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A central valued identity with automorphisms in prime rings

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Abstract. In the present paper, we prove that a prime ring R with center $Z(R)$ satisfies s_4, the standard identity in four variables, if R admits a non-identity automorphism η such that $([(x^p)^\eta, x^q]x^r + x^r[(x^p)^\eta, x^q])^n \in Z(R)$ for all $x \in R$, whenever either $char(R) > n$ or $char(R) = 0$, where p, q, r, n are fixed positive integers.

1 Introduction

Throughout this article, R is a prime ring with center $Z(R)$. For given $x, y \in R$, the Lie commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. A ring R is said to be prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$. The standard identity s_4 in four variables is defined as follows:

$$s_4 = \sum (-1)^\tau X_{\tau(1)} X_{\tau(2)} X_{\tau(3)} X_{\tau(4)}$$

where $(-1)^\tau$ is the sign of a permutation τ of the symmetric group of degree 4.

The theory of commuting and centralizing maps on (semi-)prime rings was motivated by the result of Posner [20] and was developed by Brešar [3, 4, 5]. Posner's second theorem states that if there exists a nonzero centralizing derivation on a prime ring R, then R is commutative. Later, Mayne [18] obtained an analogous result for automorphisms of prime rings. Many people have extended Posner's result in various ways, obtaining numerous powerful results. In [16], Lee and Lee generalized Posner's result by showing that if $char(R) \neq 2$ and $[d(x), x] \in Z(R)$ for all x in a non-central Lie ideal of R, then R is commutative. In [15], Lanski proved that if $[d(x), x]_n = 0$ for all x in a non-commutative Lie ideal of R, then $char(R) = 2$ and $R \subseteq M_2(\mathbb{F})$ for a field $\mathbb{F}$. Mayne obtained a similar extension [19] for automorphisms on Lie ideals.

In 2000, Carini and De Filippis [6] investigated power-centralising derivations on non-central Lie ideals of prime rings. More precisely, they proved that if $char(R) \neq 2$ and $[d(x), x]^n \in Z(R)$ for all x in a non-central Lie ideal of R, then R satisfies s_4, the standard identity in four variables. Recently, Wang [24] obtained a similar result for automorphisms of prime rings. To be more specific, Wang proved the following: Let R be a prime ring with center $Z(R)$, L be a non-central Lie ideal of R and η be a nontrivial automorphism of R such that $[u^\eta, u]^n \in Z(R)$ for all $u \in L$. If either $char(R) > n$ or $char(R) = 0$, then R satisfies s_4, the standard identity in four variables.

Moreover, Herstein [11] proved that if there exists a nonzero derivation d on a prime ring R such that the map $x \mapsto d(x)$ is commuting on R, then R may be non-commutative, i.e., the following relation $[d(x), x]d(x) + d(x)[d(x), x] = 0$ for all $x \in R$ does not imply that $d = 0$. Motivated by the above result Cheng [9] proved the following theorem which can be considered as an extension of Posner's second theorem: If R is a 2-torsion free non-commutative prime ring and d be a derivation of R such that $[d(x), x]d(x) = 0$ for all $x \in R$, then $d = 0$. In [22] Vukman established that R must be commutative if $char(R) \neq 2$ and $[d(x), x]x - x[d(x), x] = 0$ for all $x \in R$ (see [1, 12, 21, 23] and references therein). Motivated by above results, Lanski [14] proved that if $[d(x), x]y - y[d(x), x] = 0$ for all x in a non-commutative Lie ideal and $y \in R$, then either R is commutative or $char(R) \neq 2$ and R satisfies s_4, the standard identity in four variables.

Inspired by the above-mentioned works, this paper examines the behaviour of a non-identity automorphism on a prime ring, in the spirit of theorems such as Posner's second theorem and Herstein's theorem on derivations with central values. More precisely, we investigate the situation when a non-identity automorphism η satisfies $([(x^p)^\eta, x^q]x^r + x^r[(x^p)^\eta, x^q])^n \in Z(R)$ for all $x \in R$.

