline{Y} := v[hX, JY]$ and $D^\circ{}_{\gamma\overline{X}}\,\rho Y := \rho[\gamma\overline{X}, \beta\overline{Y}]$ ([31, Proposition 4.4]), taking into account Theorem 5.6, and the facts that the map $\gamma : \pi^{-1}(TM) \to VTM$ is an isomorphism, the Berwlad (v)v-curvature $\widetilde{S}^\circ = 0$, $[JX, JY] = J[X, JY] + J[JX, Y]$, $vJ = J$ and $Jv = 0$.
In more details.

$$\begin{aligned}
\gamma\widetilde{D^\circ}_{hX}\,\rho Y &= \widetilde{v}[\widetilde{h}X, JY] = (v - \mathbb{L})[hX + \mathbb{L}X, JY]\\
&= v[hX, JY] + v[\mathbb{L}X, JY] - \mathbb{L}[hX, JY] - \mathbb{L}[\mathbb{L}X, JY]\\
&= \gamma D^\circ{}_{hX}\overline{Y} - \frac{\gamma}{2}\{\lambda_1\,K[JX, JY] + d_J\lambda_1(X)\,K[\gamma\overline{\eta}, JY] - d_J\lambda_2(X)\,K[\gamma\overline{p}, JY]\}\\
&\quad + \frac{\gamma}{2}\{(JY\cdot\lambda_1)\,\rho X + (JY\cdot d_J\lambda_1(X))\,\overline{\eta} - (JY\cdot d_J\lambda_2(X))\,\overline{p}\}\\
&\quad + \frac{\gamma}{2}\{\lambda_1\,\rho([hX, JY]) + d_J\lambda_1([hX, JY])\,\overline{\eta} - d_J\lambda_2([hX, JY])\,\overline{p}\}\\
&= \gamma D^\circ{}_{hX}\rho Y - \frac{\gamma}{2}\{\lambda_1\,D^\circ_{JX}\,\rho Y + d_J\lambda_1(X)\,D^\circ_{\gamma\overline{\eta}}\,\rho Y\\
&\quad - d_J\lambda_1(X)\,\rho Y - d_J\lambda_1(Y)\,\rho X - d_J\lambda_2(X)\,D^\circ_{\gamma\overline{p}}\,\rho Y\}\\
&\quad + \frac{\gamma}{2}\{dd_J\lambda_1(JY, X)\,\overline{\eta} - dd_J\lambda_2(JY, X))\,\overline{p}\}.
\end{aligned}$$

Consequently,

$$\widetilde{D^\circ}_{\beta\overline{X}}\overline{Y} = D^\circ{}_{\beta\overline{X}}\overline{Y} - \frac{1}{2}\{\lambda_1\,D^\circ_{\gamma\overline{X}}\overline{Y} + d_J\lambda_1(\beta\overline{X})\,D^\circ_{\gamma\overline{\eta}}\overline{Y}$$

$$-d_J\lambda_1(\beta\overline{X})\,\overline{Y} - d_J\lambda_1(\beta\overline{Y})\,\overline{X} - d_J\lambda_2(\beta\overline{X})\,D^\circ_{\gamma\overline{p}}\,\overline{Y}\}$$
$$+\frac{1}{2}\left\{dd_J\lambda_1(\gamma\overline{Y},\beta\overline{X})\,\overline{\eta} - dd_J\lambda_2(\gamma\overline{Y},\beta\overline{X}))\,\overline{p}\right\}.$$

This completes the proof. □

We now present an example of a Finsler metric that admits a concurrent π-vector field and proceed to compute the associated π-form. It is worth mentioning that the presence of concurrent vector fields on Finsler manifolds was initially explored by Tachibana [29]. This concept has since been extended through the introduction of semi-concurrent vector fields; for further information, see [38].

Complete calculations for the following example are available in the accompanying PDF and Maple files, which are provided at:
`https://github.com/salahelgendi/-Generalized-Matsumoto-metric_Example1.git`

Example 5.8. *Let* $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq 0\}$ *and* L *be a conic Finsler metric given by*

$$L = \sqrt{\frac{x_1^2 y_3^3}{y_2} + y_1^2},$$

where $(x_1, x_2, x_3; y_1, y_2, y_3) \in \mathcal{T}M \subset \mathbb{R}^3 \times \mathbb{R}^3$.

