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Analytical Solution of Fractional RLC, RL, and RC Electrical Circuits Containing Caputo Derivative

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Abstract. This research focuses on establishing the fractional differential equations (FDEs) for RLC, RC, and RL circuits involving the Caputo derivative. The general form of these models is given, and the existence and uniqueness of their solutions are established. The models are solved using two analytical methods: the Adomian decomposition method (ADM) and the Picard method (PM). The convergence of the series solution is proven, and the maximum value of the error is estimated. We discuss the modeling process of these electrical circuits and demonstrate the existence and uniqueness of solutions within the Caputo fractional derivative framework, with main results showing that accurate and unique solutions can be found for fractional RLC, RC, and RL circuits using simple analytical methods. Overall, this work helps us understand and predict how these electrical circuits behave due to fractional effects.

1 Introduction

Many research fields use fractional calculus [1]-[5], including automatic control [6], medical applications [7], civil engineering [8], time series, long memory effect modeling, and deep machine learning [9]-[13]. Several papers provide additional information [14]-[16]. One of the most commonly used definitions of fractional calculus is the Caputo derivative. Two analytical methods are used to solve an important model of FDEs. These two methods are the PM and the ADM, which are used to solve the different models and equations. In 2024, Ziada used them to solve the Chandrasekhar quadratic functional integral equation [17]. In 2025, Ziada used them to solve the fractional mathematical model of brain metabolite variations in the circadian rhythm containing Caputo-Fabrizio [18].

The proposed approach offers a comprehensive framework for accurately modeling and analyzing the behavior of circuit variables, such as currents and voltages, while considering the fractional-order dynamics of the system. This paper focuses on the implementation of the Caputo derivative specifically within the context of RLC, RC, and RL circuits [19]-[22], which are integral components of electrical systems.

Applying fractional calculus to electrical circuits provides a more accurate way to model their complex behavior, and our main findings show that the solutions are guaranteed to exist, be unique, and can be effectively obtained using analytical methods with reliable accuracy. By examining the unique dynamics of these circuits through fractional calculus, we offer the existence and uniqueness of solutions to the resulting FDEs [23]-[26].

Definition and properties:

The Caputo derivative of order α of the function $f(t)$ is defined as [9],

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{(\alpha+1-n)}} ds, \quad n-1 < \alpha \leq n. $$

Or

$$ {}_a^cD_t^\alpha f(t) = I_\alpha^{n-\alpha}\frac{d^n}{dt^n}f(t), \;\; n = 1, 2, \ldots. $$

2 Applications

Here, we will present the construction of models for conventional electrical RLC, RC, and RL circuits. To derive the corresponding fractional forms of each model using a series of straightforward steps guided by the following principles [4]:

$i-$ Ohm's law:

The voltage drop across a resistor, denoted as V_R, is directly proportional to the current passing through it, represented as $i(t)$.

$ii-$ Kirchhoff's law:

1. Kirchhoff's Current Law (KCL): KCL states that the sum of currents entering a node (or junction) in an electrical circuit is equal to the sum of currents leaving that node.

2. Kirchhoff's Voltage Law (KVL): KVL states that the sum of voltage drops (or potential differences) around any closed loop in a circuit is equal to zero.

Kirchhoff's laws are used to analyze and solve complex electrical circuits by writing and solving a set of simultaneous equations based on the laws.

2.1 RLC Electrical Circuit

The RLC circuit consists of the resistance (R), inductance (L) and capacitance (C), which are connected to the voltage source (V_S) and taken as positive constants; see Figure (1). They can be connected in different ways, but here we study the following series RLC circuit. The RLC circuit is a complex circuit used for signal processing. The special electronic component inside the circuit allows it to store and release energy, separate potential or current, conduct waves, etc. [27].

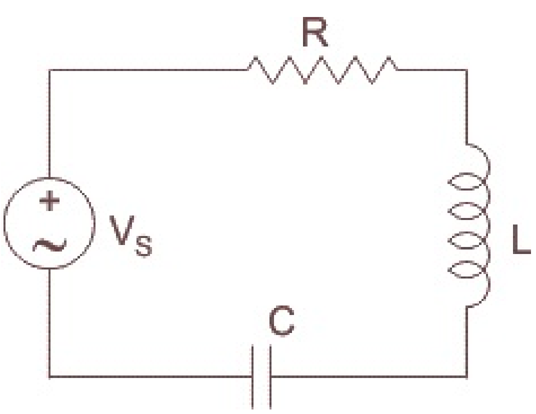

Figure 1. RLC Circuit

Using Kirchhoff's voltage law, the sum of all voltage drops around any closed loop in an electrical circuit is equal to zero. We have:

$$ V_L(t) + V_R(t) + V_C(t) = V_S(t). $$

Or

$$ \begin{aligned} L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau + v_c(0) &= V_S(t), \\ i(0) &= I_0. \end{aligned} \tag{2.1} $$

In this paper, we express the system in terms of the Caputo fractional derivatives as,

$$ D^\alpha i(t) + \frac{R}{L}i(t) + \frac{1}{LC}I^\alpha i(t) + \frac{v_c(0)}{L} = \frac{V_S(t)}{L}, $$

$$I^{1-\alpha}i(t)+\frac{R}{L}i(t)+\frac{1}{LC}I^{\alpha}i(t)+\frac{v_c(0)}{L}=\frac{V_S(t)}{L}. \tag{2.2}$$

Operating with I^{α} to both sides of (2.2), we obtain

$$I\frac{di(t)}{dt}+I^{\alpha}\frac{R}{L}i(t)+\frac{1}{LC}I^{2\alpha}i(t)+I^{\alpha}\frac{v_c(0)}{L}=I^{\alpha}\frac{V_S(t)}{L}. \tag{2.3}$$

Then,

$$\begin{aligned} i(t) &= I_0+\frac{1}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}V_S(s)ds-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i(s)ds \\ &\quad -\frac{1}{LC}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}i(s)ds-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{v_c(0)}{L}. \end{aligned} \tag{2.4}$$

Methods of Solution

First method: ADM

ADM solution algorithm Applying ADM to (2.4), the recursive relations of the ADM algorithm will be:

$$i_0(t)=I_0+\frac{1}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}V_S(s)ds+\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{v_c(0)}{L}, \tag{2.5}$$

$$i_n(t)=-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i(s)ds-\frac{1}{LC}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}i(s)ds. \tag{2.6}$$

Finally, the ADM solution of (2.1) is

$$i(t)=\sum_{n=0}^{\infty}i_n(t). \tag{2.7}$$

Convergence Analysis

Solution existence and uniqueness Define the mapping $F:E\to E$ where E is the Banach space, $(C[I],\|\cdot\|)$ is the space that consists of all continuous functions defined to the interval I with the norm

$$\|i(t)\|=\max_{t\in I}|i(t)|,\ \forall\, 0\le s\le t\le T.$$

Theorem 2.1. *The problem (2.1) has a unique solution whenever* $0<\mu_1<1$ *where*

$$\mu_1=\frac{T^{\alpha}}{L}\left[\frac{R}{\Gamma(\alpha+1)}+\frac{T^{\alpha}}{C\Gamma(2\alpha+1)}\right].$$

Proof. The mapping $F:E\to E$ can be defined as

$$\begin{aligned} Fi(t) &= I_0-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i(s)ds \\ &\quad -\frac{1}{LC}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}i(s)ds+\frac{(V_S(t)-v_c(0))t^{\alpha}}{L\Gamma(1+\alpha)}. \end{aligned}$$