2 Preliminaries

For the sake of completeness, we shall touch upon a few preliminary notions required for the exposition of the main theorem. Some of these notions are classical, and we present them briefly. Let R be a prime ring with centre $Z(R)$ and $Q = Q_{mr}(R)$ is the maximal right ring of quotients of R. Note that Q is also a prime ring and the center C of Q, which is called the extended centroid of R is a field. Moreover, $Z(R) \subseteq C$ (for a more detailed explanation, we refer to [2]). It is well known that any automorphism of R can be uniquely extended to an automorphism of Q. An automorphism η of R is called Q-inner if there exists an invertible element $g \in Q$ such that $x^\eta = gxg^{-1}$ for all $x \in R$. Otherwise, η is called Q-outer. We denote by G the group of all automorphisms of R and by A_i the group consisting of all Q-inner automorphisms of R. Recall that a subset $\mathfrak{A}$ of G is said to be independent (modulo A_i) if for any $a_1, a_2 \in \mathfrak{A}$, $a_1 a_2^{-1} \in A_i$ implies $a_1 = a_2$. For instance, if a is an outer automorphism of R, then 1 and a are independent (modulo A_i). We present some well-known facts which will be used in the sequel.

Suppose that R is a prime ring and $\mathfrak{A}$ is an independent subset of G modulo A_i. Let $\phi = \chi(x_i^{a_j}) = 0$ be a generalized identity with automorphisms of R reduced with respect to $\mathfrak{A}$. If for all $x_i \in X$, $a_j \in \mathfrak{A}$, the $x_i^{a_j}$-word degree of $\phi = \chi(x_i^{a_j})$ is strictly less than $char(R)$ when $char(R) \neq 0$, then $\chi(z_{ij}) = 0$ is also a generalized polynomial identity of R.

Let R be a prime ring and L be a non-central Lie ideal of R. If $char(R) \neq 2$, then there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If $char(R) = 2$ and $dim_c RC > 4$, then there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus either $char(R) \neq 2$ or $dim_C RC > 4$, then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

Let R be a prime ring with extended centroid C. Then the following conditions are equivalent:

(i) $dim_C RC \leq 4$.

(ii) R satisfies s_4, the standard identity in four variables.

(iii) R is commutative or R embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field.

(iv) R is algebraic of bounded degree 2 over C.

(v) R satisfies $[[x^2, y], [x, y]]$.

3 The results

We begin with the following proposition, which is crucial for proving our main result.

Proposition 3.1. *Let R be a prime ring and η be a non-identity automorphism of R such that $([[x, y]^\eta, [x_1, y_1]][[x_2, y_2] + [x_2, y_2][[x, y]^\eta, [x_1, y_1]])^n = 0$ for all $x, y, x_1, y_1, x_2, y_2 \in R$, where n is a fixed positive integer. If either $char(R) > n$ or $char(R) = 0$, then R satisfies s_4, the standard identity in four variables.*

Proof. First assume that $dim_C RC > 4$. By the hypothesis, we have

$$([[x, y]^\eta, [x_1, y_1]][x_2, y_2] + [x_2 y_2][[x, y]^\eta, [x_1, y_1]])^n = 0 \tag{3.1}$$

for all $x, y \in R$. If η is an inner automorphism, then there exists an invertible element $q \in Q$ such that $x^\eta = qxq^{-1}$ for all $x \in R$. By Theorem 2 in [7],

$$([q[x,y]q^{-1},[x_1,y_1]][x_2,y_2]+[x_2,y_2][q[x,y]q^{-1},[x_1,y_1]])^n=0$$

is also an identity for RC. By Martindale's theorem [17], RC is a primitive ring with a nonzero socle. Since RC is primitive, then there exists a vector space $\mathcal{V}$ over a division ring $\mathcal{D}$ such that RC is a dense ring of $\mathcal{D}$-linear transformations over $\mathcal{V}$. We divide the proof into two steps:

Step 1. Our aim is to show that for any $v \in \mathcal{V}$, v and vq are linearly $\mathcal{D}$-dependent. If v and vq are linearly $\mathcal{D}$-independent for some $v \in \mathcal{V}$, then we consider the following cases.

