

# Characterization of $\lambda$ -Constacyclic and Skew- $\lambda$ -Constacyclic Codes over $\frac{\mathbb{Z}_4[w]}{\langle w^2 - 1 \rangle}$

Dolly Thoudam and Okram Ratnabala Devi

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**Corresponding Author: Dolly Thoudam**

**Abstract.** This paper presents an analysis of  $\lambda$ -constacyclic codes and skew- $\lambda$ -constacyclic codes defined over the ring  $\mathbb{Z}_4 + w\mathbb{Z}_4$ , with  $w^2 = 1$  for the two units  $\lambda = 3$  and  $3w$ . Three Gray maps are introduced from the ring to the tuples of  $\mathbb{Z}_4$ . For constacyclic codes, we examined the properties of the Gray images. Moreover, we determine the generating polynomials of  $\lambda$ -constacyclic codes over this ring. Using Nechaev's permutation,  $\lambda$ -constacyclic codes are examined for odd length. Further, skew- $\lambda$ -constacyclic codes are studied and the properties of their Gray images are analysed. Some examples of  $\lambda$ -constacyclic codes are constructed based on obtained results over the ring.

## 1 Introduction

Codes over finite rings have drawn a lot of interest for over twenty years in algebraic coding theory. One of the main studies that provides insight into coding theory is Berlekamp's work [4]. The algebraic structures of these rings have inspired extensive analysis of both linear and cyclic codes. Constacyclic codes are generalisations of cyclic codes, which were used in [4, 5]. Constacyclic codes over a variety of rings have been the subject of intense research in the last several years. The class of quasi-cyclic (QC) codes, which was initially examined in [2] is an additional extension of cyclic codes. Cyclic codes are QC codes of index 1.

In coding theory, the ring of integers modulo 4, denoted by  $\mathbb{Z}_4$  holds a significant position among all the finite rings. The work of Hammons in [8] describes certain binary non-linear codes obtained on applying the Gray maps to linear codes over  $\mathbb{Z}_4$ . Further research demonstrates that the Gray images of linear codes over  $\mathbb{Z}_4$  exhibit important connections to areas like low correlation sequences, lattices and combinatorial designs.

Rings of order 16, which are constructed as extensions of  $\mathbb{Z}_4$ , are quite interesting in coding theory. In 2014, Yildiz and Aydin [15] considered the commutative ring  $\mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$  and provided a study of cyclic codes with this structure. Their work included the determination of the algebraic properties and generators of cyclic codes over this ring, leading to the discovery of numerous previously unknown linear codes. In 2018, Cengellenmis *et al.* [6] extended the research over the same ring by examining  $\lambda$ -constacyclic codes for the two units  $1 + 3u$  and  $3 + u$ , presenting the Gray images as QC codes over  $\mathbb{Z}_4$ .

Özen *et al.* [12] analysed cyclic and constacyclic codes for the unit  $2 + u$  over the ring  $\mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 1$  in the year 2016. The spanning set and the generators of the cyclic codes were identified. In their work, they obtained cyclic codes as the Gray image of an odd length  $(2 + u)$ -constacyclic codes over  $\mathbb{Z}_4$ . In addition to this, they illustrated examples of cyclic codes over this ring. In 2017, Shi *et al.* [13] considered the ring  $\mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 1$  and examined properties

of  $(1 + 2u)$ -constacyclic codes by establishing Gray maps, which yield cyclic codes over  $\mathbb{Z}_4$ . They also provided new tables of cyclic  $\mathbb{Z}_4 + u\mathbb{Z}_4$ -codes. In 2018, Aydin *et al.* [1] studied  $\lambda$ -constacyclic codes by defining two new Gray maps over the same ring  $\mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 1$  for the units  $3 + 2u$  and  $2 + 3u$ . They identified new  $\lambda$ -constacyclic codes whose  $\mathbb{Z}_4$ -images exhibit improved parameters. In 2016, Gupta *et al.* [7] examined the relationship between module theory and coding theory establishing connections with structural properties of linear codes. Additionally, as an application of codes, Kumar [11] investigated the construction of DNA codes over the ring  $\mathbb{F}_2[\mu, \nu] / \langle \mu_2, \nu_2, \mu\nu = \nu\mu \rangle$ .

In this work, we also take the same ring  $\mathbb{Z}_4 + w\mathbb{Z}_4, w^2 = 1$  and focus on investigating  $\lambda$ -constacyclic codes and skew- $\lambda$ -constacyclic codes. The structure of this paper is listed here: section 2 examines the algebraic structures of the ring, lists basic definitions and the result that follows. For section 3, three new Gray maps are stated and analyse the properties of the  $\lambda$ -constacyclic codes over the ring, giving focus to the units 3 and  $3w$ . Also, this section derives the generator polynomial corresponding to the Gray images of  $\lambda$ -constacyclic codes. In section 4, odd length  $\lambda$ -constacyclic codes are explored using Nechaev's permutation over the ring. We study Gray images derived from skew- $\lambda$ -constacyclic codes, by examining their structural properties in section 5. In section 6, for the ring  $R$ , we provide new examples of  $\lambda$ -constacyclic codes. We give a concluding remark in section 7.

## 2 Preliminaries

As viewed in [12], the ring  $R = \mathbb{Z}_4 + w\mathbb{Z}_4, w^2 = 1$  is a finite commutative ring having cardinality 16 and characteristic 4. Thus,  $R \cong \frac{\mathbb{Z}_4[w]}{\langle w^2 - 1 \rangle}$ . Elements of  $R$  can be written as  $e + wf$ , for all  $p, q \in \mathbb{Z}_4$ , out of these the set of units is  $A = \{1, 3, w, 3w, 2 + w, 1 + 2w, 3 + 2w, 2 + 3w\}$ . For all  $\lambda \in A$  it satisfies the relation  $\lambda^2 = 1$ . Here, the ring  $R$  is finite non-chain ring as the 7 ideals of this ring is the set  $\{\langle 0 \rangle, R, \langle 2w \rangle, \langle 1 + w \rangle, \langle 3 + w \rangle, \langle 2 + 2w \rangle, \langle 2w, 1 + w \rangle\}$ . Furthermore,  $R$  is local ring having unique maximal ideal  $\langle 2w, 1 + w \rangle$ . This research focuses on the study of both  $\lambda$ -constacyclic codes and skew- $\lambda$ -constacyclic codes over the ring  $R$  particularly for the units 3 and  $3w$ . Additionally, three new Gray maps are constructed and they send elements from the ring  $R$  to the tuples of  $\mathbb{Z}_4$ .

A linear code  $C$  of length  $n$  over  $R$  is an  $R$ -submodule of  $R^n$  and is cyclic if  $\rho(C) = C$ , where  $\rho : R^n \rightarrow R^n$  defined as  $\rho(g_0, g_1, g_2, g_3, \dots, g_{n-1}) = (g_{n-1}, g_0, g_1, g_2, \dots, g_{n-2})$ . For a unit element  $\lambda$  in  $R$ , a linear code  $C$  over  $R$  is called  $\lambda$ -constacyclic code if and only if  $\tau_\lambda(C) \subseteq C$ , where  $\tau_\lambda$  is the shift operator on  $R^n$  defined as  $\tau_\lambda(g_0, g_1, \dots, g_{n-1}) = (\lambda g_{n-1}, g_0, \dots, g_{n-2})$ . The constacyclic code is cyclic for  $\lambda = 1$ .