Let $i(t)$ and $z(t)\in E$:

$$\begin{aligned} \|Fi-Fz\| &= \max_{t\in I}\left|-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i(s)ds-\frac{1}{LC}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}i(s)ds\right. \\ &\quad \left.+\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+\frac{1}{LC}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}z(s)ds\right|. \\ &\le \max_{t\in I}\left|-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[i(s)-z(s)]ds\right. \\ &\quad \left.-\frac{1}{LC}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}[i(s)-z(s)]ds\right| \end{aligned}$$

$$\begin{aligned}\|Fi - Fz\| &\leq \frac{R}{L}\max_{t\in I}|i(t)-z(t)|\frac{T^{\alpha}}{\Gamma(\alpha+1)}\\ &\quad -\frac{1}{CL}\max_{t\in I}|i(t)-z(t)|\frac{T^{2^{\alpha}}}{\Gamma(2\alpha+1)}\\ &\leq \frac{T^{\alpha}}{L}\left[\frac{R}{\Gamma(\alpha+1)}+\frac{T^{\alpha}}{C\Gamma(2\alpha+1)}\right]\|i-z\|\\ &\leq \mu_1\|i-z\|.\end{aligned}$$

If and only if $0 < \mu_1 < 1$, the mapping F is a contraction, then there exists a unique solution to the problem (2.1). □

Convergence proof

Theorem 2.2. *The series solution (2.7) to the problem (2.1) using the ADM converges if* $|i_1| < \infty$ *and* $0 < \mu_1 < 1$ *where:*

$$\mu_1 = \frac{T^{\alpha}}{L}\left[\frac{R}{\Gamma(\alpha+1)}+\frac{T^{\alpha}}{C\Gamma(2\alpha+1)}\right].$$

Proof. Define the sequence $\{S_n\}$ such that $S_n = \sum_{k=0}^{n} i_k(t)$ is the sequence of partial sums from the series solution. Let S_n and S_m be two arbitrary partial sums with $n > m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned}\|S_n - S_m\| &= \max_{t\in I}|S_n - S_m|\\ &= \max_{t\in I}\left|\sum_{k=m+1}^{n} i_k(t)\right|\\ &= \max_{t\in I}\left|\sum_{k=m+1}^{n} -\int_0^t \frac{R}{L}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i_k(s)ds - \frac{1}{LC}\int_0^t\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}i_k(s)ds\right|\end{aligned}$$

$$\begin{aligned}\|S_n - S_m\| &\leq \max_{t\in I}\left|-\frac{R}{L\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[S_{n-1}-S_{m-1}]ds\right.\\ &\quad\left.-\frac{1}{CL\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}[S_{n-1}-S_{m-1}]ds\right|\\ &\leq \max_{t\in I}\frac{T^{\alpha}R}{L\Gamma(\alpha+1)}\int_0^t |S_{n-1}-S_{m-1}|\,ds\\ &\quad+\max_{t\in I}\frac{T^{2\alpha}}{CL\Gamma(2\alpha+1)}\int_0^t |S_{n-1}-S_{m-1}|\,ds\\ &\leq \frac{T^{\alpha}}{L}\left[\frac{R}{\Gamma(\alpha+1)}+\frac{T^{\alpha}}{C\Gamma(2\alpha+1)}\right]\|S_{n-1}-S_{m-1}\|.\end{aligned}$$

Let $n = m+1$, then

$$\|S_{m+1}-S_m\| \leq \mu_1\|S_m - S_{m-1}\| \leq \mu_1^2\|S_{m-1}-S_{m-2}\| \leq .... \leq \mu_1^m\|S_1 - S_0\|.$$

From the triangle inequality, we have

$$\begin{aligned}\|S_n - S_m\| &\leq \|S_{m+1}-S_m\| + \|S_{m+2}-S_{m+1}\| + ... + \|S_n - S_{n-1}\|\\ &\leq [\mu_1^m + \mu_1^{m+1} + ... + \mu_1^{n-1}]\|S_1 - S_0\|\\ &\quad \mu_1^m[\frac{1-\mu_1^{n-m}}{1-\mu_1}]\|i(t)\|.\end{aligned}$$

As $0 < \mu_1 < 1$, and $n > m$, then $\left(1 - \mu_1^{n-m}\right) \le 1$. Therefore:

$$\begin{aligned} \|S_n - S_m\| &\le \frac{\mu_1^m}{1-\mu_1}\|i(t)\| \\ &\le \frac{\mu_1^m}{1-\mu_1} \max_{t\in I} |i_1(t)| . \end{aligned}$$

But $|i_1(t)| < \infty$ and as $m \to \infty$, $\|S_n - S_m\| \to 0$, hence $\{S_n\}$ is a Cauchy sequence in this Banach space, so the series $\sum_{n=0}^{\infty} i_n(t)$ converges. □

Error analysis For the ADM, we can assess the maximum absolute truncation error of the series solution as outlined in the subsequent theorem.

Theorem 2.3. *The maximum absolute truncation error of the series solution (2.7) to the problem (2.1) is estimated to be:*

$$\max_{t\in I}\left| i(t) - \sum_{k=0}^{m} i_k(t)\right| \le \frac{\mu_1^m}{1-\mu_1} \max_{t\in I} |i_1(t)| .$$

Proof. From Theorem 2.2, we have

$$\|S_n - S_m\| \le \frac{\mu_1^m}{1-\mu_1} \max_{t\in I} |i_1(t)| .$$

But, $S_n = \sum_{k=0}^{n} i_k(t)$ as $n \to \infty$, then $S_n \to i(t)$, so

$$\|i(t) - S_m\| \le \frac{\mu_1^m}{1-\mu_1} \max_{t\in I} |i_1(t)| .$$

Therefore, the maximum absolute truncation error in the interval I is

$$\max_{t\in I}\left| i(t) - \sum_{k=0}^{m} i_k(t)\right| \le \frac{\mu_1^m}{1-\mu_1} \max_{t\in I} |i_1(t)| .$$

□

Second Method: PM

PM solution algorithm Applying PM to (2.4), the solution will be

$$i_0(t) = I_0 + \frac{1}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} V_S(s)ds - \frac{t^\alpha}{\Gamma(1+\alpha)}\frac{v_c(0)}{L}, \tag{2.8}$$

$$i_n(t) = i_0(t) - \frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds - \frac{1}{LC}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} i(s)ds \tag{2.9}$$

All the functions $i_n(t)$ are continuous functions, and $i_n(t)$ is the sum of successive differences

$$i_n(t) = i_0(t) + \sum_{k=1}^{n}[i_n(t) - i_{n-1}(t)].$$

This means that the sequence $i_n(t)$ convergence is equivalent to the infinite series convergence.
The final PM solution takes the form:

$$i(t) = \lim_{n\to\infty} i_n(t). \tag{2.10}$$

Convergence Analysis We can deduce that if the series $\sum_{k=1}^{n}[i_n(t) - i_{n-1}(t)]$ is convergent, then the sequence $\{i_n(t)\}$ will converge to $i(t)$.

To prove the convergence of the sequence $\{i_n(t)\}$, consider the related series:

$$\sum_{k=0}^{\infty}[i_k(t) - i_{k-1}(t)]$$

For $k = 1$, we get

$$\begin{aligned}
|i_1(t) - i_0(t)| &= \left|-\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_0(s)ds - \frac{1}{LC}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} i_0(s)ds\right| \\
&\leq |i_0(t)|\left[\frac{R}{L}\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{1}{LC}\frac{T^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\
&\leq \frac{T^\alpha}{L}\left[\frac{R}{\Gamma(\alpha+1)} + \frac{T^\alpha}{C\Gamma(2\alpha+1)}\right]\eta_1 \leq \varphi_1,
\end{aligned}$$

where $|i_0(t)| \leq \eta_1$ and $\varphi_1 = \frac{T^\alpha}{L}\left[\frac{R}{\Gamma(\alpha+1)} + \frac{T^\alpha}{C\Gamma(2\alpha+1)}\right]\eta_1$.