If $vq^{-1} \notin Span_{\mathcal{D}}\{v, vq\}$, then the set $\{v, vq, vq^{-1}\}$ is linearly $\mathcal{D}$-independent. By the density of RC there exist $x, y, x_1, y_1, x_2, y_2 \in RC$ such that

$$\begin{array}{llllll} vx=v, & vqx=v, & vq^{-1}x=-v, & vx_1=0, & vqx_1=v, & vq^{-1}x_1=v, \\ vx_2=0, & vqx_2=v, & vq^{-1}x_2=0, & vy=v, & vqy=0, & vq^{-1}y=0, \\ vy_1=vq, & vqy_1=0, & vq^{-1}y_1=0, & vy_2=v, & vqy_2=0, & vq^{-1}y_2=0. \end{array}$$

Then

$$v[x,y]=0,\ vq[x,y]=v,\ vq^{-1}[x,y]=0,\ v[x_1,y_1]=-v,$$
$$vq^{-1}[x_1,y_1]=vq,\ v[x_2,y_2]=v,\ vq[x_2,y_2]=v,\ vq^{-1}[x_2,y_2]=0$$

and hence we can easily see that

$$0=v([q[x,y]q^{-1},[x_1,y_1]][x_2,y_2]+[x_2,y_2][q[x,y]q^{-1},[x_1,y_1]])^n=v\neq\ 0,$$

a contradiction.

On the other hand if $vq^{-1} \in Span_{\mathcal{D}}\{v, vq\}$, then $vq^{-1} = v\lambda + vb\delta$ for some $\lambda, \delta \in \mathcal{D}$. If $\delta = 0$, then $vq^{-1} = \lambda v$ and $v = \lambda vq$, this yields $\{v, vq\}$ is D-independent, a contradiction. Thus $\delta \neq 0$. In view of the density of RC, there exist $x, y, x_1, y_1, x_2, y_2 \in RC$ such that

$$\begin{array}{llll} vx=v, & vqx=v, & vx_1=0, & vqx_1=v, \\ vx_2=0, & vqx_2=v, & vy=v, & vqy=0, \\ vy_1=vq, & vqy_1=0, & vy_2=v, & vqy_2=0. \end{array}$$

By computation, we see that

$$0=v([q[x,y]q^{-1},[x_1,y_1]][x_2,y_2]+[x_2,y_2][q[x,y]q^{-1},[x_1,y_1]])^n=2^n\delta^n v\neq 0,$$

a contradiction, for some $\gamma \in \mathcal{D}$. So, v and vq are $\mathcal{D}$-dependent for every $v \in \mathcal{V}$.

Step 2. As v and qv are $\mathcal{D}$-dependent for every $v \in \mathcal{V}$, then for each $v \in \mathcal{V}$, we write $vq = v\lambda_v$ where $\lambda_v \in \mathcal{D}$. Now, for fix $0 \neq u \in \mathcal{V}$, let $0 \neq v \in \mathcal{V}$ and we write $vq = v\lambda_v$. Suppose first that v and u are $\mathcal{D}$-independent. Then $(u+v)\lambda_{u+v} = (u+v)q = uq + vq = u\lambda_u + v\lambda_v$. Moreover, $u(\lambda_{u+v} - \lambda_u) = v(\lambda_v - \lambda_{u+v})$, and hence $\lambda_{u+v} = \lambda_u = \lambda_v$. Suppose next that u and v are $\mathcal{D}$-dependent. Indeed, for any $w \in \mathcal{V}$, w and u are $\mathcal{D}$-independent and using the same arguments as above, we have $\lambda_w = \lambda_v$. Clearly, w and v are $\mathcal{D}$-independent. So $\lambda_w = \lambda_v$, implying that $\lambda_u = \lambda_v$. Thus λ_v is the independent choice of $v \in \mathcal{V}$. Consequently, $vq = v\lambda$ for all $v \in \mathcal{V}$, where $\lambda = \lambda_v$. By standard arguments, we see that $q \in C$, a contradiction. Thus $dim_C RC \leq 4$, and by Fact 2, R satisfies s_4, the standard identity in four variables.