For  $z \in \mathbb{Z}_4^{mn}$  with  $(z^1 | z^2 | \dots | z^m), z^k \in \mathbb{Z}_4^n, \forall k = 1, 2, 3, \dots, m$ . Here,  $\rho$  and “|” are cyclic shift from  $\mathbb{Z}_4^n$  to  $\mathbb{Z}_4^n$  and usual vector concatenation respectively. Suppose  $\eta_m : \mathbb{Z}_4^{mn} \rightarrow \mathbb{Z}_4^{mn}$  be a map defined as  $\eta_m(a) = (\rho(a^1) | \rho(a^2) | \dots | \rho(a^m))$ , then a code of length  $mn$  over  $\mathbb{Z}_4$  is called quasi-cyclic (QC) code of index  $m$  if  $\eta_m(C) = C$ . A linear code  $C$  over the ring  $R$  is called s-quasi-cyclic if it remains invariant under the cyclic shift  $\rho^s$  i.e.  $\rho^s(C) = C$ , where  $\rho$  denotes cyclic shift on  $R^n$ .

We use the polynomial representation of codewords of  $R^n$  in  $R[x]$ . For each codeword  $\bar{g} = (g_0, g_1, g_2, \dots, g_{n-1}) \in R^n$ , we can make the correspondence  $\bar{g}(x) = g_0 + g_1x + g_2x^2 + \dots + g_{n-1}x^{n-1} \in \frac{R[x]}{\langle x^n - 1 \rangle}$ .

**Proposition 2.1.** [14] *A subset  $C$  of  $R^n$  is a cyclic code of length  $n$  if and only if its polynomial representation is an ideal of  $R_n = R[x] / \langle x^n - 1 \rangle$ .*

**Proposition 2.2.** [14] *A subset  $C$  of  $R^n$  is a  $\lambda$ -constacyclic code of length  $n$  if and only if its polynomial representation is an ideal of  $R_{n,\lambda} = R[x] / \langle x^n - \lambda \rangle$ .*

## 3 Gray images of $\lambda$ -constacyclic codes over $R$ and its structures

Three Gray maps on  $R$  are introduced in this section. The  $\mathbb{Z}_4$ -images of  $\lambda$ -constacyclic codes over  $R$  are investigated for the units  $\lambda = 3$  and  $3w$ . We will discuss the permutation version of the

Gray maps. Further, we obtain the generator polynomial of the Gray images of  $\lambda$ -constacyclic codes.

**$\lambda$ -constacyclic codes over  $R$**

We introduce the following three Gray maps for the units  $\lambda = 3$  and  $3w$  and examine the Gray-images. The Gray maps are defined as follows :

- $\phi_1 : R \rightarrow \mathbb{Z}_4^2$  by  $\phi_1 (e + wf) = (e + 3f, 3e + f)$ ,
- $\phi_2 : R \rightarrow \mathbb{Z}_4^2$  by  $\phi_2 (e + wf) = (2f, 2e)$ ,
- $\phi_3 : R \rightarrow \mathbb{Z}_4^3$  by  $\phi_3 (e + wf) = (e + f, 3e + 3f, 2e + 2f)$ ,

where  $e+wf \in R$ , for all  $e, f \in \mathbb{Z}_4$ . They are all linear maps and can be extended componentwise as,

- $\Phi_1 : R^n \rightarrow \mathbb{Z}_4^{2n}$  by

$$\begin{aligned} \Phi_1 (\bar{g}) &= \Phi_1 (g_0, g_1, g_2, \dots, g_{n-1}) \\ &= (e_0 + 3f_0, e_1 + 3f_1, \dots, e_{n-1} + 3f_{n-1}, 3e_0 + f_0, 3e_1 + f_1, \dots, 3e_{n-1} \\ &\quad + f_{n-1}), \end{aligned}$$

- $\Phi_2 : R^n \rightarrow \mathbb{Z}_4^{2n}$  given by

$$\Phi_2 (\bar{g}) = (2f_0, 2f_1, \dots, 2f_{n-1}, 2e_0, 2e_1, \dots, 2e_{n-1}).$$

- $\Phi_3 : R^n \rightarrow \mathbb{Z}_4^{3n}$  by

$$\begin{aligned} \Phi_3 (\bar{g}) &= (e_0 + f_0, e_1 + f_1, \dots, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, 3e_1 + 3f_1, \dots, 3e_{n-1} + \\ &\quad 3f_{n-1}, 2e_0 + 2f_0, 2e_1 + 2f_1, \dots, 2e_{n-1} + 2f_{n-1}). \end{aligned}$$

where  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$  and  $g_i = e_i + wf_i, \forall e_i, f_i \in \mathbb{Z}_4$  and  $i = 0, 1, \dots, n - 1$ .

The Gray maps defined above are not bijective. We examine the images of  $\lambda$ -constacyclic codes over  $R$  using the Gray maps defined above. For  $z \in \mathbb{Z}_4$ , the Lee weight of  $z$  is defined by  $\min\{|z|, |4 - z|\}$ , i.e. the Lee weights of 0, 1, 2, 3 are 0, 1, 2, 1 respectively. So, the Lee weight of  $g \in R$  is defined as  $w_L(g) = w_L(\phi_i(g))$ , for  $i = 1, 2, 3$ . Thus for  $\bar{g} = (g_0, g_1, g_2, \dots, g_{n-1}) \in R^n, w_L(\bar{g}) = \sum_{i=0}^{n-1} w_L(g_i)$ . The Lee distance for the code  $\mathcal{C}$  is  $d(\mathcal{C}) = \min\{d_L(\bar{j}, \bar{k})\} : \bar{j} \neq \bar{k}, \bar{j}, \bar{k} \in \mathcal{C}\}$ , where  $d_L(\bar{j}, \bar{k}) = w_L(\bar{j} - \bar{k})$ . Thus  $d_L(\bar{j}, \bar{k}) = w_L(\bar{j} - \bar{k}) = w_L(\Phi_i(\bar{j} - \bar{k})) = w_L(\Phi_i(\bar{j}) - \Phi_i(\bar{k})) = d_L(\Phi_i(\bar{j}), \Phi_i(\bar{k}))$ , for all  $\bar{j}, \bar{k} \in R^n, i = 1, 2, 3$ . We analyse the relationship between these Gray maps with the cyclic, QC and  $\lambda$ -constacyclic shift operations.

Recall that  $\rho$  is a cyclic shift operator,  $\tau_\lambda$  is a  $\lambda$ -constacyclic shift operator and  $\eta$  is QC shift operator. Here, we use the permutation  $\delta$  of  $\mathbb{Z}_4^{3n}$  mentioned in [10] defined by  $\delta (z_1, z_2, \dots, z_n, \dots, z_{2n}, \dots, z_{3n}) = (z_{\beta(1)}, \dots, z_{\beta(n)}, \dots, z_{\beta(2n)}, \dots, z_{\beta(3n)})$  with the permutation  $\beta = (1, n + 1)$  of  $\{1, 2, \dots, 3n\}$ .

**Proposition 3.1.** For any  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$  and  $\lambda = 3$ , considering

- (i) The Gray map  $\Phi_1$ , we have  $\Phi_1\tau_\lambda(\bar{g}) = \rho\Phi_1(\bar{g})$ .
- (ii) The Gray map  $\Phi_2$ , we get  $\Phi_2\tau_\lambda(\bar{g}) = \eta_2\Phi_2(\bar{g})$ .
- (iii) The Gray map  $\Phi_3$ , we get  $\Phi_3\tau_\lambda(\bar{g}) = \delta\eta_3\Phi_3(\bar{g})$ , with the permutation  $\delta$  defined above.

