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WEAKLY 2-ABSORBING IDEALS IN NONCOMMUTATIVE SEMIRINGS

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Abstract. In this paper, we define and study the notion of weakly 2-absorbing ideals of noncommutative semirings. Let S be a noncommutative semiring with $1 \neq 0$. A proper ideal P of S is called a weakly 2-absorbing ideal of S if for all $a, b, c \in S$, $0 \neq aSbSc \subseteq P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$. Several properties of weakly 2-absorbing ideals of noncommutative semirings are proved as a generalization of the results for those over rings. For example, we show that if (S, M) is a local semiring such that $M^3 = 0$, then every proper ideal of S is weakly 2-absorbing. We show that if P_1 and P_2 are distinct weakly prime ideals of S, then $P_1 \cap P_2$ is a weakly 2-absorbing ideal of S. Also, we show that a proper ideal P of S is weakly 2-absorbing if whenever $0 \neq IJK \subseteq P$ for some ideals I, J, K of S, then $IJ \subseteq P$ or $IK \subseteq P$ or $JK \subseteq P$.

1 Introduction

Let R be a commutative ring with $1 \neq 0$ and P be a proper ideal of R. P is called a 2-absorbing ideal of R if for all $a, b, c \in R$, $abc \in P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$ [8]. The notion of a 2-absorbing ideal of a commutative ring, which is a generalization of the notion of a prime ideal, was introduced and investigated by A. Badawi [8]. P is called a weakly prime ideal of R if for all $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$ [5]. P is called a weakly 2-absorbing ideal of R if for all $a, b, c \in R$, $0 \neq abc \in P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$ [9]. The notion of a weakly 2-absorbing ideal of a commutative ring, which is a generalization of the notion of a weakly prime ideal, was introduced by A. Badawi and D. A. Yousefian [9].

Let R be a noncommutative ring with $1 \neq 0$ and P be a proper ideal of R. P is called a (weakly) 2-absorbing ideal of R if for all $a, b, c \in R$, ($0 \neq aRbRc \subseteq P$) $aRbRc \subseteq P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$. The concept of a (weakly) 2-absorbing ideal of a noncommutative ring was introduced by N. J. Groenewald ([18]) [16]. For more details on (weakly) 2-absorbing ideals in (non)commutative rings, we refer to [1, 3, 4, 5, 8, 9, 10, 13, 16, 17, 18].

A semiring is a commutative monoid $(S, +, 0)$ and a monoid $(S, \cdot, 1)$ with $1 \neq 0$ such that $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$, and $a \cdot 0 = 0 \cdot a = 0$ for all $a, b, c \in S$. The notion of semirings was introduced by H. S. Vandiver [22] in 1935. A semiring S is called commutative if $xy = yx$ for any $x, y \in S$, and otherwise S is called noncommutative. An ideal I of a semiring S is called subtractive if whenever $x + y \in I$ and $x \in I$, then $y \in I$ for any $x, y \in S$. A proper ideal P of a semiring S is called weakly prime if whenever $0 \neq IJ \subseteq P$ for some ideals I, J of S, then $I \subseteq P$ or $J \subseteq P$ [14]. For more details on semirings, we may refer to [6, 15, 19].

Let S be a commutative semiring with $1 \neq 0$. A proper ideal P of S is called (weakly) 2-absorbing ideal of S if for all $a, b, c \in S$, ($0 \neq abc \in P$) $abc \in P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$. The notions of 2-absorbing ideal and weakly 2-absorbing ideal of a commutative semiring were introduced by A. Y. Darani [12]. For more details on (weakly) 2-absorbing ideals in commutative semirings, we refer to [11, 12, 20, 21].

Recently, M. Adarbeh and M. Saleh [2] introduced and studied the notion of 2-absorbing ideals of noncommutative semirings. A proper ideal P of a noncommutative semiring S is said to be a 2-absorbing ideal if $aSbSc \subseteq P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$ for all $a, b, c \in S$. In this paper, we define the notion of a weakly 2-absorbing ideal of a noncommutative semiring as a generalization of the notion of a weakly 2-absorbing ideal of a noncommutative ring. A proper ideal P of a noncommutative semiring S is said to be a weakly 2-absorbing ideal if $0 \neq aSbSc \subseteq P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$ for all $a, b, c \in S$.