Next we assume that η is not an inner automorphism, then by Chuang [9, Main Theorem], R satisfies

$$([[x^\eta,y^\eta],[x_1,y_1]][x_2,y_2]+[x_2,y_2][[x^\eta,y^\eta],[x_1,y_1]])^n=0.$$

Since either $char(R) > n$ or $char(R) = 0$, it follows from Fact 2 that $([[r,s],[x_1,y_1]][x_2,y_2] + [x_2,y_2][[w,z],[x_1,y_1]])^n = 0$ for all $r,s,x_1,y_1,x_2,y_2 \in R$. Note that this is a polynomial identity, and thus there exists a field $\mathbb{F}$ such that $R \subseteq M_k(\mathbb{F})$, the ring of $k \times k$ matrices over a field $\mathbb{F}$, where $k \geq 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity [15, Lemma 1], that is $([[r,s],[x_1,y_1]][x_2,y_2] + [x_2,y_2][[w,z],[x_1,y_1]])^n = 0$ for all $r,s,x_1,y_1,x_2,y_2 \in M_k(\mathbb{F})$. But by choosing $x_1 = e_{12}$, $y_1 = e_{21}$, $r = e_{21}$, $s = e_{11}$, $x_2 = e_{12}$, $y_2 = e_{22}$ we get

$$\begin{aligned} 0 &= ([[r,s],[x_1,y_1]][x_2,y_2] + [x_2,y_2][[w,z],[x_1,y_1]])^n \\ &= ([[e_{21},e_{11}],[e_{12},e_{21}]] + [e_{12},e_{22}] + [e_{12},e_{22}][[e_{21},e_{11}],[e_{12},e_{21}]])^n \\ &= 2^n(e_{22}+e_{11}) \neq 0, \end{aligned}$$

which leads to a contradiction. Thereby, the proof is completed. □

Theorem 3.2. *Let R be a non-commutative prime ring and η is a non-identity automorphism of R such that $([(x^p)^\eta, x^q]x^r + x^r[(x^p)^\eta, x^q])^n \in Z(R)$ for all $x \in R$, where p,q,r,n are fixed positive integers. If either $char(R) > n$ or $char(R) = 0$, then R satisfies s_4, the standard identity in four variables.*

Proof. By the assumption, we have

$$([(x^p)^\eta, x^q]x^r + x^r[(x^p)^\eta, x^q])^n \in Z(R) \tag{3.2}$$

for all $x \in R$. Let Υ_1, Υ_2 and Υ_3 be the additive subgroups generated by $\{s^p \mid s \in R\}$, $\{s^q \mid s \in R\}$ and $\{s^r \mid s \in R\}$, respectively. Thus,

$$([x^\eta, y]z + z[x^\eta, y])^n \in Z(R) \tag{3.3}$$

for all $x \in \Upsilon_1$, $y \in \Upsilon_2$ and $z \in \Upsilon_3$. Therefore, either Υ_1 contains a non-central Lie ideal U_1 or $s^p \in Z(R)$ for all $s \in R$. The latter case forces that the ring must be commutative, a contradiction. Analogously, we may assume that there exist U_2 and U_3, non-central Lie ideals of R, which are contained in Υ_2 and Υ_3, respectively. In view of [10, pp 4-5], there exist non-zero ideals I_1, I_2 and I_3 of R such that $[I_1,R] \subseteq U_1$, $[I_2,R] \subseteq U_2$ and $[I_3,R] \subseteq U_3$. Hence,