*Proof.* Let  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$ , where  $g_i = e_i + wf_i, \forall e_i, f_i \in \mathbb{Z}_4$  and  $i = 0, 1, \dots, n - 1$ . Using the  $\lambda$ -constacyclic shift, we get

$$\tau_3 (\bar{g}) = (3g_{n-1}, g_0, \dots, g_{n-2}) = (3e_{n-1} + 3wf_{n-1}, e_0 + wf_0, \dots, e_{n-2} + wf_{n-2}).$$

(i) For the Gray map  $\Phi_1$ , we have

$$\begin{aligned}\Phi_1(\tau_3(\bar{g})) &= \Phi_1(3e_{n-1} + 3wf_{n-1}, e_0 + wf_0, \dots, e_{n-2} + wf_{n-2}) \\ &= (3e_{n-1} + f_{n-1}, e_0 + 3f_0, \dots, e_{n-2} + 3f_{n-2}, e_{n-1} + 3f_{n-1}, 3e_0 + \\ &\quad f_0, \dots, 3e_{n-2} + f_{n-2}) \\ &= \rho\Phi_1(\bar{g}).\end{aligned}$$

(ii) For the Gray map  $\Phi_2$ , we have

$$\begin{aligned}\Phi_2(\tau_3(\bar{g})) &= \Phi_2(3e_{n-1} + 3wf_{n-1}, e_0 + wf_0, \dots, e_{n-2} + wf_{n-2}) \\ &= (2f_{n-1}, 2f_0, \dots, 2f_{n-2}, 2e_{n-1}, 2e_0, \dots, 2e_{n-2}) \\ &= \eta_2\Phi_2(\bar{g}).\end{aligned}$$

(iii) For the Gray map  $\Phi_3$ , we have

$$\begin{aligned}\Phi_3(\tau_3(\bar{g})) &= \Phi_3(3e_{n-1} + 3wf_{n-1}, e_0 + wf_0, \dots, e_{n-2} + wf_{n-2}) \\ &= (3e_{n-1} + 3f_{n-1}, e_0 + f_0, \dots, e_{n-2} + f_{n-2}, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, \\ &\quad \dots, 3e_{n-2} + 3f_{n-2}, 2e_{n-1} + 2f_{n-1}, 2e_0 + 2f_0, \dots, 2e_{n-2} + 2f_{n-2}) \\ &= \delta\eta_3\Phi_3(\bar{g}).\end{aligned}$$

□

**Theorem 3.2.** For  $\lambda = 3$ , consider a  $\lambda$ -constacyclic code  $\mathcal{C}$  of length  $n$  over  $R$ , its Gray image of  $\mathcal{C}$

- (i) under  $\Phi_1$  corresponds to a cyclic code over  $\mathbb{Z}_4$  of length  $2n$ .
- (ii) under  $\Phi_2$  is a QC code of index 2 over  $\mathbb{Z}_4$  of length  $2n$ .
- (iii) under  $\Phi_3$  results to a permutation equivalent QC codes of index 3 over  $\mathbb{Z}_4$  of length  $3n$ .

*Proof.* Here  $\mathcal{C}$  is a  $\lambda$ -constacyclic code over  $R$  of length  $n$ , then  $\tau_\lambda(\mathcal{C}) = \mathcal{C}$ .

- (i) On applying  $\Phi_1$  to each sides, we obtain  $\Phi_1\tau_\lambda(\mathcal{C}) = \Phi_1(\mathcal{C})$ . From Prop. 3.1, we get  $\rho\Phi_1(\mathcal{C}) = \Phi_1(\mathcal{C})$  which implies  $\Phi_1(\mathcal{C})$  is a cyclic code over  $\mathbb{Z}_4$  of length  $2n$ .
- (ii) On applying  $\Phi_2$  to each sides, it follows that  $\Phi_2\tau_\lambda(\mathcal{C}) = \Phi_2(\mathcal{C})$ . From Prop. 3.1, we get  $\eta_2\Phi_2(\mathcal{C}) = \Phi_2(\mathcal{C})$  which implies  $\Phi_2(\mathcal{C})$  is a QC code of index 2 over  $\mathbb{Z}_4$  of length  $2n$ .
- (iii) By applying  $\Phi_3$  to both sides gives  $\Phi_3\tau_\lambda(\mathcal{C}) = \Phi_3(\mathcal{C})$ . From Prop. 3.1, we get  $\delta\eta_3\Phi_3(\mathcal{C}) = \Phi_3(\mathcal{C})$ . So,  $\Phi_3(\mathcal{C})$  is permutation equivalent to a QC code of index 3 over  $\mathbb{Z}_4$  of length  $3n$ .

□

**Proposition 3.3.** For any  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$  and  $\lambda = 3w$ , considering

- (i) The Gray map  $\Phi_1$ , we have  $\Phi_1\tau_\lambda(\bar{g}) = \eta_2\Phi_1(\bar{g})$ .
- (ii) The Gray map  $\Phi_2$ , we get  $\Phi_2\tau_\lambda(\bar{g}) = \rho\Phi_2(\bar{g})$ .
- (iii) The Gray map  $\Phi_3$ , we get  $\Phi_3\tau_\lambda(\bar{g}) = \delta\eta_3\Phi_3(\bar{g})$ , with the permutation  $\delta$  defined above.

*Proof.* Let  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$  where  $g_i = e_i + wf_i, \forall e_i, f_i \in \mathbb{Z}_4$  and  $i = 0, 1, \dots, n-1$ . Using the  $\lambda$ -constacyclic shift, we get

$$\tau_{3w}(\bar{g}) = (3wg_{n-1}, g_0, \dots, g_{n-2}) = (3f_{n-1} + 3we_{n-1}, e_0 + wf_0, \dots, e_{n-2} + wf_{n-2}).$$

(i) For the Gray map  $\Phi_1$ , we have

$$\begin{aligned} \Phi_1(\tau_{3w}(\bar{g})) &= \Phi_1(3f_{n-1} + 3we_{n-1}, e_0 + wf_0, \dots, e_{n-2} + wf_{n-2}) \\ &= (e_{n-1} + 3f_{n-1}, e_0 + 3f_0, \dots, e_{n-2} + 3f_{n-2}, 3e_{n-1} + f_{n-1}, 3e_0 + f_0, \\ &\quad \dots, 3e_{n-2} + f_{n-2}) \\ &= \eta_2 \Phi_1(\bar{g}). \end{aligned}$$

(ii) For the Gray map  $\Phi_2$ , we have

$$\begin{aligned} \Phi_2(\tau_{3w}(\bar{g})) &= \Phi_2(3f_{n-1} + 3we_{n-1}, e_0 + wf_0, \dots, e_{n-2} + wf_{n-2}) \\ &= (2e_{n-1}, 2f_0, \dots, 2f_{n-2}, 2f_{n-1}, 2e_0, \dots, 2e_{n-2}) \\ &= \rho \Phi_2(\bar{g}). \end{aligned}$$