The metric tensor has the following non-vanishing components g_{ij}*:*

$$g_{11} = 1, \quad g_{22} = \frac{x_1^2\, y_3^3}{y_2^3}, \quad g_{23} = -\frac{3}{2}\frac{x_1^2\, y_3^2}{y_2^2}, \quad g_{33} = \frac{3x_1^2\, y_3}{y_2}.$$

The inverse metric tensor has the following non-vanishing components g^{ij}*:*

$$g^{11} = 1, \quad g^{22} = \frac{4y_2^3}{x_1^2\, y_3^3}, \quad g^{23} = \frac{2y_2^2}{x_1^2\, y_3^2}, \quad g^{33} = \frac{4}{3}\frac{y_2}{x_1^2\, y_3}.$$

The the Cartan tensor has the following non-vanishing components C_{ijk}*:*

$$C_{222} = -\frac{3}{2}\frac{x_1^2 y_3^3}{y_2^4}, \quad C_{223} = \frac{3}{2}\frac{x_1^2 y_3^2}{y_2^3}, \quad C_{233} = -\frac{3}{2}\frac{x_1^2 y_3}{y_2^2}, \quad C_{333} = \frac{3}{2}\frac{x_1^2}{y_2}.$$

The coefficients G^i *of the geodesic spray are given by*

$$G^1 = -\frac{1}{2}\frac{x_1 y_3^3}{y_2}, \quad G^2 = \frac{y_1 y_2}{x_1}, \quad G^3 = -\frac{y_1 y_2}{x_1}.$$

By performing direct computations, or alternatively using the Finsler package [37], one can obtain the coefficients of the Cartan connection. For instance,

$$\Gamma^2_{12} = \frac{1}{x_1}, \quad \Gamma^3_{13} = \frac{1}{x_1}, \quad \Gamma^1_{11} = \Gamma^3_{33} = 0.$$

It is evident that this metric supports a concurrent π*-vector field of the form* $\overline{p} = p^i\,\overline{\partial_i}$*, where* $\overline{\partial_i}$ *are the local basis vectors of the fibers of* $\pi^{-1}(TM)$*, with components defined by* $p^2(x) = p^3(x) = 0$ *and* $p^1(x) = x_1$*. In this case, it follows that* $p^i C_{ijk} = 0$*, and for instance, we compute*

$$p^i_{|j} = \delta_j p^i + p^h\Gamma^i_{hj} = \delta_j p^i + p^1\Gamma^i_{1j}.$$

$$p^1_{|1} = \delta_1 p^1 + p^1\Gamma^1_{11} = 1, \quad p^2_{|2} = \delta_2 p^2 + p^1\Gamma^2_{12} = 1, \quad p^3_{|3} = \delta_3 p^3 + p^1\Gamma^3_{13} = 1.$$

While all other components of $p^i_{|j}$ *vanish. In addition, the corresponding* π*-form* $\mathbf{B}$ *has components* $\mathbf{B}^2 = \mathbf{B}^3 = 0$*,* $\mathbf{B}^1 = x_1$*, which implies the associated 1-form is given by* $\mathfrak{B} = x_1 y_1$*.*

Hence, we obtain

$$\widetilde{L}(x,y) = L(x,y) + \epsilon_1\,\mathfrak{B}(x,y) + \epsilon_2\,\frac{\mathfrak{B}^2(x,y)}{L(x,y)} + \epsilon_3\,\frac{\mathfrak{B}^3(x,y)}{L^2(x,y)}$$

$$= \sqrt{\frac{x_1^2 y_3^3}{y_2} + y_1^2} + \epsilon_1\, x_1 y_1 + \epsilon_2\, \frac{(x_1 y_1)^2}{\sqrt{\frac{x_1^2 y_3^3}{y_2} + y_1^2}} + \epsilon_3\, \frac{(x_1 y_1)^3}{\frac{x_1^2 y_3^3}{y_2} + y_1^2},$$

which defines a special generalized approximation Matsumoto metric over M.

6 Applications

6.1 Projectively related metric

Every two Finsler manifolds (M, L) and $(M, \widetilde{L})$ are said to be projectively related [32] if each geodesic of (M, L) is a geodesic of $(M, \widetilde{L})$ and vice versa. Here, we study some special cases of the generalized approximation Matsumoto metric *(3.1)* which are projectively related.