$$([x^\eta, y]z + z[x^\eta, y])^n \in Z(R) \tag{3.4}$$

for all $x \in [I_1,I_1]$, $y \in [I_2,I_2]$ and $z \in [I_3,I_3]$. Since I_1, I_2, I_3 and R satisfy same generalized polynomial identities with automorphisms (see [8, Theorem 1]), so we have

$$([x^\eta, y]z + z[x^\eta, y])^n \in Z(R) \tag{3.5}$$

for all $x,y,z \in [R,R]$. Next, assume that $dim_C RC > 4$. Then for x_1, y_1, x_2, y_2, x_3, y_3, $z \in R$, we have

$$[([[x_1^\eta, y_1^\eta],[x_2,y_2]][x_3,y_3] + [x_3,y_3][[x_1^\eta, y_1^\eta],[x_2,y_2]])^n, z] = 0 \tag{3.6}$$

for all $x_1,y_1,x_2,y_2,x_3,y_3,z \in R$. Since either $char(R) > n$ or $char(R) = 0$, so by Fact 2 one can have

$$[([[s,t],[x_2,y_2]][x_3,y_3] + [x_3,y_3][[s,t],[x_2,y_2]])^n, z] = 0$$

for all $s,t,x_2,y_2,x_3,y_3,z \in R$. It is well known that there exists a field $\mathcal{F}$ such that R and $\mathcal{F}_m$ satisfy the same polynomial identities [13, p. 57 and 89]. Since $dim_C RC > 4$, we see that $m > 2$. By choosing $x_1 = e_{12}$, $y_1 = e_{21}$, $s = e_{21}$, $t = e_{11}$, $x_2 = e_{12}$, $y_2 = e_{22}$, $z = e_{23}$ we get a contradiction as follows

$$\begin{aligned} 0 &= [([[s,t],[x_2,y_2]][x_3,y_3] + [x_3,y_3][[s,t],[x_2,y_2]])^n, z] \\ &= [([[e_{21},e_{11}],[e_{12},e_{21}]][e_{12},e_{22}] + [e_{12},e_{22}][[e_{21},e_{11}],[e_{12},e_{21}]])^n, e_{23}] \\ &= 2^n e_{23}. \end{aligned}$$

Secondly, we now assume that η is an inner automorphism, so there exists an invertible element $q \in Q$ such that $x^\eta = qxq^{-1}$ for all $x \in R$. Since Q and I satisfy the same generalised polynomial identities [7]. Therefore, Q satisfies the following generalised polynomial identities

$$[([[x_1^\eta, y_1^\eta], [x_2, y_2]][x_3, y_3] + [x_3, y_3][[x_1^\eta, y_1^\eta], [x_2, y_2]])^n, z] = 0 \tag{3.7}$$

for all $x_1, y_1, x_2, y_2, x_3, y_3, z \in Q$. Since $q \notin C$ and (3.7) is a nontrivial generalized polynomial identity on Q. Therefore, by Martindale's theorem [17], Q is a primitive ring. Let $\mathcal{V}_Q$ be a faithful irreducible right R-module with commuting ring $\mathcal{D} = End(\mathcal{V}_Q)$, a finite-dimensional division algebra over C. By density theorem, Q acts densely on $\mathcal{V}_\mathcal{D}$. If $\mathcal{V}_\mathcal{D}$ is infinite dimensional, then

$$([[x_1^\eta, y_1^\eta], [x_2, y_2]][x_3, y_3] + [x_3, y_3][[x_1^\eta, y_1^\eta], [x_2, y_2]])^n = 0$$

holds on H, the socle of Q, and hence it also holds on Q. Thus, by Proposition 3.1 we prove the theorem in this case. So $\mathcal{V}_\mathcal{D}$ must be finite-dimensional. Thus Q is isomorphic to $\mathcal{D}_s$, the $s \times s$ matrix ring over $\mathcal{D}$ for some s. Since $\mathcal{D}$ is finite-dimensional over C, if C is finite, then $\mathcal{D}$ is a finite division ring and thus is a field by Wedderburn's theorem. In this case, $Q = C_s$. On the other hand, if C is infinite and $\mathcal{F}$ is a maximal subfield of $\mathcal{D}$, then by a Vandermonde determinant argument, we know that the condition