(iii) For the Gray map  $\Phi_3$ , we have

$$\begin{aligned} \Phi_3(\tau_{3w}(\bar{g})) &= \Phi_3(3f_{n-1} + 3we_{n-1}, e_0 + wf_0, \dots, e_{n-2} + wf_{n-2}) \\ &= (3e_{n-1} + 3f_{n-1}, e_0 + f_0, \dots, e_{n-2} + f_{n-2}, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, \\ &\quad \dots, 3e_{n-2} + 3f_{n-2}, 2e_{n-1} + 2f_{n-1}, 2e_0 + 2f_0, \dots, 2e_{n-2} + 2f_{n-2}) \\ &= \delta \eta_3 \Phi_3(\bar{g}). \end{aligned}$$

□

**Theorem 3.4.** For  $\lambda = 3w$ , consider a  $\lambda$ -constacyclic code  $\mathcal{C}$  of length  $n$  over  $R$ , its Gray images of  $\mathcal{C}$

- (i) under  $\Phi_1$  is a QC code of index 2 over  $\mathbb{Z}_4$  of length  $2n$ .
- (ii) under  $\Phi_2$  results to a cyclic code over  $\mathbb{Z}_4$  of length  $2n$ .
- (iii) under  $\Phi_3$  is permutation equivalent to QC codes of index 3 over  $\mathbb{Z}_4$  of length  $3n$ .

*Proof.* Proof is omitted as it is similar to that of Theorem 3.2. □

**Permutation version of Gray maps**

For  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$  where  $g_i = e_i + wf_i$ , with  $e_i, f_i \in \mathbb{Z}_4$  for  $0 \leq i \leq n - 1$ . We consider the permutation form of  $\Phi_1$ , denoted by  $\Phi_{\pi,1}$ , is given by

$$\Phi_{\pi,1}(\bar{g}) = (e_0 + 3f_0, 3e_0 + f_0, e_1 + 3f_1, 3e_1 + f_1, \dots, e_{n-1} + 3f_{n-1}, 3e_{n-1} + f_{n-1})$$

Also, the permutation version  $\Phi_{\pi,2}$  of  $\Phi_2$  is defined as

$$\Phi_{\pi,2}(\bar{g}) = (2f_0, 2e_0, 2f_1, 2e_1, \dots, 2f_{n-1}, 2e_{n-1}).$$

**Proposition 3.5.** For any  $\bar{g} \in R^n$ ,

- (i)  $\Phi_{\pi,1} \rho(\bar{g}) = \rho^2 \Phi_{\pi,1}(\bar{g})$ .
- (ii)  $\Phi_{\pi,2} \rho(\bar{g}) = \rho^2 \Phi_{\pi,2}(\bar{g})$ .

*Proof.* Let  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$  where  $g_i = e_i + wf_i, \forall e_i, f_i \in \mathbb{Z}_4$  and  $0 \leq i \leq n - 1$ .

(i) Here,

$$\begin{aligned} \rho^2 \Phi_{\pi,1}(\bar{g}) &= \rho^2(e_0 + 3f_0, 3e_0 + f_0, e_1 + 3f_1, \dots, e_{n-1} + 3f_{n-1}, 3e_{n-1} + f_{n-1}) \\ &= \rho(3e_{n-1} + f_{n-1}, e_0 + 3f_0, 3e_0 + f_0, e_1 + 3f_1, \dots, e_{n-1} + 3f_{n-1}) \\ &= (e_{n-1} + 3f_{n-1}, 3e_{n-1} + f_{n-1}, e_0 + 3f_0, 3e_0 + f_0, e_1 + 3f_1, 3e_1 + f_1, \\ &\quad \dots, e_{n-2} + 3f_{n-2}, 3e_{n-2} + f_{n-2}) \\ &= \Phi_{\pi}(g_{n-1}, g_0, g_1, \dots, g_{n-2}) \\ &= \Phi_{\pi} \rho(\bar{g}). \end{aligned}$$

(ii) We continue without proof as the same argument holds.  $\square$

**Theorem 3.6.** For  $i = 1, 2$  the  $\mathbb{Z}_4$  image  $\Phi_{\pi,i}(\mathcal{C})$  is a 2-quasi-cyclic code of length  $2n$  over  $\mathbb{Z}_4$ , where  $\mathcal{C}$  is a cyclic code of length  $n$  over  $R$ .

*Proof.* Here  $\rho(\mathcal{C}) = \mathcal{C}$  as  $\mathcal{C}$  is cyclic code. For  $i = 1, 2$ , applying  $\Phi_{\pi,i}$  on both sides, we get  $\Phi_{\pi,i}\rho(\mathcal{C}) = \Phi_{\pi,i}(\mathcal{C})$ . From Prop. 3.5,  $\Phi_{\pi,i}(\mathcal{C}) = \rho^2\Phi_{\pi,i}(\mathcal{C})$ . So,  $\Phi_{\pi,i}(\mathcal{C})$  is a 2-quasi-cyclic code of length  $2n$  over  $\mathbb{Z}_4$ .  $\square$

### Structures of $\lambda$ -constacyclic codes over $R$

The generator polynomial corresponding to the Gray images of the  $\lambda$ -constacyclic codes is obtained in this subsection.

**Theorem 3.7.** Let  $\mathcal{C}$  be  $\lambda$ -constacyclic code of length  $n$  over  $R$ , generated by  $e(x) + wf(x)$ , where  $e(x), f(x) \in \mathbb{Z}_4[x]$  are of degree  $\leq (n - 1)$ .

- (i) For  $\lambda = 3$ , the  $\Phi_1$ -Gray image of  $\mathcal{C}$ , generated by  $(e(x) + 3f(x)) + x^n(3e(x) + f(x))$ , is a cyclic code of length  $2n$  over  $\mathbb{Z}_4$ .
- (ii) For  $\lambda = 3w$ , the  $\Phi_1$ -Gray image of  $\mathcal{C}$ , generated by  $(e(x) + 3f(x)) + x^n(3e(x) + f(x))$ , is a QC code of index 2 over  $\mathbb{Z}_4$  of length  $2n$ .

*Proof.* The polynomial version of the Gray map  $\Phi_1$  is defined as follows

$$\Phi_1 : \frac{R[x]}{\langle x^n - \lambda \rangle} \rightarrow \frac{\mathbb{Z}_4[x]}{\langle x^n - 1 \rangle} \times \frac{\mathbb{Z}_4[x]}{\langle x^n - 1 \rangle},$$

by  $\Phi_1(e(x) + wf(x)) = (e(x) + 3f(x), 3e(x) + f(x)), \forall e(x), f(x) \in \mathbb{Z}_4[x]$ .

For  $p_i \in \mathbb{Z}_4[x], i = 1, 2$ , we get

$$\begin{aligned} & \Phi_1[(p_1(x) + wp_2(x))(e(x) + wf(x))] \\ &= p_1(x)(e(x) + 3f(x), 3e(x) + f(x)) + p_2(x)(3e(x) + f(x), e(x) + 3f(x)). \end{aligned}$$

For a vector  $(g, h) \in \frac{\mathbb{Z}_4[x]}{\langle x^n - 1 \rangle} \times \frac{\mathbb{Z}_4[x]}{\langle x^n - 1 \rangle}$ , it corresponds to the vector  $g + hx^n \in \frac{\mathbb{Z}_4[x]}{\langle x^{2n} - 1 \rangle}$ .