Now, we will get an estimate for $i_n(t) - i_{n-1}(t), n \geq 2$

$$\begin{aligned}
|i_n(t) - i_{n-1}(t)| &= \left|-\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_{n-1}(s)ds - \frac{1}{LC}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} i_{n-1}(s)ds\right. \\
&\quad \left. + \frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_{n-2}(s)ds + \frac{1}{LC}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} i_{n-2}(s)ds\right| \\
&\leq \left[\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \frac{1}{LC}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}ds\right]|i_{n-1}(t) - i_{n-2}(t)| \\
&\leq [\frac{T^\alpha}{L}\left[\frac{R}{\Gamma(\alpha+1)} + \frac{T^\alpha}{C\Gamma(2\alpha+1)}\right]|i_{n-1}(t) - i_{n-2}(t)| \\
&\leq \mu_1|i_{n-1}(t) - i_{n-2}(t)|.
\end{aligned}$$

In the above equation, if we put $n = 2$:

$$\begin{aligned}
|i_2(t) - i_1(t)| &\leq \frac{T^\alpha}{L}\left(\frac{R}{\Gamma(\alpha+1)} + \frac{T^\alpha}{C\Gamma(2\alpha+1)}\right)|i_1(t) - i_0(t)| \\
&\leq \mu_1\varphi_1.
\end{aligned}$$

Doing the same for $n = 3, 4, \ldots$

$$|i_3(t) - i_2(t)| \leq \mu_1|i_2(t) - i_1(t)| \leq \mu_1^2\varphi_1,$$

$$|i_4(t) - i_3(t)| \leq \mu_1|i_2(t) - i_1(t)| \leq \mu_1^3\varphi_1,$$

$$\vdots$$

Then the general solution will be:

$$|i_n(t) - i_{n-1}(t)| \leq \mu_1^{n-1}\varphi_1.$$

Since $\mu_1 < 1$ then, the sequence $\{i_n(t)\}$ will be convergent.

$$i(t) = \lim_{n\to\infty}\left(-\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds - \frac{1}{LC}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} i(s)ds\right)$$

$$i(t) = -\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds - \frac{1}{LC}\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} i(s)ds.$$

Example (1): *Consider the following circuit shown in Figure (2), i(0)=5 A, v(0)=2.5 V, and 0.5. Ω The resistor represents the resistance of the inductor:*

Solution:

1- ADM Solution:

From (2.5) and (2.6), we get

$$i_0(t) = 5 + \frac{(12500t^{\alpha})}{\Gamma(1+\alpha)},$$

$$i_n(t) = -500\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds - 60000\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} i(s)ds.$$

Hence,

$$i(t) = \sum_{n=0}^{\infty} i_n(t).$$

2- PM Solution:

From (2.5) and (2.6), we get

$$\begin{aligned} i_0(t) &= 5 + \frac{(12500t^{\alpha})}{\Gamma(1+\alpha)}, \\ i_n(t) &= i_0(t) - 500\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds - 60000\int_0^t \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} i(s)ds. \end{aligned}$$

Hence,

$$i(t) = \lim_{n\to\infty} i_n(t).$$

Figures (3) and (4) illustrate ADM and PM solutions at multiple values of α, while Figure (5) illustrates a comparison between ADM and PM solutions at the same value of α. All calculations and graphical representations in the paper were performed using MATHEMATICA 5.2 software for the examples presented.

From these figures, we see that when we increase the number of the terms n, the solution will be more accurate.

Table (1.1) shows the absolute difference (AD) between ADM and PM solutions, while Table (1.2) shows a time comparison between them.

Table 1.1: Absolute difference

t	$\lvert i_{ADM} - i_{PM}\rvert$
0.0001	1.5763×10^{-9}
0.0002	3.00484×10^{-9}
0.0003	4.37953×10^{-9}
0.0004	5.71967×10^{-9}
0.0005	7.03434×10^{-9}
0.0006	8.32883×10^{-9}
0.0007	9.60663×10^{-9}
0.0008	1.08702×10^{-8}

Table 1.2: Time comparison

ADM time	PM time
1520.17	5480.12

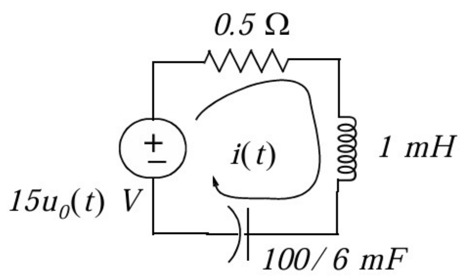

Figure 2. Example (1) Fig.

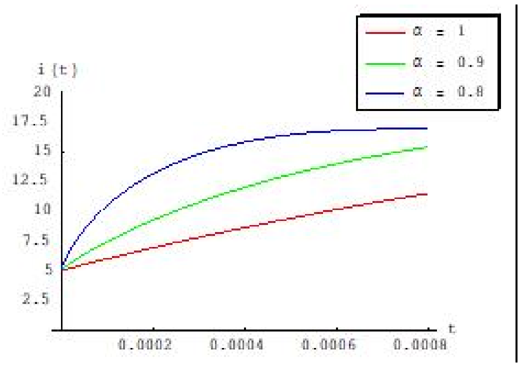

Figure 3. ADM solution for various α of eq. 7

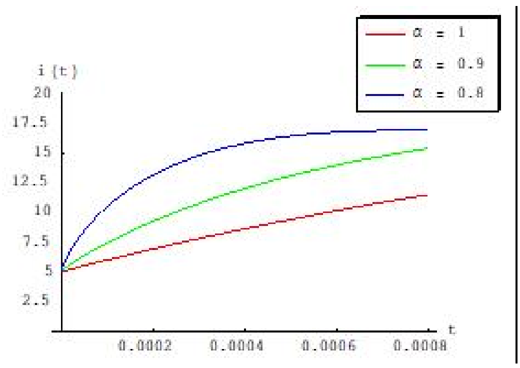

Figure 4. PM solution for various α of eq. 10

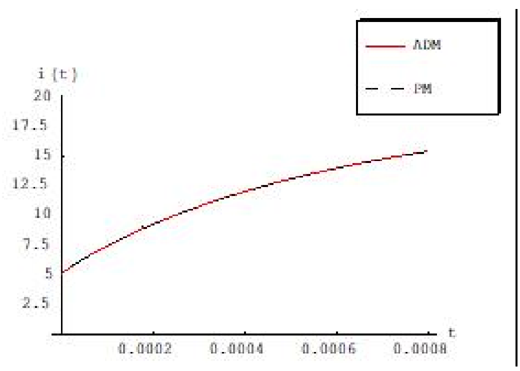

Figure 5. ADM and PM solutions for α = 0.9

From table 1.2 we deduce that ADM gives results faster than PM.

2.2 RC Electrical Circuit

The RC circuit consists of R and C connected in series with a voltage source (V_S), see Figure (6). When a voltage is applied to an RC circuit, the capacitor either charges or discharges through the resistor, resulting in a time-dependent response. The behavior of the circuit is characterized by exponential charging and discharging curves, which depend on the resistance and capacitance values.