$$([[x_1^\eta, y_1^\eta], [x_2, y_2]][x_3, y_3] + [x_3, y_3][[x_1^\eta, y_1^\eta], [x_2, y_2]])^n \in C$$

for all $x_1, y_1, x_2, y_2, x_3, y_3 \in Q$ carries over to

$$([[x_1^\eta, y_1^\eta], [x_2, y_2]][x_3, y_3] + [x_3, y_3][[x_1^\eta, y_1^\eta], [x_2, y_2]])^n \in \mathcal{F}$$

for all $x_1, y_1, x_2, y_2, x_3, y_3 \in Q \otimes_C \mathcal{F}$. But $Q \otimes_C \mathcal{F} = \mathcal{D}_s \otimes \mathcal{F} = (\mathcal{D} \otimes_C \mathcal{F})_s = \mathcal{F}_m$ for some m. In either case, we may suppose that $Q = \mathcal{F}_m$ for some $m > 1$. Since $dim_C Q > 4$, we see that $m > 2$. Let $\mathcal{V}$ be an m-dimensional vector space over $\mathcal{F}$, and Q can be realised as a ring consisting of all $\mathcal{F}$-linear transformations of $\mathcal{V}$. For any given $v \in \mathcal{V}$, we claim that v and vq are $\mathcal{F}$-dependent. Suppose, on the contrary, that v and vq are $\mathcal{F}$-independent.

Assuming first that v, vq, vq^{-1} are $\mathcal{F}$-independent, we extend v, vq, vq^{-1} to $\mathcal{F}$-base $v, vq, vq^{-1}, v_i, \cdots, v_m$ of $\mathcal{V}_\mathcal{F}$, where $i = 4, 5, \cdots, m$ if it exists. Then by density theorem, there exist $x, y, x_1, y_1 \in Q$ such that

$$\begin{array}{lllll} vx = v, & vqx = v, & vq^{-1}x = -v, & vx_1 = 0, & vqx_1 = v, \\ vq^{-1}x_1 = v, & v_ix_1 = 0, & vx_2 = 0, & vqx_2 = v, & vq^{-1}x_2 = 0, \\ vy = v, & vqy = 0, & vq^{-1}y = 0, & v_ix_1 = 0, & vy_1 = vq, \\ vqy_1 = 0, & vq^{-1}y_1 = 0, & vy_2 = v, & vqy_2 = 0, & vq^{-1}y_2 = 0 \end{array}$$

for $i = 4, 5, \cdots, m$. Thus, we have

$$v[x_1, y_1] = 0,\ vq[x_1, y_1] = v,\ vq^{-1}[x_1, y_1] = 0,\ v[x_i, y_i] = 0,\ v[x_2, y_2] = -v,$$
$$vq^{-1}[x_2, y_2] = vq,\ v[x_3, y_3] = v,\ vq[x_3, y_3] = v,\ vq^{-1}[x_3, y_3] = 0.$$

Since rank of $[x_1, y_1]$ is 1, therefore

$$([[x_1^\eta, y_1^\eta], [x_2, y_2]][x_3, y_3] + [x_3, y_3][[x_1^\eta, y_1^\eta], [x_2, y_2]])^n$$

is of rank at most 2. Being in $\mathcal{F}$, we get that

$$([[x_1^\eta, y_1^\eta], [x_2, y_2]][x_3, y_3] + [x_3, y_3][[x_1^\eta, y_1^\eta], [x_2, y_2]])^n = 0$$

and hence we get a contradiction as

$$0 = v([q[x_1, y_1]q^{-1}, [x_2, y_2]][x_3, y_3] + [x_3, y_3][q[x_1, y_1]q^{-1}, [x_2, y_2]])^n = v.$$