Thus,

- (i) For  $\lambda = 3$  and using Theorem 3.2 we get, the  $\Phi_1$ -Gray image of  $\mathcal{C}$ , generated by the polynomial  $(e(x) + 3f(x)) + x^n(3e(x) + f(x))$ , is a cyclic code of length  $2n$  over  $\mathbb{Z}_4$ .
- (ii) For  $\lambda = 3w$  and using Theorem 3.4 we have, the  $\Phi_1$ -Gray image of  $\mathcal{C}$ , generated by the polynomial  $(e(x) + 3f(x)) + x^n(3e(x) + f(x))$ , is a QC code of length  $2n$  and index 2 over  $\mathbb{Z}_4$ .

Similar to this result, the following theorems hold.

**Theorem 3.8.** Let  $\mathcal{C}$  be  $\lambda$ -constacyclic code of length  $n$  over  $R$  generated by  $e(x) + wf(x)$ , where  $e(x), f(x) \in \mathbb{Z}_4[x]$  are of degree  $\leq (n - 1)$ ,

- (i) For  $\lambda = 3$ ,  $\Phi_2(\mathcal{C})$ , generated by the polynomials  $2e(x) + 2f(x)x^n$  and  $2e(x) + 2f(x)x^n$ , is a QC code of index 2 over  $\mathbb{Z}_4$  of length  $2n$ .
- (ii) For  $\lambda = 3w$ ,  $\Phi_2(\mathcal{C})$ , generated by the polynomials  $2f(x) + 2e(x)x^n$  and  $2e(x) + 2f(x)x^n$ , is a cyclic code of length  $2n$  over  $\mathbb{Z}_4$ .

**Theorem 3.9.** For the two units  $\lambda = 3$  and  $3w$ , let  $\mathcal{C}$  be  $\lambda$ -constacyclic code of length  $n$  over  $R$  generated by  $e(x) + wf(x)$ , where  $e(x), f(x) \in \mathbb{Z}_4[x]$  are of degree  $\leq (n - 1)$ . Then  $\Phi_3(\mathcal{C})$ , generated by the polynomials  $(e(x) + f(x)) + x^n(3e(x) + 3f(x)) + x^{2n}(2e(x) + 2f(x))$ , is permutation equivalent to a QC codes of index 3 over  $\mathbb{Z}_4$  of length  $3n$ .

### 4 Odd length Constacyclic Codes over $R$ with Nechaev's Permutation

In this section, we evaluate  $\lambda$ -constacyclic codes over  $R$  of odd length  $n$ . When  $\lambda$  is any one of the units 3 and  $3w$ , it satisfies the relation  $\lambda^2 = 1$ . So  $\lambda^n = \lambda$ , if  $n$  is odd. We study about cyclic, QC and  $\lambda$ -constacyclic codes with respect to Nechaev's permutation  $\pi$  over the copies of  $\mathbb{Z}_4$  and together with the images of the two Gray maps. Similar to the results achieved in [1, 9, 10, 12, 15], the following results are presented without proofs.

**Definition 4.1.** [14] For an odd integer  $n$ , consider the permutation  $\zeta$  of the set  $\{0, 1, 2, \dots, 2n - 1\}$  given by  $\zeta = (1, n + 1)(3, n + 3) \cdots (2i + 1, n + 2i + 1) \cdots (n - 2, 2n - 2)$  as a product of transpositions. The Nechaev's permutation  $\pi$  is defined as  $\pi(z_0, z_1, z_2, \dots, z_{2n-1}) = (z_{\zeta(0)}, z_{\zeta(1)}, \dots, z_{\zeta(2n-1)})$  where  $(z_0, z_1, \dots, z_{2n-1}) \in R^{2n}$  and  $z_i = e_i + wf_i$  for  $e_i, f_i \in \mathbb{Z}_4$  and  $i = 0, 1, \dots, 2n - 1$ .

**Proposition 4.2.** For an odd integer  $n$  and  $\lambda = 3$  and  $3u$ , consider the map  $\Gamma : R_n = \frac{R[x]}{\langle x^n - 1 \rangle} \rightarrow R_{n,\lambda} = \frac{R[x]}{\langle x^n - \lambda \rangle}$  defined by  $\Gamma(p(x)) = p(\lambda x)$ , for  $p(x) \in R_n$ , is a ring isomorphism.

**Corollary 4.3.** For an odd integer  $n$ .  $I$  is an ideal of  $R_n$  if and only if  $\Gamma(I)$  is an ideal of  $R_{n,\lambda}$ .

**Corollary 4.4.** Let  $\mu : R_n \rightarrow R_n$  defined by  $\mu(g_0, g_1, g_2, \dots, g_{n-1}) = (g_0, \lambda g_1, \lambda^2 g_2, \dots, \lambda^{n-1} g_{n-1})$ . Then,  $\mathcal{C}$  is a cyclic code over  $R$  of odd length  $n$  if and only if  $\mu(\mathcal{C})$  is a  $\lambda$ -constacyclic code of length  $n$  over  $R$ .

Also, we use the permutation  $\chi$  of  $\mathbb{Z}_4^{3n}$  mentioned in [3] defined by  $\chi(g_0, g_1, \dots, g_{3n-1}) = (g_{\gamma(0)}, g_{\gamma(1)}, \dots, g_{\gamma(3n-1)})$ , with the permutation  $\gamma = (1, n + 1)(3, n + 3)(5, n + 5) \cdots (2i + 1, n + 2i + 1) \cdots (n - 2, 2n - 2)$  of  $\{0, 1, 2, \dots, 3n - 1\}$ .

**Proposition 4.5.** For an odd integer  $n$  and  $\lambda = 3$ , using the Nechaev's permutation  $\pi$  and the map  $\mu$  defined in Cor. 4.4,

- (i)  $\Phi_1\mu = \pi\Phi_1$ .
- (ii)  $\Phi_3\mu = \chi\Phi_3$ .

*Proof.* Here  $\lambda^{2k} = 1, \forall k \in \mathbb{N}$ . Let  $\bar{g} = (g_0, g_1, g_2, \dots, g_{n-1}) \in R^n$ , where  $g_i = e_i + uf_i$ , for  $e_i, f_i \in \mathbb{Z}_4$  and  $i = 0, 1, 2, \dots, n - 1$ .

So, for an odd integer  $n$  and  $\lambda = 3$ ,

$$\mu(\bar{g}) = (g_0, \lambda g_1, g_2, \dots, \lambda g_{n-2}, g_{n-1}) = (e_0 + uf_0, 3e_1 + 3uf_1, \dots, e_{n-1} + uf_{n-1}).$$

(i) Applying  $\Phi_1$  on the above equation, we get

$$\begin{aligned} \Phi_1\mu(\bar{g}) &= (e_0 + 3f_0, 3e_1 + f_1, e_2 + 3f_2, 3e_3 + f_3, \dots, e_{n-1} + 3f_{n-1}, \\ &\quad 3e_0 + f_0, e_1 + 3f_1, 3e_2 + f_2, e_3 + 3f_3, \dots, 3e_{n-1} + f_{n-1}). \end{aligned}$$

And

$$\begin{aligned} \Phi_1(\bar{g}) &= \Phi_1(g_0, g_1, g_2, \dots, g_{n-1}) \\ &= (e_0 + 3f_0, e_1 + 3f_1, e_2 + 3f_2, e_3 + 3f_3, \dots, e_{n-1} + 3f_{n-1}, \\ &\quad 3e_0 + f_0, 3e_1 + f_1, 3e_2 + f_2, 3e_3 + f_3, \dots, 3e_{n-1} + f_{n-1}). \end{aligned}$$

Applying the permutation  $\pi$  on  $\Phi_1$ , we get the result i.e.  $\Phi_1\mu = \pi\Phi_1$ .