In Section 2, we study some properties of a weakly 2-absorbing ideal of a noncommutative semiring S. For example, we prove in Theorem 2.17 that if P is a weakly 2-absorbing subtractive ideal of S that is not a 2-absorbing ideal, then $P^3 = 0$. In Theorem 2.6, we show that if P_1 and P_2 are distinct weakly prime ideals of S, then $P_1 \cap P_2$ is a weakly 2-absorbing ideal of S. In Theorem 2.10, we show that a proper ideal P of S is a weakly 2-absorbing ideal if whenever $0 \neq IJK \subseteq P$ for some ideals I, J, K of S, then $IJ \subseteq P$ or $IK \subseteq P$ or $JK \subseteq P$. At the end of this section, we define the notion of a strongly weakly 2-absorbing ideal of a semiring S. A proper ideal P of S is a strongly weakly 2-absorbing ideal of S if whenever $0 \neq IJK \subseteq P$ for some ideals I, J, K of S, then $IJ \subseteq P$ or $IK \subseteq P$ or $JK \subseteq P$. We show in Theorem 2.24 that if P is a proper subtractive ideal of S such that $P^3 \neq 0$, then P is a strongly weakly 2-absorbing ideal if and only if P is a weakly 2-absorbing ideal if and only if P is a 2-absorbing ideal.

In Section 3, we study weakly 2-absorbing ideals in certain semiring constructions, such as idealization and the product of semirings. Let S be a semiring and M be an S-S-bisemimodule. The idealization of M is the semiring $S \ltimes M$ with component-wise addition and multiplication given by $(s, m)(t, n) = (st, sn + mt)$. The ideal $0 \ltimes M \cong M$ of $S \ltimes M$ is nilpotent of index 2, and $S \cong S \ltimes 0$ is a subring of $S \ltimes M$. For an ideal I of S and an S-S-bi-subsemimodule N of M, $I \ltimes N$ is an ideal of $S \ltimes M$ if and only if $IM + MI \subseteq N$. We show in Theorem 3.1, that the ideal $I \ltimes M$ of $S \ltimes M$ is a weakly 2-absorbing ideal of $S \ltimes M$ if and only if I is a weakly 2-absorbing ideal of S and $aSbSM = MSbSc = aMc = 0$ for every triple-zero (a, b, c) of I. Throughout this paper, S will always denote a noncommutative semiring with $1 \neq 0$ unless otherwise stated.

2 Weakly 2-absorbing ideals

In this section, we define and study some properties of weakly 2-absorbing ideals of noncommutative semirings. We begin this section with the following definition:

Definition 2.1. A proper ideal P of a semiring S is said to be a weakly 2-absorbing ideal if $0 \neq aSbSc \subseteq P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$ for all $a, b, c \in S$.

Recall that a semiring S is called local if it has a unique maximal left ideal. Equivalently, the sum of two nonunits of S is a nonunit of S.

Example 2.2. Let (S, M) be a local semiring such that $M^3 = 0$. Then every proper ideal of S is weakly 2-absorbing. Let I be a proper ideal of S. If $I = 0$, then clearly, I is weakly 2-absorbing. So we may assume $I \neq 0$ and let $0 \neq aSbSc \subseteq I$. If a, b, c are nonunits in S, then $a, b, c \in M$ but then $0 \neq aSbSc \subseteq SaSbSc \subseteq M^3 = 0$, a contradiction. Hence, either a, b, or c is a unit in S. If a is a unit, then $bc = aa^{-1}b1c \in aSbSc \subseteq I$. If b is a unit, then $ac = ab^{-1}b1c \in aSbSc \subseteq I$. If c is a unit, then $ab = a1bc^{-1}c \in aSbSc \subseteq I$. Thus I is a weakly 2-absorbing ideal of S.

Proposition 2.3. *Let P be an ideal of a semiring S with identity. Consider the following statements:*

(i) P is a weakly prime ideal.

(ii) If $0 \neq IJ \subseteq P$ for some left (right) ideals I, J of S, then $I \subseteq P$ or $J \subseteq P$.

(iii) If $0 \neq aSb \subseteq P$ for some $a, b \in S$, then $a \in P$ or $b \in P$.

Then $(1) \Rightarrow (2) \Rightarrow (3)$. Moreover, if P is subtractive, then $(3) \Rightarrow (1)$, that is, all the above statements are equivalent.

Proof. The proof as in [14, Theorem 2.4]. □

Lemma 2.4. *Let P be a weakly prime ideal of a semiring S. If $0 \neq aSbSc \subseteq P$, then either $a \in P$ or $b \in P$ or $c \in P$.*

Proof. Suppose that $0 \neq aSbSc \subseteq P$. Then we have $0 \neq SaSbSc \subseteq P$. By Proposition 2.3, we have $a \in Sa \subseteq P$ or $SbSc \subseteq P$. Since $aSbSc \neq 0$, then $SbSc \neq 0$. So $0 \neq SbSc \subseteq P$. Again by Proposition 2.3, we have $b \in Sb \subseteq P$ or $c \in Sc \subseteq P$. Thus $a \in P$ or $b \in P$ or $c \in P$. □

Proposition 2.5. *Let S be a semiring. If P is a weakly prime ideal of S, then it is weakly 2-absorbing.*

Proof. Let $a, b, c \in S$ and suppose $0 \neq aSbSc \subseteq P$. Then by Lemma 2.4, $a \in P$ or $b \in P$ or $c \in P$. So $ab \in P$ or $ac \in P$ or $bc \in P$. It follows that P is weakly 2-absorbing. □

The following result proves that $P_1 \cap P_2$ is a weakly 2-absorbing ideal of S for any distinct weakly prime ideals P_1, P_2 of S.