Also, we have

Definition 6.1. [32] Two Finsler manifolds (M, L) and $(M, \widetilde{L})$ are projectively related if, and only if, the associated canonical sprays G and $\widetilde{G}$ are related by

$$\widetilde{G} = G - 2\lambda(x, y)\mathcal{C},$$

for some function $\lambda(x, y)$ on TM, positively homogenous of degree 1 in y and called the projective factor.

In view of the above definition and Theorem 5.4, we obtain the following important result.

Theorem 6.2. *Let (M, L) and $(M, \widetilde{L})$ be two Finsler manifolds are related by the special generalized β-change* (3.1)*. If the Finsler structure L satisfies the condition $3\mathfrak{B}\epsilon_3 + L\epsilon_2 = 0$, then the two Finsler structures L and $\widetilde{\mathrm{L}}$ are projectively related with projective factor $\lambda(x, y)$ given by*

$$\lambda \quad := \quad \frac{L^2\left(\mathfrak{B}^2\left(12\mathfrak{B}^3\epsilon_3^2 + 3L\epsilon_3\left(5\mathfrak{B}^2\epsilon_2 + L^2\right) + 4\mathfrak{B}L^2\epsilon_2^2\right) + L^2\epsilon_1\left(8\mathfrak{B}^3\epsilon_3 - L^3 + 3\mathfrak{B}^2L\epsilon_2\right)\right)}{\left(\mathfrak{B}^3\epsilon_3 + L^3 + \mathfrak{B}L\left(\mathfrak{B}\epsilon_2 + L\epsilon_1\right)\right)\left(\epsilon_2\left(2L^3p^2 - 3\mathfrak{B}^2L\right) + L^3 + \epsilon_3\left(6\mathfrak{B}L^2p^2 - 8\mathfrak{B}^3\right)\right)}.$$

Definition 6.3. The special generalized approximation Matsumoto metric *(3.1)* which satisfies the condition $3\mathfrak{B}\epsilon_3 + L\epsilon_2 = 0$ is called the special generalized conditionally approximation Matsumoto metric.

In view of the above definition 6.3, Theorem 6.2 and using the results of [32], we have

Theorem 6.4. *Under the special generalized conditionally approximation Matsumoto metric, we obtain*

(a) *The associated Barthel connections Γ and $\widetilde{\Gamma}$ are related by*

$$\widetilde{\Gamma} = \Gamma - 2\{\lambda J + d_J\lambda \otimes \mathcal{C}\}.$$

(b) *The curvature tensors $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ of the associated Barthel connections Γ and $\widetilde{\Gamma}$ are related by*

$$\widetilde{\mathfrak{R}} = \mathfrak{R} + (d_h\lambda - \lambda\, d_J\lambda) \wedge J + d_h d_J\lambda \otimes \mathcal{C}.$$

(c) *The associated Berwald connections D° and $\widetilde{D^\circ}$ are related by*

$$\widetilde{D}^\circ_X\overline{Y} = D^\circ_X\overline{Y} + \omega(\overline{Y})\rho X + \omega(\rho X)\overline{Y} + (\overset{v}{D}{}^\circ\omega)(\overline{Y}, \rho X)\overline{\eta},$$

where λ is the projective factor given by Theorem 6.2 and ω is the π-form defined by

$$\omega(\overline{X}) := d_J\lambda(\beta\overline{X}).$$

Theorem 6.5. *Under the special generalized conditionally approximation Matsumoto metric* (3.1)*, the h-curvature and hv-curvature tensor fields associated to the Berwald connection are given by* [1]

$$\widetilde{R^\circ}(\overline{X}, \overline{Y})\overline{Z} \quad = \quad R^\circ(\overline{X}, \overline{Y})\overline{Z} + (D^\circ_{\gamma\overline{Z}}Q)(\overline{Y})\overline{X} - (D^\circ_{\gamma\overline{Z}}Q)(\overline{X})\overline{Y} + \varepsilon(\overline{X}, \overline{Y})\overline{Z} + (D^\circ_{\gamma\overline{Z}}\,\varepsilon)(\overline{X}, \overline{Y})\overline{\eta}.$$