Using Kirchhoff's voltage law, the sum of all voltage drops around any closed loop in an electrical circuit is equal to zero. We have:

$$V_R(t) + V_C(t) = V_S(t).$$

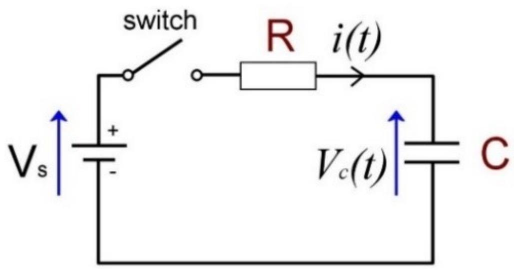

Figure 6. RC circuit

Or

$$\begin{aligned} Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau + v_c(0) &= V_S(t), \\ i(0) &= I_0. \end{aligned} \tag{2.11}$$

In this paper, we express the RC circuit in terms of the Caputo fractional derivatives as

$$Ri(t) + \frac{1}{C} I^\alpha i(t) + v_c(0) = V_S(t). \tag{2.12}$$

Then,

$$i(t) = \frac{V_S(t)}{R} - \frac{v_c(0)}{R} - \frac{1}{RC}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds. \tag{2.13}$$

Methods of Solution

First method: ADM

ADM solution algorithm Applying ADM to (2.13), the recursive relations of the ADM algorithm will be:

$$i_0(t) = \frac{V_S(t)}{R} - \frac{v_c(0)}{R}, \tag{2.14}$$

$$i_n(t) = -\frac{1}{RC}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds. \tag{2.15}$$

Finally, the ADM solution of (2.11) is

$$i(t) = \sum_{n=0}^{\infty} i_n(t). \tag{2.16}$$

Convergence Analysis

Solution existence and uniqueness

Theorem 2.4. *The problem (2.11) has a unique solution whenever* $0 < \mu_2 < 1$ *where*

$$\mu_2 = \frac{1}{RC}\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\right].$$

Proof. The mapping $F : E \to E$ can be defined as,

$$Fi(t) = \frac{V_S(t)}{R} - \frac{v_c(0)}{R} - \frac{1}{RC}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds.$$

Let $i(t)$ and $z(t) \in E$

$$\begin{aligned}
\|Fi - Fz\| &= \max_{t\in I}\left|-\frac{1}{RC}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds + \frac{1}{RC}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds\right| \\
&\leq \frac{1}{RC}\max_{t\in I}|i(t)-z(t)|\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right| \\
&\leq \frac{1}{RC}\max_{t\in I}|i(t)-z(t)|\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\right] \\
&\leq \frac{1}{RC}\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\right]\|i-z\| \\
&\leq \mu_2\|i-z\|.
\end{aligned}$$

If and only if $0 < \mu_2 < 1$, the mapping F is a contraction, then there exists a unique solution to the problem (2.11). □

Proof of convergence

Theorem 2.5. *The series solution (2.16) of the problem (2.11) using ADM converges if $|i_1| < \infty$ and $0 < \mu_2 < 1$ where:*

$$\mu_2 = \frac{1}{RC}\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\right].$$

Proof. Define the sequence $\{S_n\}$ such that $S_n = \sum_{k=0}^{n} i_k(t)$ is the sequence of partial sums from the series solution. Let S_n and S_m be two arbitrary partial sums with $n > m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned}
\|S_n - S_m\| &= \max_{t\in I}|S_n - S_m| \\
&= \max_{t\in I}\left|\sum_{k=m+1}^{n} i_k(t)\right| \\
&= \max_{t\in I}\left|\sum_{k=m+1}^{n}\left(\frac{1}{RC}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_k(s)ds\right)\right| \\
&\leq \max_{t\in I}\left|-\frac{1}{RC\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[S_{n-1}-S_{m-1}]ds\right| \\
&\leq \max_{t\in I}\frac{T^\alpha}{RC\Gamma(\alpha+1)}\int_0^t |S_{n-1}-S_{m-1}|\,ds \\
&\leq \frac{1}{RC}[\frac{T^\alpha}{\Gamma(\alpha+1)}]\,\|S_{n-1}-S_{m-1}\|.
\end{aligned}$$

Let $n = m+1$ then,

$$\|S_{m+1}-S_m\| \leq \mu_2\|S_m - S_{m-1}\| \leq \mu_2^2\|S_{m-1}-S_{m-2}\| \leq .... \leq \mu_2^m\|S_1 - S_0\|$$

From the triangle inequality we have,

$$\begin{aligned}
\|S_n - S_m\| &\leq \|S_{m+1}-S_m\| + \|S_{m+2}-S_{m+1}\| + ... + \|S_n - S_{n-1}\| \\
&\leq [\mu_2^m + \mu_2^{m+1} + ... + \mu_2^{n-1}]\,\|S_1 - S_0\| \\
&\leq \mu_2^m[\frac{1-\mu_2^{n-m}}{1-\mu_2}]\|i(t)\|.
\end{aligned}$$

As $0 < \mu_2 < 1$, and $n > m$, then $\left(1-\mu_2^{n-m}\right) \leq 1$. Therefore,

$$\begin{aligned}
\|S_n - S_m\| &\leq \frac{\mu_2^m}{1-\mu_2}\|i(t)\| \\
&\leq \frac{\mu_2^m}{1-\mu_2}\max_{t\in I}|i_1(t)|.
\end{aligned}$$

But $|i_1(t)| < \infty$ and as $m \to \infty$, $\|S_n - S_m\| \to 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space, so the series $\sum_{n=0}^{\infty} i_n(t)$ converges. □

Error analysis

Theorem 2.6. *The maximum absolute truncation error of the series solution (2.16) to the problem (2.11) is estimated to be:*

$$\max_{t\in I}\left|i(t) - \sum_{k=0}^{m} i_k(t)\right| \leq \frac{\mu_2^m}{1-\mu_2}\max_{t\in I}|i_1(t)|.$$

Proof. From Theorem 2.5 we have,

$$\|S_n - S_m\| \leq \frac{\mu_2^m}{1-\mu_2}\max_{t\in I}|i_1(t)|.$$

But, $S_n = \sum_{k=0}^{n} i_k(t)$ as $n \to \infty$, then $S_n \to i(t)$, so

$$\|i(t) - S_m\| \leq \frac{\mu_2^m}{1-\mu_2}\max_{t\in I}|i_1(t)|.$$

Therefore, the maximum absolute truncation error in the interval I is

$$\max_{t\in I}\left|i(t) - \sum_{k=0}^{m} i_k(t)\right| \leq \frac{\mu_2^m}{1-\mu_2}\max_{t\in I}|i_1(t)|.$$

□

Second Method: PM

PM solution algorithm Applying PM to IDE (2.11), the solution will be

$$i_0(t) = \frac{V_S(t)}{R} - \frac{v_c(0)}{R}, \tag{2.17}$$

$$i_n(t) = i_0(t) - \frac{1}{RC}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds. \tag{2.18}$$

All the functions $i_n(t)$ are continuous functions, and $i_n(t)$ is the sum of successive differences,

$$i_n(t) = i_0(t) + \sum_{k=1}^{n}[i_k(t) - i_{k-1}(t)].$$

This means that the sequence $i_n(t)$ convergence is equivalent to the infinite series convergence.

The final PM solution takes the form

$$i(t) = \lim_{n\to\infty} i_n(t). \tag{2.19}$$

Convergence Analysis We can deduce that if the series $\sum_{k=1}^{n}[i_k(t) - i_{k-1}(t)]$ is convergent, then the sequence $\{i_n(t)\}$ will converge to $i(t)$.