(ii) Applying  $\Phi_3$  on  $\mu(\bar{g})$ , we have

$$\begin{aligned} \Phi_3\mu(\bar{g}) &= (e_0 + f_0, 3e_1 + 3f_1, e_2 + f_2, \dots, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, e_1 + f_1, \\ &\quad 3e_2 + 3f_2, \dots, 3e_{n-1} + 3f_{n-1}, 2e_0 + 2f_0, 2e_1 + 2f_1, 2e_2 + 2f_2, \\ &\quad \dots, 2e_{n-1} + 2f_{n-1}). \end{aligned}$$

Also,

$$\begin{aligned} \Phi_3(\bar{g}) &= \Phi_3(g_0, g_1, g_2, \dots, g_{n-1}) \\ &= (e_0 + f_0, e_1 + f_1, e_2 + f_2, \dots, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, 3e_1 + 3f_1, \\ &\quad \dots, 3e_{n-1} + 3f_{n-1}, 2e_0 + 2f_0, 2e_1 + 2f_1, \dots, 2e_{n-1} + 2f_{n-1}). \end{aligned}$$

On applying the permutation  $\chi$  on  $\Phi_3$  we get,  $\Phi_3\mu = \chi\Phi_3$ .

□

**Theorem 4.6.** For an odd integer  $n$  and  $\lambda = 3$ , let  $\pi$  denote the Nechaev's permutation and  $C$  be a cyclic code over  $R$ ,

- (i) If  $\theta$  denotes  $\Phi_1$ -Gray image of  $C$  then, the permuted code  $\pi(\theta)$  is also cyclic code.
- (ii) If  $\alpha$  denotes the Gray image of  $C$  under  $\Phi_3$ ,  $\alpha$  is permutation equivalent to a QC codes of index 3 of length  $3n$  over  $\mathbb{Z}_4$  with respect to the permutation  $\chi$ .

*Proof.* Let  $C$  be a cyclic code over  $R$ .

- (i) Here  $\theta = \Phi_1(C)$ . From Prop. 4.5,  $\Phi_1\mu(C) = \pi\Phi_1(C) = \pi(\theta)$ . Since  $C$  is cyclic, using Cor. 4.4 and applying Theorem 3.2, we obtain  $\Phi_1\mu(C)$  is cyclic code of length  $2n$  over  $\mathbb{Z}_4$ . Hence,  $\pi(\theta)$  is cyclic code of length  $2n$  over  $\mathbb{Z}_4$ .
- (ii) Here  $\alpha = \Phi_3(C)$ , from Prop. 4.5,  $\Phi_3\mu(C) = \chi\Phi_3(C) = \chi\alpha$ . Since  $C$  is cyclic, using Cor. 4.4 and applying Theorem 3.2, it follows that  $\Phi_3\mu(C)$  is permutation equivalent to a QC code of index 3 over  $\mathbb{Z}_4$  of length  $3n$ . Consequently,  $\alpha$  is permutation equivalent to a QC code over  $\mathbb{Z}_4$  of index 3 of length  $3n$ .

□

**Proposition 4.7.** For an odd integer  $n$  and  $\lambda = 3w$ , using the Nechaev's permutation  $\pi$  and the map  $\mu$  defined in Cor. 4.4,

- (i)  $\Phi_2\mu = \pi\Phi_2$ .
- (ii)  $\Phi_3\mu = \chi\Phi_3$ .

*Proof.* Here  $\lambda^{2k} = 1, \forall k \in \mathbb{N}$ . Let  $\bar{g} = (g_0, g_1, g_2, \dots, g_{n-1}) \in R^n$ , where  $g_i = e_i + wf_i$ , for  $e_i, f_i \in \mathbb{Z}_4$  and  $i = 0, 1, 2, \dots, n - 1$ .

So, for an odd integer  $n$  and  $\lambda = 3w$ ,

$$\begin{aligned} \mu(\bar{g}) &= (g_0, \lambda g_1, g_2, \dots, \lambda g_{n-2}, g_{n-1}) \\ &= (e_0 + wf_0, 3f_1 + 3we_1, e_2 + wf_2, 3f_3 + 3we_3, \dots, e_{n-1} + wf_{n-1}). \end{aligned}$$

- (i) Applying  $\Phi_2$  on the above equation, we get

$$\Phi_2\mu(\bar{g}) = (2f_0, 2e_1, 2f_2, 2e_3, \dots, 2f_{n-1}, 2e_0, 2f_1, 2e_2, 2f_3, \dots, 2e_{n-1}).$$

And

$$\Phi_2(\bar{g}) = (2f_0, 2f_1, 2f_2, 2f_3, \dots, 2f_{n-1}, 2e_0, 2e_1, 2e_2, 2e_3, \dots, 2e_{n-1}).$$

Applying the permutation  $\pi$  on  $\Phi_2$ , we get the result i.e.  $\Phi_2\mu = \pi\Phi_2$ .

- (ii) Applying  $\Phi_3$  on  $\mu(\bar{g})$ , we have

$$\begin{aligned} \Phi_3\mu(\bar{g}) &= (e_0 + f_0, 3e_1 + 3f_1, e_2 + f_2, \dots, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, e_1 + \\ &\quad f_1, 3e_2 + 3f_2, \dots, 3e_{n-1} + 3f_{n-1}, 2e_0 + 2f_0, 2e_1 + 2f_1, 2e_2 + \\ &\quad 2f_2, \dots, 2e_{n-1} + 2f_{n-1}). \end{aligned}$$

Also,

$$\begin{aligned} \Phi_3(\bar{g}) &= \Phi_3(g_0, g_1, g_2, \dots, g_{n-1}) \\ &= (e_0 + f_0, e_1 + f_1, e_2 + f_2, \dots, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, 3e_1 + 3f_1, \\ &\quad \dots, 3e_{n-1} + 3f_{n-1}, 2e_0 + 2f_0, 2e_1 + 2f_1, \dots, 2e_{n-1} + 2f_{n-1}). \end{aligned}$$

Applying the permutation  $\chi$  on  $\Phi_3$ , we get  $\Phi_3\mu = \chi\Phi_3$ . □

**Theorem 4.8.** For an odd integer  $n$  and  $\lambda = 3w$ , let  $\pi$  denote the Nechaev's permutation and  $C$  be a cyclic code over  $R$ ,

- (i) If  $\theta$  denotes  $\Phi_2$ -Gray image of  $C$  then, the permuted code  $\pi(\theta)$  is also cyclic code.
- (ii) If  $\alpha$  denotes the Gray image of  $C$  under  $\Phi_3$ ,  $\alpha$  is permutation equivalent to a QC codes of index 3 of length  $3n$  over  $\mathbb{Z}_4$  with respect to the permutation  $\chi$ .

*Proof.* Proof is omitted as it is similar to that of Theorem 4.6. □

### 5 $\mathbb{Z}_4$ -images of Skew $\lambda$ -constacyclic codes over $R$

We examine skew  $\lambda$ -constacyclic codes of length  $n$  over the ring  $R = \mathbb{Z}_4 + w\mathbb{Z}_4$ . The automorphism  $\theta : R \rightarrow R$ , is specified as  $\theta(e + wf) = (e + 2f) + w(3f)$  i.e.  $\theta(e) = e, \forall e \in \mathbb{Z}_4, \theta(w) = 2 + 3w$ , which gives  $\theta^2(e + wf) = e + wf, \forall e + wf \in R$ .