---

[1] $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}$ means the cyclic sum over $\overline{X}, \overline{Y}, \overline{Z}$.

$$\widetilde{P}^\circ(\overline{X},\overline{Y})\overline{Z} \;=\; P^\circ(\overline{X},\overline{Y})\overline{Z} + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{((\overset{v}{D}{}^\circ\,\omega)(\overline{Y},\overline{Z}))\overline{X}\} + (\overset{v}{D}{}^\circ\overset{v}{D}{}^\circ\,\omega)(\overline{X},\overline{Y}.\overline{Z})\overline{\eta},$$

Moreover, the (v)h-torsion $\widehat{R}^\circ$ *and the deviation tensor* $H := i_{\overline{\eta}}\widehat{R}^\circ$ *are related respectively by*

$$\begin{aligned}\widetilde{\widehat{R}}{}^\circ(\overline{X},\overline{Y}) &= \widehat{R}^\circ(\overline{X},\overline{Y}) + Q(\overline{Y})\overline{X} - Q(\overline{X})\overline{Y} + \varepsilon(\overline{X},\overline{Y})\overline{\eta}.\\ \widetilde{H}(\overline{X}) &= H(\overline{X}) - Q(\overline{\eta})\overline{X} + \{Q(\overline{X}) + \varepsilon(\overline{\eta},\overline{X})\}\overline{\eta},\end{aligned}$$

where ω *is the* π*-form defined above,* Q *and* ε *are* π*-forms defined respectively by*

$$\begin{aligned}Q(\overline{X}) &:= \beta\overline{X}\cdot\lambda - \lambda\,\omega(\overline{X}),\\ \varepsilon(\overline{X},\overline{Y}) &:= (D^\circ_{\gamma\overline{X}}Q)(\overline{Y}) - (D^\circ_{\gamma\overline{Y}}Q)(\overline{X}).\end{aligned}$$

Adopting to the results of [32], together with Theorems 6.2, 6.4 and 6.5, we are in a position to obtain the following invariant objects.

Theorem 6.6. *Under the special generalized conditionally approximation Matsumoto metric* (3.1)*, if the factor of projectivity* λ *has the property that the* π*-form* Q *vanishes, then the deviation tensor* H*, the (v)h-torsion tensor* $\widehat{R^\circ}$*, the (h)h-curvature tensor* R° *and the curvature tensor* $\mathfrak{R}$ *of Barthel connection are invariant.*

Theorem 6.7. *Under the special generalized conditionally approximation Matsumoto metric* (3.1)*, the following tensor field on* π*-tensor field, provided the* $\dim M > 2$*, is projectively invariant*:

$$\begin{aligned}W(\overline{X},\overline{Y})\overline{Z} \;:=\;& R^\circ(\overline{X},\overline{Y})\overline{Z} + \frac{1}{n+1}\mathfrak{U}_{\overline{X},\overline{Y}}\{(\overset{v}{D}{}^\circ\,R_1)(\overline{Z},\overline{Y})\overline{X}\\ &+(\overset{v}{D}{}^\circ\,R_1)(\overline{X},\overline{Y})\overline{Z} + (\overset{v}{D}{}^\circ\overset{v}{D}{}^\circ\,R_1)(\overline{Z},\overline{X},\overline{Y})\overline{\eta}\},\end{aligned}$$

where

$$\begin{aligned}R_2(\overline{X},\overline{Y}) &:= Tr^c{}_{\overline{Z}}\{R^\circ(\overline{X},\overline{Z})\overline{Y}\},\\ R_1(\overline{X}) &:= \frac{1}{n-1}\{nR_2(\overline{X},\overline{\eta}) + R_2(\overline{\eta},\overline{X})\}\;;\; n>2.\end{aligned}$$

In this case, the π*-tensor field* W*, defined above is called the Weyl curvature tensor.*

Theorem 6.8. *Under the special generalized conditionally approximation Matsumoto metric* (3.1)*, the following linear connection is invariant*

$$\nabla_X\overline{Y} = D^\circ_X\overline{Y} - \frac{1}{n+1}\{\varpi(\overline{Y})\rho X + \varpi(\rho X)\overline{Y} + (p\,(\rho X,\overline{Y}))\overline{\eta}\},$$

where $\varpi(\overline{Y}) := Tr^c{}_X\,\{D^\circ_X\overline{Y}\}$ *and* $p\,(\overline{X},\overline{Y}) := Tr^c{}_{\overline{Z}}\,\{P^\circ(\overline{X},\overline{Y})\overline{Z}\}$.