To prove the convergence of the sequence $\{i_n(t)\}$, consider the related series,

$$\sum_{k=0}^{\infty}[i_k(t) - i_{k-1}(t)]$$

For $k = 1$, we get

$$\begin{aligned} |i_1(t) - i_0(t)| &= \left| -\frac{1}{RC} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_0(s) ds \right| \\ &\leq |i_0(t)| \left[\frac{1}{RC} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\ &\leq |i_0(t)| \left[\frac{1}{RC} \frac{T^\alpha}{\Gamma(\alpha+1)} \right] \\ &\leq \left[\frac{1}{RC} \frac{T^\alpha}{\Gamma(\alpha+1)} \right] \eta_2 \leq \varphi_2, \end{aligned}$$

where $|i_0(t)| \leq \eta_2$ and $\varphi_2 = \left[\frac{1}{RC} \frac{T^\alpha}{\Gamma(\alpha+1)} \right] \eta_2$.
Now, we will get an estimate for $i_n(t) - i_{n-1}(t), n \geq 2$

$$\begin{aligned} |i_n(t) - i_{n-1}(t)| &= \left| -\frac{1}{RC} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_{n-1}(s) ds + \frac{1}{RC} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_{n-2}(s) ds \right| \\ &\leq [\frac{1}{RC} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds] |i_{n-1}(t) - i_{n-2}(t)| \\ &\leq [\frac{1}{RC} \frac{T^\alpha}{\Gamma(\alpha+1)}] |i_{n-1}(t) - i_{n-2}(t)| \\ &\leq \mu_2 |i_{n-1}(t) - i_{n-2}(t)|. \end{aligned}$$

In the above equation, if we put $n = 2$:

$$\begin{aligned} |i_2(t) - i_1(t)| &\leq \left[\frac{1}{RC} \frac{T^\alpha}{\Gamma(\alpha+1)} \right] |i_1(t) - i_0(t)| \\ &\leq \mu_2 \varphi_2. \end{aligned}$$

Doing the same for $n = 3, 4, \ldots$

$$|i_3(t) - i_2(t)| \leq \mu_2 |i_2(t) - i_1(t)| \leq \mu_2^2 \varphi_2,$$

$$\begin{aligned} |i_4(t) - i_3(t)| &\leq \mu_2 |i_2(t) - i_1(t)| \leq \mu_2^3 \varphi_2, \\ &\vdots \end{aligned}$$

Then the general solution will be,

$$|i_n(t) - i_{n-1}(t)| \leq \mu_2^{n-1} \varphi_2.$$

Since $\mu_2 < 1$, so the sequence $\{i_n(t)\}$ will be convergent.

$$i(t) = \lim_{n\to\infty} \left(-\frac{1}{RC} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s) ds \right)$$

$$i(t) = -\frac{1}{RC} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s) ds.$$

Example (2): Consider the following circuit shown in Figure (7),

$$V_S = 1V, R = 1000\Omega, C = 0.1mF, v_c(0) = 1V$$

We will solve it when:
(1) S1 is closed and S2 is open (charging the capacitor).
(2) S1 is open and S2 is closed (discharging the capacitor).
Solution:

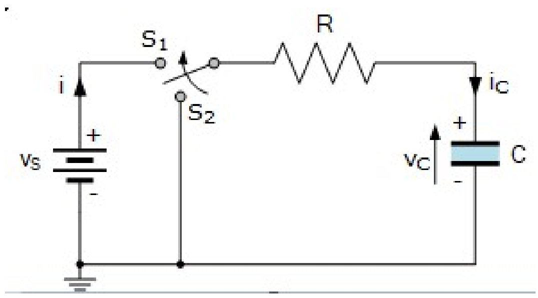

Figure 7. Example (2) Fig.

(1) When S1 is closed and S2 is open (charging the capacitor)
$i-$ ADM Solution:
From (2.14) and (2.15), we get

$$i_0(t) = \frac{V_S}{R} = 0.001,$$
$$i_n(t) = -10\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds.$$

Hence,

$$i(t) = \sum_{n=0}^{\infty} i_n(t).$$

$ii-$ PM Solution:
From (2.17) and (2.18), we get

$$i_0(t) = \frac{V_S}{R} = 0.001,$$
$$i_n(t) = i_0(t) - 10\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds.$$

Hence,

$$i(t) = \lim_{n\to\infty} i_n(t).$$

Figures (8) and (9) illustrate ADM and PM solutions at multiple values of α, while Figure (10) illustrates a comparison between ADM and PM solutions at the same value of α.

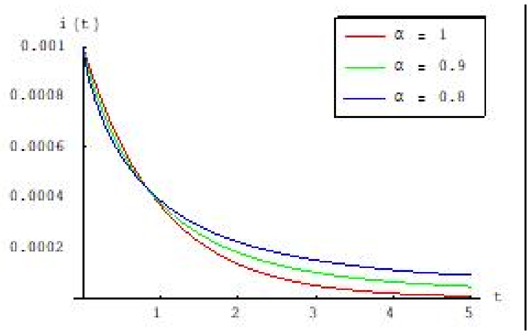

Figure 8. ADM solution for various α of eq. 16

From these figures, we see that when we increase the number of the terms n, the solution will be more accurate.

Table (2.1) shows the absolute difference (AD) between ADM and PM solutions, while Table (2.2) shows a time comparison between them.

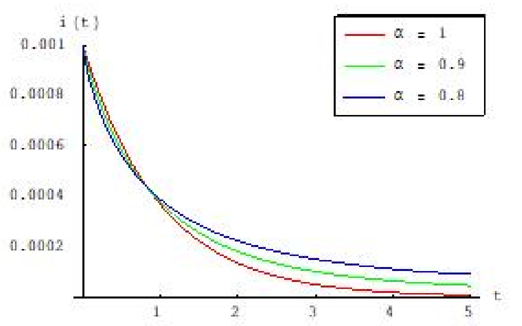

Figure 9. PM solution for various α of eq. 19

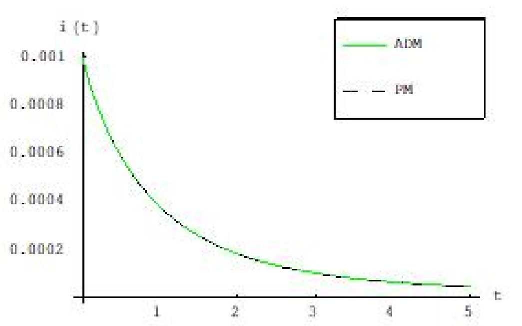

Figure 10. ADM and PM solutions for α = 0.9

Table 2.1: *Absolute difference*

t	$\lvert i_{ADM} - i_{PM}\rvert$
0.1	1.04863×10^{-22}
0.2	6.63859×10^{-22}
0.3	1.9366×10^{-21}
0.4	4.11605×10^{-21}
0.5	7.3571×10^{-21}
0.6	1.17884×10^{-20}
0.7	1.75189×10^{-20}
0.8	2.46429×10^{-20}
0.9	3.32421×10^{-20}
1.0	4.33895×10^{-20}

Table 2.2: *Time comparison*

ADM time	PM time
1.374	249.859

(2) When S1 is open and S2 is closed (discharging the capacitor)
$i-$ ADM Solution:
From (2.14) and (2.15), we get

$$\begin{aligned} i_0(t) &= \frac{-v_c(0)}{R} = -0.001, \\ i_n(t) &= -10\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds. \end{aligned}$$

Hence,

$$i(t) = \sum_{n=0}^{\infty} i_n(t).$$

$ii-$ PM Solution:

From (2.17) and (2.18), we get

$$\begin{aligned} i_0(t) &= -0.001, \\ i_n(t) &= i_0(t) - 10\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds. \end{aligned}$$

Hence,

$$i(t) = \lim_{n\to\infty} i_n(t).$$

Figures (11) and (12) illustrate ADM and PM solutions at multiple values of α, while Figure (13) illustrates a comparison between ADM and PM solutions at the same value of α.