The set  $R[x; \theta] = \{e_0 + e_1x + e_2x^2 + \dots + e_{n-1}x^{n-1}\}$  forms a ring known as skew polynomial ring w.r.t. usual addition and multiplication is given by the condition  $(ex^i)(fx^j) = e\theta^i(f)x^{i+j}$ .

**Definition 5.1.** [9] A non-empty subset  $C$  of  $R^n$  is called a **skew  $\lambda$ -constacyclic code** of length  $n$  over  $R$  if  $C$  satisfies the following conditions:

- (a)  $C$  is an  $R$ -submodule of  $R^n$ .
- (b) If  $\bar{g} = (g_0, g_1, g_2, \dots, g_{n-1}) \in C$ , then  $\sigma_{\theta, \lambda}(\bar{g}) = (\theta(\lambda g_{n-1}), \theta(g_0), \dots, \theta(g_{n-2})) \in C$ .

**Theorem 5.2.** [9] Let  $C$  be a linear code of length  $n$  over  $R$ . Then  $C$  is a skew  $\lambda$ -constacyclic code over  $R$  if and only if  $C$  is a left  $R[x; \theta]$  - submodule of  $\frac{R[x; \theta]}{\langle x^n - \lambda \rangle}$ .

**Proposition 5.3.** For  $\lambda = 3$ , consider skew  $\lambda$ -constacyclic shift  $\sigma_{\theta, \lambda}$  on  $R^n$ ,

- (i) For the Gray map  $\Phi_1 : R^n \rightarrow \mathbb{Z}_4^{2n}$ , we have  $\rho\Phi_1 = \Phi_1\sigma_{\theta, \lambda}$ .
- (ii) For the Gray map  $\Phi_2 : R^n \rightarrow \mathbb{Z}_4^{2n}$ , we obtain  $\eta_2\Phi_2 = \Phi_2\sigma_{\theta, \lambda}$ .
- (iii) For the Gray map  $\Phi_3 : R^n \rightarrow \mathbb{Z}_4^{3n}$ , we have  $\delta\eta_3\Phi_3 = \Phi_3\sigma_{\theta, \lambda}$ .

*Proof.* Let  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$  where  $g_i = e_i + wf_i, \forall e_i, f_i \in \mathbb{Z}_4$  and  $i = 0, 1, \dots, n-1$ . For  $\lambda = 3$  and using the skew- $\lambda$ -constacyclic shift, we get

$$\begin{aligned} \sigma_{\theta, 3}(\bar{g}) &= (\theta(3g_{n-1}), \theta(g_0), \dots, \theta(g_{n-2})) \\ &= ((3e_{n-1} + 2f_{n-1}) + wf_{n-1}, (e_0 + 2f_0) + w3f_0, \dots, (e_{n-2} + 2f_{n-2}) + \\ &\quad w3f_{n-2}). \end{aligned}$$

(i) Applying  $\Phi_1$  on above equation, we get

$$\begin{aligned} \Phi_1(\sigma_{\theta, 3}(\bar{g})) &= \Phi_1((3e_{n-1} + 2f_{n-1}) + wf_{n-1}, (e_0 + 2f_0) + w3f_0, \dots, (e_{n-2} + \\ &\quad 2f_{n-2}) + w3f_{n-2}) \\ &= (3e_{n-1} + f_{n-1}, e_0 + 3f_0, \dots, e_{n-2} + 3f_{n-2}, e_{n-1} + 3f_{n-1}, 3e_0 + \\ &\quad f_0, \dots, 3e_{n-2} + f_{n-2}) \\ &= \rho\Phi_1(\bar{g}). \end{aligned}$$

(ii) Applying  $\Phi_2$  on  $\sigma_{\theta,3}(\bar{g})$ , we get

$$\begin{aligned} \Phi_2(\sigma_{\theta,3}(\bar{g})) &= \Phi_2((3e_{n-1} + 2f_{n-1}) + wf_{n-1}, (e_0 + 2f_0) + w3f_0, \dots, (e_{n-2} + 2f_{n-2}) + w3f_{n-2}) \\ &= (2f_{n-1}, 2f_0, \dots, 2f_{n-2}, 2e_{n-1}, 2e_0, \dots, 2e_{n-2}) \\ &= \eta_2 \Phi_2(\bar{g}). \end{aligned}$$

(iii) Applying  $\Phi_3$  on  $\sigma_{\theta,3}(\bar{g})$ , we get

$$\begin{aligned} \Phi_3(\sigma_{\theta,3}(\bar{g})) &= \Phi_3((3e_{n-1} + 2f_{n-1}) + wf_{n-1}, (e_0 + 2f_0) + w3f_0, \dots, (e_{n-2} + 2f_{n-2}) + w3f_{n-2}) \\ &= (3e_{n-1} + 3f_{n-1}, e_0 + f_0, \dots, e_{n-2} + f_{n-2}, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, \dots, 3e_{n-2} + 3f_{n-2}, 2e_{n-1} + 2f_{n-1}, 2e_0 + 2f_0, \dots, 2e_{n-2} + 2f_{n-2}) \\ &= \delta\eta_3 \Phi_3(\bar{g}). \end{aligned}$$

□

**Theorem 5.4.** For  $\lambda = 3$ , consider a skew- $\lambda$ -constacyclic code  $\mathcal{C}$  of length  $n$  over  $R$ , its Gray image

- (i) under  $\Phi_1$  is a cyclic code over  $\mathbb{Z}_4$  with length  $2n$ .
- (ii) under  $\Phi_2$  results to a QC code of index 2 over  $\mathbb{Z}_4$  of length  $2n$ .
- (iii) under  $\Phi_3$  is permutation equivalent to a QC code of index 3 over  $\mathbb{Z}_4$  of length  $3n$ .

*Proof.* Consider a skew- $\lambda$ -constacyclic code  $\mathcal{C}$  of length  $n$  over  $R$ , so  $\sigma_{\theta,\lambda}(\mathcal{C}) = \mathcal{C}$ .

- (i) Applying  $\Phi_1$  on both sides, we get  $\Phi_1\sigma_{\theta,\lambda}(\mathcal{C}) = \Phi_1(\mathcal{C})$ . From Prop. 5.3, we get  $\rho\Phi_1(\mathcal{C}) = \Phi_1(\mathcal{C})$ , which gives  $\Phi_1(\mathcal{C})$  is a cyclic code of length  $2n$  over  $\mathbb{Z}_4$ .
- (ii) Applying  $\Phi_2$  on both sides, we get  $\Phi_2\sigma_{\theta,\lambda}(\mathcal{C}) = \Phi_2(\mathcal{C})$ . By Prop. 5.3, we get  $\eta_2\Phi_2(\mathcal{C}) = \Phi_2(\mathcal{C})$ , which gives  $\Phi_2(\mathcal{C})$  is a QC codes of index 2 over  $\mathbb{Z}_4$  of length  $2n$ .
- (iii) Applying  $\Phi_3$  on both sides, we get  $\Phi_3\sigma_{\theta,\lambda}(\mathcal{C}) = \Phi_3(\mathcal{C})$ . From Prop. 5.3, we get  $\delta\eta_3\Phi_3(\mathcal{C}) = \Phi_3(\mathcal{C})$ , which gives  $\Phi_3(\mathcal{C})$  is permutation equivalent to a QC code of index 3 over  $\mathbb{Z}_4$  of length  $3n$ .