Theorem 6.9. *Under the special generalized conditionally approximation Matsumoto metric* (3.1)*, the* π*-tensor field*

$$\mathbb{P}(\overline{X},\overline{Y})\overline{Z} := P^\circ(\overline{X},\overline{Y})\overline{Z} - \frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{(p\,(\overline{X},\overline{Y}))\overline{Z}\} - \frac{1}{n+1}\{(D^\circ_{\gamma\overline{Y}}p)(\overline{X},\overline{Z})\}\overline{\eta}.$$

is invariant. In this case, the π*-tensor field* $\mathbb{P}$*, defined above, is called the Douglas tensor associated with the change* (3.1)*.*

A Finsler manifold (M,L) [32] is said to be a Berwald manifold if the $(h)hv$-torsion tensor T of the Cartan connection ∇ is horizontally parallel ($\nabla_{\beta\overline{X}}T = 0$). Also, a Finsler manifold (M,L) is said to be a Douglas manifold if its Douglas tensor vanishes identically ($\mathbb{P} = 0$). Furthermore, every Berwald manifold is a Douglas manifold. Consequently, we have

Theorem 6.10. *Under the special generalized conditionally approximation Matsumoto metric* (3.1)*, if the transformed Finsler manifold* $(M,\widetilde{L})$ *is Berwald, then it is a Douglas manifold.*

6.2 Special cases

Here, we construct and investigate examples as special cases of a special generalized approximation Matsumoto metric (3.1). Within this section, we assume that (M, L) be a Finsler manifold admitting concurrent π-vector field $\overline{p}(x)$ with the associated π-form $\mathbf{B} := i_{\overline{p}}\, g$ and the corresponding one form $\mathfrak{B} := \mathbf{B}(\overline{\eta})$, where g is the Finsler metric associated with L. Consider the change

$$\widetilde{L}(x,y) = L(x,y) + \epsilon_1 \mathfrak{B}(x,y) + \epsilon_2\, \frac{\mathfrak{B}^2(x,y)}{L(x,y)} + \epsilon_3\, \frac{\mathfrak{B}^3(x,y)}{L^2(x,y)},$$

where $\epsilon_i; i = 1,2,3$ are positive constants and $\mathfrak{B}(x,y) := g(\overline{p},\overline{\eta}) =: \mathbf{B}(\overline{\eta})$.

Example 6.11. **(Special generalized Randers metric [26]):**
Assume that $\epsilon_1 = 1$ and $\epsilon_2 = 0 = \epsilon_3$.
Hence, we obtain the following Finsler metric (**the special GRM**):

$$\widetilde{L}(x,y) = L + \mathfrak{B}.$$

In this case, we have the following corresponding geometric objects:

- The supporting form $\widetilde{\ell}$:

$$\widetilde{\ell}(\overline{X}) := \ell(\overline{X}) + \mathbf{B}(\overline{X}).$$

- The angular metric tensor $\widetilde{\hbar}$:

$$\widetilde{\hbar}(\overline{X},\overline{Y}) \;=\; \frac{L+\mathfrak{B}}{L}\hbar(\overline{X},\overline{Y}).$$

- The Finsler metric tensor $\widetilde{g}$:

$$\widetilde{g}(\overline{X},\overline{Y}) \;=\; \frac{L+\mathfrak{B}}{L}\, g(\overline{X},\overline{Y}) + \mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X}) - L^{-1}\,\mathfrak{B}\ell(\overline{X})\,\ell(\overline{Y}).$$

- The associated spray $\widetilde{G}$ is given by

$$\widetilde{G} = G + \frac{L^2}{L+\mathfrak{B}}\,\mathcal{C}.$$

Example 6.12. **(Special generalized Shen square metric [23]):**
Assume that $\epsilon_1 = 2$, $\epsilon_2 = 1$, and $\epsilon_3 = 0$.
Hence, we obtain the following Finsler metric (**the special GSSM**):

$$\widetilde{L}(x,y) = \frac{(L+\mathfrak{B})^2}{L}.$$

In this case, we have the following corresponding geometric objects:

- The supporting form $\widetilde{\ell}$:

$$\widetilde{\ell}(\overline{X}) = (1 - \frac{\mathfrak{B}^2}{L^2})\,\ell(\overline{X}) + \frac{2(L+\mathfrak{B})}{L}\,\mathbf{B}(\overline{X}).$$

- The angular metric tensor $\widetilde{\hbar}$:

$$\begin{aligned}\widetilde{\hbar}(\overline{X},\overline{Y}) \;&=\; \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}\hbar(\overline{X},\overline{Y}) + \frac{2(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+\frac{2\mathfrak{B}^2(L+\mathfrak{B})^2}{L^3}\,\ell(\overline{X})\,\ell(\overline{Y}) - \frac{2\mathfrak{B}(L+\mathfrak{B})^2}{L^3}\,\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\}\,.\end{aligned}$$

- The Finsler metric tensor $\widetilde{g}$:

$$\begin{aligned}\widetilde{g}(\overline{X},\overline{Y}) &= \frac{(L-\mathfrak{B})(L+\mathfrak{B})^3}{L^4}g(\overline{X},\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) \\ &\quad + \frac{2\mathfrak{B}(2\mathfrak{B}-L)(L+\mathfrak{B})^2}{L^4}\,\ell(\overline{X})\,\ell(\overline{Y}) + \frac{6(L+\mathfrak{B})^2}{L^2}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

  Moreover, the non-degenerate condition for $\widetilde{g}$ becomes $L^2(1+2p^2) - 3\mathfrak{B}^2 \neq 0$.

- The associated spray $\widetilde{G}$ is given by

$$\widetilde{G} = G - \frac{2L^2(2\mathfrak{B}-L)}{L^2(1+2p^2)-3\mathfrak{B}^2}\,\mathcal{C} + \frac{2L^4}{L^2(1+2p^2)-3\mathfrak{B}^2}\,\gamma\overline{p}.$$

Example 6.13. **(Special generalized first approximation Matsumoto metric [4]):**
Assume that $\epsilon_1 = 1 = \epsilon_2$, and $\epsilon_3 = 0$.
Hence, we obtain the following Finsler metric (**the special GFAMM**):

$$\widetilde{L}(x,y) = L(x,y) + \mathfrak{B}(x,y) + \frac{\mathfrak{B}^2(x,y)}{L(x,y)}.$$

In this case, we have the following corresponding geometric objects:

- The supporting form $\widetilde{\ell}$:

$$\widetilde{\ell}(\overline{X}) = (1 - \frac{\mathfrak{B}^2}{L^2})\,\ell(\overline{X}) + (1 + \frac{2\mathfrak{B}}{L})\,\mathbf{B}(\overline{X})..$$

- The angular metric tensor $\widetilde{\hbar}$:

$$\begin{aligned}\widetilde{\hbar}(\overline{X},\overline{Y}) &= \frac{(L^2-\mathfrak{B}^2)(\mathfrak{B}^2+\mathfrak{B}L+L^2)}{L^4})\hbar(\overline{X},\overline{Y}) + \frac{2}{L}(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L})\mathbf{B}(\overline{X})\mathbf{B}(\overline{Y}) \\ &\quad + \frac{2\mathfrak{B}^2}{L^3}(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L})\ell(\overline{X})\ell(\overline{Y}) - \frac{2\mathfrak{B}}{L^2}(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L})\left\{\mathbf{B}(\overline{X})\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\ell(\overline{X})\right\}.\end{aligned}$$

- The Finsler metric tensor $\widetilde{g}$:

$$\begin{aligned}\widetilde{g}(\overline{X},\overline{Y}) &= \frac{(L^2-\mathfrak{B}^2)(\mathfrak{B}^2+L\mathfrak{B}+L^2)}{L^4}g(\overline{X},\overline{Y}) + \frac{6\mathfrak{B}(L+\mathfrak{B})+3L^2}{L^2}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) \\ &\quad + \frac{4\mathfrak{B}^4+3\mathfrak{B}^3L-\mathfrak{B}L^3}{L^4}\ell(\overline{X})\ell(\overline{Y}) + \frac{(L^3-3\mathfrak{B}^2L-4\mathfrak{B}^3)}{L^3}\left\{\mathbf{B}(\overline{X})\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\ell(\overline{X})\right\}.\end{aligned}$$

  Moreover, the non-degenerate condition for $\widetilde{g}$ becomes $L^2(1+2p^2) - 3\mathfrak{B}^2 \neq 0$.