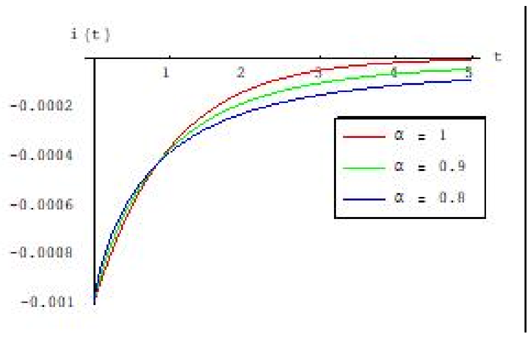

Figure 11. ADM solution for various α of eq. 16

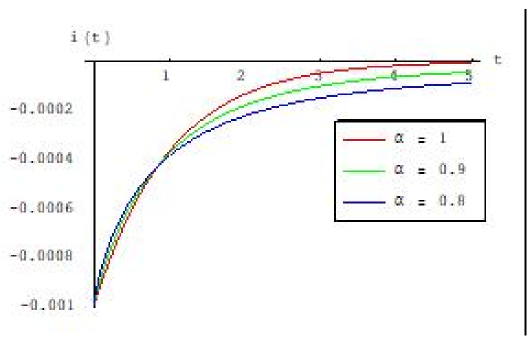

Figure 12. PM solution for various α of eq. 19

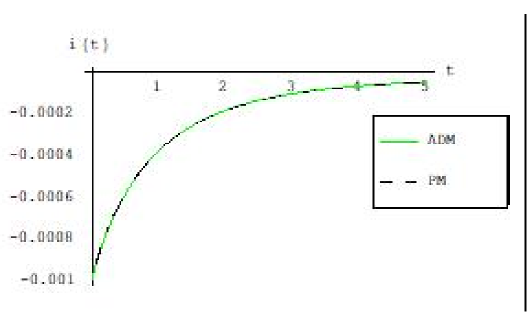

Figure 13. ADM and PM solutions for α = 0.9

From these figures, we see that when we increase the number of the terms n, the solution will be more accurate.

Table (2.3) shows the AD between ADM and PM solutions, while table (2.4) shows a time comparison between them.

Table 2.3: Absolute difference

t	$\lvert i_{ADM} - i_{PM} \rvert$
0.1	1.04863×10^{-22}
0.2	6.63859×10^{-22}
0.3	1.9366×10^{-21}
0.4	4.11605×10^{-21}
0.5	7.3571×10^{-21}
0.6	1.17884×10^{-20}
0.7	1.75189×10^{-20}
0.8	2.46429×10^{-20}
0.9	3.32421×10^{-20}
1.0	4.33895×10^{-20}

Table 2.4: Time comparison

ADM time	PM time
1.595	257.11

2.3 RL Electrical Circuit

The RL circuit is an electrical network composed of R and L; see Figure (14). These elements can be arranged in series or parallel configurations and are powered by either voltage or current sources. The RL circuit is an important component in many electronic systems and is used in various applications, including filters, oscillators, and power supplies [4]. These circuits are notable for their ability to store and dissipate energy through their resistor and inductor components.

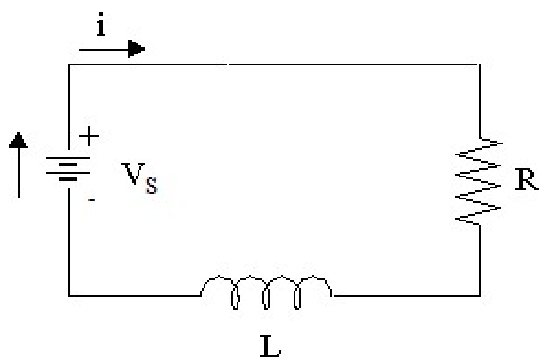

Figure 14. RL circuit

Using Kirchhoff's voltage law, the sum of all voltage drops around any closed loop in an electrical circuit is equal to zero. We have:

$$V_R(t) + V_L(t) = V_S(t).$$

Or

$$\begin{aligned} Ri(t) + L\frac{di(t)}{dt} &= V_S(t), \\ i(0) &= I_0. \end{aligned} \tag{2.20}$$

Here, we express the model for RL circuit in the form of Caputo FDE as,

$$Ri(t) + L\frac{di(t)}{dt} = V_S(t).$$

From (2.20), we have

$$Ri(t) + LD^{\alpha}i(t) = V_S(t),$$

$$Ri(t) + LI^{1-\alpha}\frac{di(t)}{dt} = V_S(t). \tag{2.21}$$

Operating with I^{α} to both sides of (2.21), we obtain

$$I\frac{di(t)}{dt}+\frac{R}{L}I^{\alpha}i(t)=I^{\alpha}\frac{V_S(t)}{L}. \tag{2.22}$$

Then,

$$i(t)=I_0+\frac{1}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}V_S(s)ds-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i(s)ds. \tag{2.23}$$

Methods of Solution

First method: ADM

ADM solution algorithm Applying ADM to (2.23), the recursive relations of the ADM algorithm will be:

$$i_0(t)=I_0+\frac{1}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}V_S(s)ds, \tag{2.24}$$

$$i_n(t)=-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i(s)ds. \tag{2.25}$$

Finally, the ADM solution of (2.20) is

$$i(t)=\sum_{n=0}^{\infty}i_n(t). \tag{2.26}$$

Convergence Analysis

Solution existence and uniqueness

Theorem 2.7. *The problem (2.20) has a unique solution whenever* $0<\mu_3<1$ *where*

$$\mu_3=\frac{R}{L}\left[\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right].$$

Proof. The mapping $F:E\to E$ can be defined as,

$$Fi(t)=I_0+\frac{1}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}V_S(s)ds-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i(s)ds.$$

Let $i(t)$ and $z(t)\in E$:

$$\begin{aligned}
\|Fi-Fz\| &= \max_{t\in I}\left|-\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}i(s)ds+\frac{R}{L}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds\right|\\
&\le \frac{R}{L}\max_{t\in I}|i(t)-z(t)|\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right|\\
&\le \frac{R}{L}\max_{t\in I}|i(t)-z(t)|\left[\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right]\\
&\le \frac{R}{L}\left[\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right]\|i-z\|\\
&\le \mu_3\|i-z\|.
\end{aligned}$$

If and only if $0<\mu_3<1$, the mapping F is a contraction, then there exists a unique solution to the problem (2.20). □

Proof of convergence

Theorem 2.8. *The series solution (2.26) of the problem (2.20) using ADM converges if* $|i_1| < \infty$ *and* $0 < \mu_3 < 1$ *where:*

$$\mu_3 = \frac{R}{L}\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\right].$$

Proof. Define the sequence $\{S_n\}$ such that $S_n = \sum_{k=0}^{n} i_k(t)$ is the sequence of partial sums from the series solution. Let S_n and S_m be two arbitrary partial sums with $n > m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned}
\|S_n - S_m\| &= \max_{t\in I}|S_n - S_m| \\
&= \max_{t\in I}\left|\sum_{k=m+1}^{n} i_k(t)\right| \\
&\le \max_{t\in I}\left|\sum_{k=m+1}^{n}\left(\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_k(s)ds\right)\right| \\
&\le \max_{t\in I}\left|-\frac{R}{L}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[S_{n-1} - S_{m-1}]ds\right| \\
&\le \max_{t\in I}\frac{R}{L}\frac{T^\alpha}{\Gamma(\alpha+1)}\int_0^t |S_{n-1} - S_{m-1}|\, ds \\
&\le \frac{R}{L}\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\right]\|S_{n-1} - S_{m-1}\|.
\end{aligned}$$