On the hand if v, vq, vq^{-1} are $\mathcal{F}$-dependent. Since v and vq are $\mathcal{F}$-independent, we extend v, vq to be an $\mathcal{F}$-base $v, vq, v_i, \cdots, v_m$ of $\mathcal{V}_\mathcal{F}$, where $i = 3, 4, \cdots, m$. We have that $vq^{-1} = v\alpha + vq\beta$

for some $\alpha, \beta \in \mathcal{F}$. Moreover, we claim that $\beta \neq 0$. Indeed, if $\beta = 0$, then $vq^{-1} = \alpha v$ and $v = \alpha vq$, a contradiction. By density of Q, there exist $x, y \in Q$ such that

$$\begin{array}{llll} vx_1 = v, & vqx_1 = v, & vx_2 = 0, & vqx_2 = v, \\ vx_3 = 0, & vqx_3 = v, & vy_1 = v, & vqy_1 = 0, \\ vy_2 = vq, & vqy_2 = 0, & vy_3 = v, & vqy_3,\ vx_i = 0,\ vy_i = 0. \end{array}$$

We can easily see that

$$v[x_1, y_1] = 0, vq[x_1, y_1] = v, vq[x_2, y_2] = vq, v[x_i, y_i] = 0$$

for $i = 3, 4, \cdots, m$. Moreover, we see that $[x_1, y_1]$ is of rank 1. So

$$([q[x_1, y_1]q^{-1}, [x_2, y_2]][x_3, y_3] + [x_3, y_3][q[x_1, y_1]q^{-1}, [x_2, y_2]])^n$$

is of rank at most 2. Being in $\mathcal{F}$, we get

$$([q[x_1, y_1]q^{-1}, [x_2, y_2]][x_3, y_3] + [x_3, y_3][q[x_1, y_1]q^{-1}, [x_2, y_2]])^n = 0.$$

We also get a contradiction as follows:

$$\begin{aligned} 0 &= v([q[x_1, y_1]q^{-1}, [x_2, y_2]][x_3, y_3] + [x_3, y_3][q[x_1, y_1]q^{-1}, [x_2, y_2]])^n \\ &= 2^n \lambda^n v \neq 0. \end{aligned}$$

From the above, we have proven that $vq = \lambda(v)v$ for all $v \in \mathcal{V}$, where $\lambda(v) \in \mathcal{D}$ depends on $v \in \mathcal{V}$. A standard argument shows that $q \in C$, a contradiction. Thus $dim_C RC \leq 4$, so by Fact 2, R satisfies s_4, the standard identity in four variables. This completes the proof of the theorem. □

We immediately write the following corollaries in view of the above theorem.

Corollary 3.3. *Let R be a non-commutative prime ring and η is a non-identity automorphism of R such that $([x^\eta, x]x + x[x^\eta, x])^n \in Z(R)$ for all $x \in R$, where n is a fixed positive integer. If either $char(R) > n$ or $char(R) = 0$, then R satisfies s_4, the standard identity in four variables.*

Corollary 3.4. *Let R be a non-commutative prime ring and η be a non-identity automorphism of R such that $([(x^p)^\eta, x^q]x^r + x^r[(x^p)^\eta, x^q]) \in Z(R)$ for all $x \in R$, where p, q, r are fixed positive integers. If $char(R) \neq 2$, then R satisfies s_4, the standard identity in four variables.*

Corollary 3.5. *Let R be a non-commutative prime ring and η be a non-identity automorphism of R such that $([u^\eta, v]w + w[u^\eta, v])^n \in Z(R)$ for all $u, v, w \in L$, where L a non-central Lie ideal R and n is a fixed positive integer. If $char(R) > n$ or $char(R) = 0$, then R satisfies s_4, the standard identity in four variables.*