- The associated spray $\widetilde{G}$ is given by

$$\widetilde{G} = G - \frac{L^2(4\mathfrak{B}^3+3\mathfrak{B}^2L-L^3)}{(\mathfrak{B}^2+\mathfrak{B}L+L^2)(L^2(1+2p^2)-3\mathfrak{B}^2)}\,\mathcal{C} + \frac{2L^4}{L^2(1+2p^2)-3\mathfrak{B}^2}\,\gamma\overline{p}.$$

Example 6.14. **(Special generalized second approximation Matsumoto metric [24]):**
Assume that $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$.
Hence, we obtain the following Finsler metric (**the special GSAMM**):

$$\widetilde{L}(x,y) = L(x,y) + \mathfrak{B}(x,y) + \frac{\mathfrak{B}^2(x,y)}{L(x,y)} + \frac{\mathfrak{B}^3(x,y)}{L^2(x,y)}.$$

In this case, we have the following corresponding geometric objects:

- The supporting form $\widetilde{\ell}$:

$$\widetilde{\ell}(\overline{X}) = \frac{L^3-\mathfrak{B}^2L-2\mathfrak{B}^3}{L^3}\,\ell(\overline{X}) + \frac{L^2+2\mathfrak{B}L+3\mathfrak{B}^2}{L^2}\,\mathbf{B}(\overline{X}).$$

- The angular metric tensor $\widetilde{\hbar}$:

$$\begin{aligned}\widetilde{\hbar}(\overline{X},\overline{Y}) &= \frac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\,\hbar(\overline{X},\overline{Y})\\ &+\frac{2(3\mathfrak{B}+L)}{L^2}(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L}+\frac{\mathfrak{B}^3}{L^2})\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+\frac{2\mathfrak{B}^2(3\mathfrak{B}+L)}{L^4}(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L}+\frac{\mathfrak{B}^3}{L^2})\,\ell(\overline{X})\,\ell(\overline{Y})\\ &-\frac{2\mathfrak{B}(3\mathfrak{B}+L)}{L^3}(L+\mathfrak{B}+\frac{\mathfrak{B}^2}{L}+\frac{\mathfrak{B}^3}{L^2})\,\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

- The Finsler metric tensor $\widetilde{g}$:

$$\begin{aligned}\widetilde{g}(\overline{X},\overline{Y}) &= \frac{(L+\mathfrak{B})(L^2+\mathfrak{B}^2)(L^3-\mathfrak{B}^2L-2\mathfrak{B}^3)}{L^6}\,g(\overline{X},\overline{Y})\\ &+\frac{15\mathfrak{B}^4+20\mathfrak{B}^3L+18\mathfrak{B}^2L^2+12\mathfrak{B}L^3+3L^4}{L^4}\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &+\frac{12\mathfrak{B}^6+15\mathfrak{B}^5L+12\mathfrak{B}^4L^2+6\mathfrak{B}^3L^3-\mathfrak{B}L^5}{L^6}\,\ell(\overline{X})\,\ell(\overline{Y})\\ &+\frac{L^5-12\mathfrak{B}^5-15\mathfrak{B}^4L-12\mathfrak{B}^3L^2-6\mathfrak{B}^2L^3}{L^5}\,\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y})+\mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

Moreover, the non-degenerate condition for $\widetilde{g}$ becomes:

$$2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3\neq 0.$$

- The associated spray $\widetilde{G}$ is given by

$$\begin{aligned}\widetilde{G} &= G-\frac{L^2(12\mathfrak{B}^5+15\mathfrak{B}^4L+\mathfrak{B}^3L^2+6\mathfrak{B}^2\,L^3-L^5)}{(\mathfrak{B}+L)(\mathfrak{B}^2+L^2)(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}\,\mathcal{C}\\ &+\frac{2L^4(3\mathfrak{B}+L)}{(2L^2(3\mathfrak{B}+L)p^2+L^3-3\mathfrak{B}^2L-8\mathfrak{B}^3)}\,\gamma\overline{p}.\end{aligned}$$

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Received: 2025-07-28
Accepted: 2025-12-11