Let $n = m+1$ then,

$$\|S_{m+1} - S_m\| \le \mu_3\|S_m - S_{m-1}\| \le \mu_3^2\|S_{m-1} - S_{m-2}\| \le .... \le \mu_3^m\|S_1 - S_0\|.$$

From the triangle inequality we have,

$$\begin{aligned}
\|S_n - S_m\| &\le \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + ... + \|S_n - S_{n-1}\| \\
&\le [\mu_3^m + \mu_3^{m+1} + ... + \mu_3^{n-1}]\|S_1 - S_0\| \\
&\le \mu_3^m[\frac{1-\mu_3^{n-m}}{1-\mu_3}]\|i(t)\|.
\end{aligned}$$

As $0 < \mu_3 < 1$, and $n > m$, then $\left(1 - \mu_3^{n-m}\right) \le 1$. Therefore,

$$\begin{aligned}
\|S_n - S_m\| &\le \frac{\mu_3^m}{1-\mu_3}\|i(t)\| \\
&\le \frac{\mu_3^m}{1-\mu_3}\max_{t\in I}|i_1(t)|.
\end{aligned}$$

But $|i_1(t)| < \infty$ and as $m \to \infty$, $\|S_n - S_m\| \to 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space, so the series $\sum_{n=0}^{\infty} i_n(t)$ converges. □

Error analysis

Theorem 2.9. *The maximum absolute truncation error of the series solution (2.26) to the problem (2.20) is estimated to be:*

$$\max_{t\in I}\left|i(t) - \sum_{k=0}^{m} i_k(t)\right| \le \frac{\mu_3^m}{1-\mu_3}\max_{t\in I}|i_1(t)|.$$

Proof. From Theorem 2.8 we have,

$$\|S_n - S_m\| \leq \frac{\mu_3^m}{1-\mu_3} \max_{t\in I} |i_1(t)| .$$

But, $S_n = \sum_{k=0}^{n} i_k(t)$ as $n \to \infty$, then $S_n \to i(t)$, so

$$\|i(t) - S_m\| \leq \frac{\mu_3^m}{1-\mu_3} \max_{t\in I} |i_1(t)| .$$

Therefore, the maximum absolute truncation error in the interval I is

$$\max_{t\in I} \left| i(t) - \sum_{k=0}^{m} i_k(t) \right| \leq \frac{\mu_3^m}{1-\mu_3} \max_{t\in I} |i_1(t)| .$$

□

Second Method: PM

PM solution algorithm Applying PM to IDE (2.23), the solution will be

$$i_0(t) = I_0 + \frac{1}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} V_S(s)ds, \tag{2.27}$$

$$i_n(t) = i_0(t) - \frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds. \tag{2.28}$$

All the functions $i_n(t)$ are continuous functions, and $i_n(t)$ is the sum of successive differences,

$$i_n(t) = i_0(t) + \sum_{k=1}^{n} [i_k(t) - i_{k-1}(t)].$$

This means that the sequence $i_n(t)$ convergence is equivalent to the infinite series convergence.

The final PM solution takes the form

$$i(t) = \lim_{n\to\infty} i_n(t). \tag{2.29}$$

Convergence Analysis We can deduce that if the series $\sum_{k=1}^{n} [i_k(t) - i_{k-1}(t)]$ is convergent, then the sequence $\{i_n(t)\}$ will converge to $i(t)$.

To prove the convergence of the sequence $\{i_n(t)\}$, consider the related series,

$$\sum_{k=0}^{\infty} [i_k(t) - i_{k-1}(t)].$$

For $k = 1$, we get

$$\begin{aligned}
|i_1(t) - i_0(t)| &= \left| -\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_0(s)ds \right| \\
&\leq |i_0(t)| \left[\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\
&\leq |i_0(t)| \left[\frac{R}{L} \frac{T^\alpha}{\Gamma(\alpha+1)} \right] \\
&\leq \left[\frac{R}{L} \frac{T^\alpha}{\Gamma(\alpha+1)} \right] \eta_3 \leq \varphi_3,
\end{aligned}$$

where $|i_0(t)| \leq \eta_3$ and $\varphi_3 = \left[\frac{R}{L}\frac{T^\alpha}{\Gamma(\alpha+1)}\right]\eta_3$.
Now, we will get an estimate for $i_n(t) - i_{n-1}(t), n \geq 2$

$$\begin{aligned}|i_n(t) - i_{n-1}(t)| &= \left|-\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_{n-1}(s)ds + \frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i_{n-2}(s)ds\right| \\ &\leq \left[\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]|i_{n-1}(t) - i_{n-2}(t)| \\ &\leq \left[\frac{R}{L}\frac{T^\alpha}{\Gamma(\alpha+1)}\right]|i_{n-1}(t) - i_{n-2}(t)| \\ &\leq \mu_3|i_{n-1}(t) - i_{n-2}(t)|.\end{aligned}$$

In the above equation, if we put $n = 2$:

$$\begin{aligned}|i_2(t) - i_1(t)| &\leq \left[\frac{R}{L}\frac{T^\alpha}{\Gamma(\alpha+1)}\right]|i_1(t) - i_0(t)| \\ &\leq \mu_3\varphi_3.\end{aligned}$$

Doing the same for $n = 3, 4, \ldots$

$$|i_3(t) - i_2(t)| \leq \mu_3|i_2(t) - i_1(t)| \leq \mu_3^2\varphi_3,$$

$$|i_4(t) - i_3(t)| \leq \mu_3|i_2(t) - i_1(t)| \leq \mu_3^3\varphi_3,$$
$$\vdots$$

Then the general solution will be,

$$|i_n(t) - i_{n-1}(t)| \leq \mu_3^{n-1}\varphi_3.$$

Since $\mu_3 < 1$, so the sequence $\{i_n(t)\}$ will be convergent.

$$i(t) = \lim_{n\to\infty}\left(-\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds\right),$$

$$i(t) = -\frac{R}{L}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds.$$

Example (3): Consider the following circuit of Figure (15),

$$V_S = 6.3V, R = 50\Omega, L = 10H.$$

Solution:
$i-$ ADM Solution:
From (2.24) and (2.25), we get

$$\begin{aligned}i_0(t) &= \frac{6.3}{10}\frac{t^\alpha}{\Gamma(1+\alpha)}, \\ i_n(t) &= i_0(t) - 5\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds.\end{aligned}$$

Hence,

$$i(t) = \sum_{n=0}^{\infty} i_n(t).$$

PM Solution:

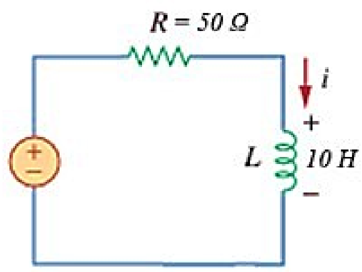

Figure 15. Example (3) Fig.

From (2.27) and (2.28), we get

$$i_0(t) = \frac{6.3}{10}\frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$i_n(t) = i_0(t) - 5\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds.$$

Hence,

$$i(t) = \lim_{n\to\infty} i_n(t).$$

Figures (16) and (17) illustrate ADM and PM solutions at multiple values of α, while Figure (18) illustrates a comparison between ADM and PM solutions at the same value of α.