Proof. By the hypothesis, we have

$$([u^\eta, v]w + w[u^\eta, v])^n \in Z(R) \text{ for all } u, v, w \in L.$$

In view of Fact 2, there exists an ideal I of R such that $[I, R] \subseteq L$. This implies that

$$([x^\eta, y]z + z[x^\eta, y])^n \in Z(R) \text{ for all } x, y, z \in [I, I].$$

Since I and R satisfies same generalized polynomial identities with automorphisms (see [8, Theorem 1]), so we have

$$([x^\eta, y]z + z[x^\eta, y])^n \in Z(R) \text{ for all } x, y, z \in [R, R],$$

which is same as (3.5). Therefore, using the same arguments, we get the required result. □

References

[1] S. Ali and S. Huang, *On derivations in semiprime rings*, Algebr. Represent. Theo., **15**, 1023–1033, (2012).

[2] K. I. Beidar, W. S. Martindale III and A. V. Mikhalev, *Rings with Generalized Identities*, Marcel Dekker, **196**, New York, 1996.

[3] M. Brešar, *Centralizing mappings and derivations in prime ring*, J. Algebra, **156**, 385–394, (1993).

[4] M. Brešar, *Commuting traces of biadditive mappings, commutativity preserving mappings and Lie mappings*, Trans. Amer Math. Soc., **335(2)**, 525–546, (1993).

[5] M. Brešar, *On a generalization of the notion of centralizing mappings*, Proc. Amer. Math. Soc., **114**, 641–649, (1992).

[6] L. Carini and V. De Filliippis, *Commutators with power central values on a Lie ideals*, Pacific J. Math., **193**, 269–278, (2000).

[7] C. L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., **103**, 723–728, (1988).

[8] C. L. Chuang, *Differential identities with automorphism and anti-automorphism-II*, J. Algebra, **160**, 291–335, (1993).

[9] H. Cheng, *Some results about derivations of prime rings*, J. Math. Reser. Expos., **25(4)**, 625–633, (2005).

[10] I. N. Herstein, Topics in Ring Theory, *The University of Chicago Press, Chicago*, 1969.

[11] I. N. Herstein, *Derivations of prime rings having power central values*, Contemp. Math., **13**, 163–171, (1982).

[12] S. Huang, *Derivations with Engel conditions in prime and semiprime rings*, Czechoslovak Math. J., **61(136)**, 1135–1140, (2013).

[13] N. Jacobson, PI-algebras, an introduction, *Lecture Notes in Math.,* **441***, Berlin Heidelberg, New york: Springer Verlag*, (1975).

[14] C. Lanski, *Differential identities, Lie ideals and Posner's theorems*, Pacific J. Math., **134**, 275–297, (1988).

[15] C. Lanski, *An Engel condition with derivation*, Proc. Amer. Math. Soc., **118**, 731–734, (1993).

[16] P. H. Lee and T. K. Lee, *Lie ideals of prime rings with derivations*, Bull. Inst. Math. Acad. Sin., **11**, 75–80, (1983).

[17] W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12**, 576–584, (1969).

[18] J. H. Mayne, *Centralizing automorphisms of prime rings*, Canad. Math. Bull., **19**, 113–115, (1976).

[19] J. H. Mayne, *Centralizing automorphisms of Lie ideals in prime rings*, Canad. Math. Bull., **35**, 510–514, (1992).

[20] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8**, 1093–1100, (1957).

[21] N. Rehman and T. Bano, *A note on skew-commuting automorphism in prime ring*, Kyungpook Math. J., **55**, 21–28, (2015).

[22] J. Vukman, *Commuting and centralizing mappings in prime rings*, Proc. Amer. Math. Soc., **109(1)**, 47–52, (1990).

[23] J. Vukman, *Identities with derivations and automorphisms on semiprime rings*, Int. J. Math. Math. Sci., **7**, 1031–1038, (2005).

[24] Y. Wang, *Power-centralizing automorphisms of Lie ideals in prime rings*, Comm. Algebra, **34**, 609–615, (2006).

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