□

**Proposition 5.5.** For  $\lambda = 3w$ , consider skew  $\lambda$ -constacyclic shift  $\sigma_{\theta,\lambda}$  on  $R^n$ , then

- (i) For the Gray map  $\Phi_1 : R^n \rightarrow \mathbb{Z}_4^{2n}$ , we have  $\eta_2\Phi_1 = \Phi_1\sigma_{\theta,\lambda}$ .
- (ii) For the Gray map  $\Phi_3 : R^n \rightarrow \mathbb{Z}_4^{3n}$ , we have  $\delta\eta_3\Phi_3 = \Phi_3\sigma_{\theta,\lambda}$ .

*Proof.* Let  $\bar{g} = (g_0, g_1, \dots, g_{n-1}) \in R^n$  where  $g_i = e_i + wf_i, \forall e_i, f_i \in \mathbb{Z}_4$  and  $i = 0, 1, \dots, n-1$ . For  $\lambda = 3w$  and using skew- $\lambda$ -constacyclic shift, we get

$$\begin{aligned} \sigma_{\theta,3w}(\bar{g}) &= (\theta(3wg_{n-1}), \theta(g_0), \dots, \theta(g_{n-2})) \\ &= ((2e_{n-1} + 3f_{n-1}) + we_{n-1}, (e_0 + 2f_0) + w3f_0, \dots, (e_{n-2} + 2f_{n-2}) + w3f_{n-2}). \end{aligned}$$

(i) Applying  $\Phi_1$  on above equation, we get

$$\begin{aligned} \Phi_1(\sigma_{\theta,3w}(\bar{g})) &= \Phi_1((2e_{n-1} + 3f_{n-1}) + we_{n-1}, (e_0 + 2f_0) + w3f_0, \dots, (e_{n-2} + 2f_{n-2}) + w3f_{n-2}) \\ &= (e_{n-1} + 3f_{n-1}, e_0 + 3f_0, \dots, e_{n-2} + 3f_{n-2}, 3e_{n-1} + f_{n-1}, 3e_0 + f_0, \dots, 3e_{n-2} + f_{n-2}) \\ &= \eta_2 \Phi_1(\bar{g}). \end{aligned}$$

(ii) Applying  $\Phi_3$  on  $\sigma_{\theta,3w}(\bar{g})$ , we get

$$\begin{aligned} \Phi_3(\sigma_{\theta,3w}(\bar{g})) &= \Phi_3((2e_{n-1} + 3f_{n-1}) + we_{n-1}, (e_0 + 2f_0) + w3f_0, \dots, (e_{n-2} + 2f_{n-2}) + w3f_{n-2}) \\ &= (3e_{n-1} + 3f_{n-1}, e_0 + f_0, \dots, e_{n-2} + f_{n-2}, e_{n-1} + f_{n-1}, 3e_0 + 3f_0, \dots, 3e_{n-2} + 3f_{n-2}, 2e_{n-1} + 2f_{n-1}, 2e_0 + 2f_0, \dots, 2e_{n-2} + 2f_{n-2}) \\ &= \delta \eta_3 \Phi_3(\bar{g}). \end{aligned}$$

□

**Theorem 5.6.** For  $\lambda = 3w$ , consider a skew- $\lambda$ -constacyclic code  $\mathcal{C}$  of length  $n$  over  $R$ , its Gray image

(i) under  $\Phi_1$  results to a QC code of index 2 over  $\mathbb{Z}_4$  of length  $2n$ .

(ii) under  $\Phi_3$  is permutation equivalent to QC code of index 3 over  $\mathbb{Z}_4$  of length  $3n$ .

*Proof.* Proof is omitted as it is similar to that of Theorem 5.4.

□

### 6 Computational Results

This section provides numerical results on  $\lambda$ -constacyclic codes of odd length over  $R$  and their corresponding images over  $\mathbb{Z}_4$ . Using Magma software, the calculations are done by constructing constacyclic codes over  $R$  and deriving their images through Gray maps introduced above. Each code is defined by a generator polynomial  $g(x)$ , which is a monic divisor of  $x^n - \lambda$ , where  $\lambda$  is taken to be either 3 or  $3w$ . Here, we list some selected codes over  $\mathbb{Z}_4$  and their parameters in terms of Lee weight.

For  $\lambda = 3$ , let  $g(x)$  represents the generator polynomial of  $\mathcal{C}$ ,

- (i) For  $n = 9, g(x) = 1 + 3x + x^2 + 3x^3 + x^4 + 3x^5 + x^6 + 3x^7 + x^8$ , the corresponding code  $\Phi_1(\mathcal{C})$  forms a cyclic code with parameters  $[18, 4^1 2^0, 18]$ .
- (ii) For  $n = 15, g(x) = 1 + x + 3x^5 + 3x^6 + x^{10} + x^{11}$ , the code  $\Phi_1(\mathcal{C})$  corresponds to a cyclic code characterized by the parameters  $[30, 4^4 2^0, 12]$ .
- (iii) For  $n = 15, g(x) = 1 + 3x^3 + x^6 + 3x^9 + x^{12}$ , the code  $\Phi_1(\mathcal{C})$  is a cyclic code with parameters  $[30, 4^3 2^0, 10]$ .

For  $\lambda = 3w$ , let  $g(x)$  represents the generator polynomial of  $\mathcal{C}$ ,

- (i) For  $n = 7, g(x) = 1 + 3wx + x^2 + 3wx^3 + x^4 + 3wx^5 + x^6$ , the image  $\Phi_2(\mathcal{C})$  yields a cyclic code with parameters  $[14, 4^0 2^2, 14]$ .
- (ii) For  $n = 9, g(x) = w + x + 3x^3 + 3wx^4 + wx^6 + x^7$ , the code  $\Phi_2(\mathcal{C})$  corresponds to a cyclic code with parameters  $[18, 4^0 2^4, 12]$ .

## 7 Conclusion

This paper examines  $\lambda$ -constacyclic codes over  $R = \mathbb{Z}_4 + w\mathbb{Z}_4$ ,  $w^2 = 1$ , where  $\lambda$  takes the values 3 and  $3w$ . We introduced Gray maps from  $R$  to the tuples of  $\mathbb{Z}_4$  and the Gray images of  $\lambda$ -constacyclic codes over  $R$  are found to be cyclic, QC, 2-quasi-cyclic and permutation equivalent to QC codes over  $\mathbb{Z}_4$ . The corresponding generator polynomial of the Gray images are obtained. For odd length  $n$  codes over  $R$ , applying Nechaev's permutation, it is observe that cyclic codes maps to cyclic and permutation equivalent QC codes over  $\mathbb{Z}_4$ . Similarly, skew- $\lambda$ -constacyclic codes maps to cyclic, QC and permutation equivalent to QC codes over  $\mathbb{Z}_4$ , which are similar exactly to those of  $\lambda$ -constacyclic codes. Some examples are presented to support the theoretical results.

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## Author information

Dolly Thoudam, Department of Mathematics, Manipur University, India.

E-mail: dollythoudam2401@manipuruniv.ac.in

Okram Ratnabala Devi, Department of Mathematics, Manipur University, India.

E-mail: ratnabala@manipuruniv.ac.in

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