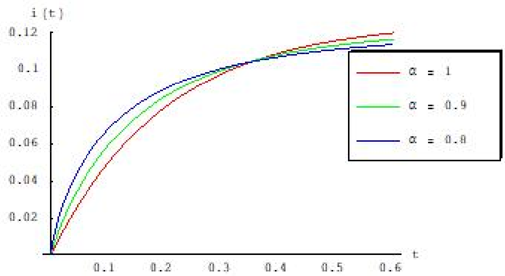

Figure 16. ADM solution for various α of eq. 26

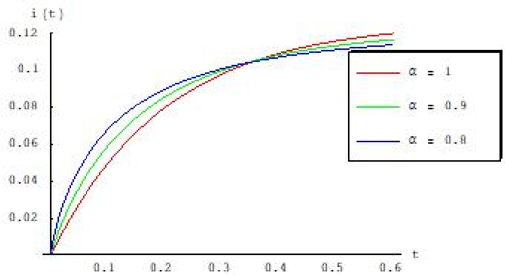

Figure 17. PM solution for various α of eq. 29

From these figures, we see that when we increase the number of the terms n, the solution will be more accurate.

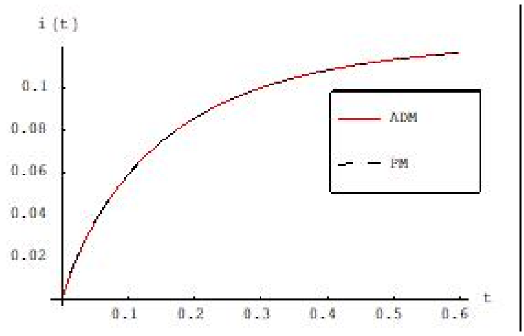

Figure 18. ADM and PM solutions for α = 0.9

Table (3.1) shows the absolute difference (AD) between ADM and PM solutions, while Table (3.2) shows a time comparison between them.

Table 3.1: *Absolute difference*

t	$\|i_{ADM} - i_{PM}\|$
0.1	5.27213×10^{-18}
0.2	1.32148×10^{-17}
0.3	1.8233×10^{-17}
0.4	1.84367×10^{-17}
0.5	1.60167×10^{-17}
0.6	1.87823×10^{-17}
0.7	4.23951×10^{-17}
0.8	1.1316×10^{-16}
0.9	2.70981×10^{-16}
1.0	5.71266×10^{-16}

Table 3.2: *Time comparison*

ADM time	PM time
135.922	1463.55

Example (4): Consider the following circuit of Figure (19), $i(0) = 0.72A$.

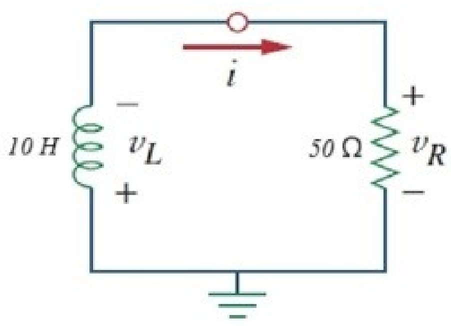

Figure 19. Example (4) Fig.

Solution:
$i-$ ADM Solution:
From (2.24) and (2.25), we get

$$\begin{aligned} i_0(t) &= 0.72, \\ i_n(t) &= i_0(t) - 5\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds. \end{aligned}$$

Hence,

$$i(t) = \sum_{n=0}^{\infty} i_n(t).$$

$ii-$ PM Solution:

From (2.27) and (2.28), we get

$$\begin{aligned} i_0(t) &= 0.72, \\ i_n(t) &= i_0(t) - 5\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} i(s)ds. \end{aligned}$$

Hence,

$$i(t) = \lim_{n\to\infty} i_n(t).$$

Figures (20) and (21) illustrate ADM and PM solutions at multiple values of α, while Figure (22) illustrates a comparison between ADM and PM solutions at the same value of α.

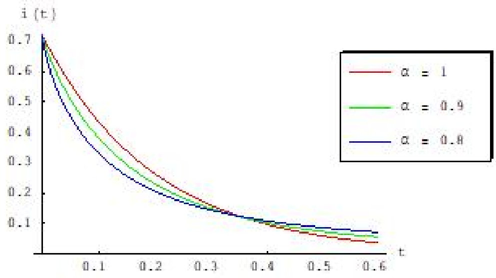

Figure 20. ADM solution for various α of eq. 24

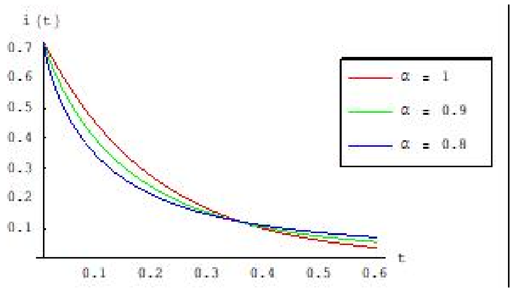

Figure 21. PM solution for various α of eq. 26

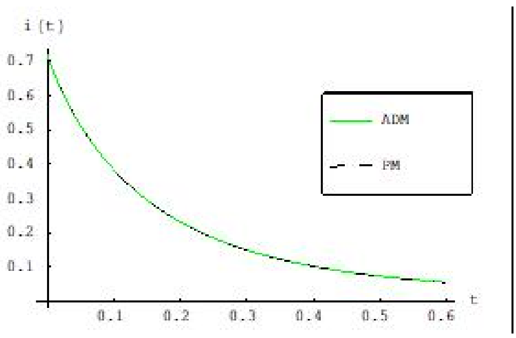

Figure 22. ADM and PM solutions for α = 0.9

From these figures, we see that when we increase the number of the terms n, the solution will be more accurate.

Table (3.3) shows the absolute difference (AD) between ADM and PM solutions, while Table (3.4) shows a time comparison between them.

$Table$ 3.3:$Absolute\ difference$

t	$\lvert i_{ADM} - i_{PM} \rvert$
0.1	8.36617×10^{-17}
0.2	1.10619×10^{-16}
0.3	9.71458×10^{-17}
0.4	4.79743×10^{-17}
0.5	3.42695×10^{-17}
0.6	1.48334×10^{-16}
0.7	2.95852×10^{-16}
0.8	4.87835×10^{-16}
0.9	7.64554×10^{-16}
1.0	1.249×10^{-15}

$Table$ 3.4:$Time\ comparison$

ADM time	PM time
135.922	1463.55

We see from all the pervious figures and tables that PM gives more accurate solution than ADM, so it gives more convergence, but ADM takes less time than PM.

3 Conclusion

This study shows how fractional calculus, especially Caputo derivatives, can be used to model RLC, RC, and RL electrical circuits more accurately. We proved that the mathematical models have unique solutions, meaning they are reliable and consistent. Using the Caputo derivative makes it easier to work with initial conditions and helps better understand the effects of long-term memory and past influences in the circuits.

We applied two methods, the Adomian Decomposition Method (ADM) and the Picard Method (PM), to solve these models. The results show that both methods are accurate and work quickly, making them useful tools for engineers and scientists. Overall, this research improves our understanding of how fractional calculus can be used to analyze and predict circuit behavior, opening new opportunities for designing better electrical systems in the future.

4 Abbreviations

CD	Caputo Derivative
ADM	Adomian Decomposition Method
PM	Picard Method
R	Resistance
C	Capacitance
L	Inductance

5 Declarations

5.1 Availability of data and material

Data can be shared

5.2 Competing interests

All financial and non-financial competing interests are declared.

5.3 Funding

There is no funding.

5.4 Authors' contributions

All authors equally participated in preparing and finishing the paper.

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