Theorem 2.6. *If P_1 and P_2 are distinct weakly prime ideals of a semiring S, then $P_1 \cap P_2$ is a weakly 2-absorbing ideal of S.*

Proof. Let P_1 and P_2 be distinct weakly prime ideals of a semiring S. If $P_1 \cap P_2 = 0$, then by definition, $P_1 \cap P_2$ is a weakly 2-absorbing ideal of S. So assume that $P_1 \cap P_2 \neq 0$. Let $a, b, c \in S$ and suppose $0 \neq aSbSc \subseteq P_1 \cap P_2$. Assume that $ac \notin P_1 \cap P_2$ and $ab \notin P_1 \cap P_2$. If $ac \notin P_1$ and $ab \notin P_1$, then $a \notin P_1$, $b \notin P_1$, and $c \notin P_1$ and so by Lemma 2.4, P_1 is not weakly prime, which is a contradiction. Similarly, if $ac \notin P_2$ and $ab \notin P_2$, then by Lemma 2.4, P_2 is not weakly prime, which is also a contradiction. Now, assume $ac \notin P_1$ and $ab \notin P_2$. Then since P_1 and P_2 are weakly prime, we have by Lemma 2.4, $b \in P_1$ and $c \in P_2$. Thus $bc \in P_1 \cap P_2$. Similarly, if $ac \notin P_2$ and $ab \notin P_1$, then again by Lemma 2.4, $b \in P_2$ and $c \in P_1$, and hence $bc \in P_1 \cap P_2$. Therefore, $P_1 \cap P_2$ is a weakly 2-absorbing ideal of S. □

Example 2.7. Let $\mathbb{Z}^\circ := \mathbb{Z}^+ \cup \{0\}$ be the semiring of nonnegative integers with the usual addition and multiplication. Then $2\mathbb{Z}^\circ$ and $3\mathbb{Z}^\circ$ are prime ideals of $\mathbb{Z}^\circ$, so they are weakly prime. Thus by Theorem 2.6, $6\mathbb{Z}^\circ = 2\mathbb{Z}^\circ \cap 3\mathbb{Z}^\circ$ is a weakly 2-absorbing ideal of $\mathbb{Z}^\circ$. But $6\mathbb{Z}^\circ$ is not a weakly prime ideal of $\mathbb{Z}^\circ$.

Proposition 2.8. *Let S and T be semirings and $f : S \to T$ be an onto homomorphism with $f(0) = 0$. If K is a weakly-2-absorbing ideal of T and* ker *f is a weakly-2-absorbing ideal of S, then $f^{-1}(K)$ is a weakly-2-absorbing ideal of S.*

Proof. Suppose $0 \neq aSbSc \subseteq f^{-1}(K)$ for $a, b, c \in S$. Then $f(a)Tf(b)Tf(c) \subseteq K$. If $f(a)Tf(b)Tf(c) \neq 0$, then since K is a weakly-2-absorbing ideal of T, we have $f(ab) = f(a)f(b) \in K$ or $f(ac) = f(a)f(c) \in K$ or $f(bc) = f(b)f(c) \in K$, and so $ab \in f^{-1}(K)$ or $ac \in f^{-1}(K)$ or $bc \in f^{-1}(K)$. If $f(a)Tf(b)Tf(c) = 0$, then $f(aSbSc) = 0$. So we have $0 \neq aSbSc \subseteq \ker f$ but $\ker f$ is a weakly-2-absorbing ideal of S, hence $ab \in \ker f$ or $ac \in \ker f$ or $bc \in \ker f$. Since $\ker f = f^{-1}(0) \subseteq f^{-1}(K)$, then $ab \in f^{-1}(K)$ or $ac \in f^{-1}(K)$ or $bc \in f^{-1}(K)$. Thus $f^{-1}(K)$ is a weakly-2-absorbing ideal of S. □

Recall that a proper ideal I of a semiring S is said to be a strong ideal if for each $i \in I$, there exists $i' \in I$ such that $i + i' = 0$ [7].

Proposition 2.9. *Let S, T be semirings, $f : S \to T$ be an onto homomorphism with $f(0) = 0$, and I be a subtractive strong ideal of S. If I is a weakly-2-absorbing ideal of S and* ker *$f \subseteq I$, then $f(I)$ is a weakly-2-absorbing ideal of T.*

Proof. Assume that $0 \neq xTyTz \subseteq f(I)$ for $x, y, z \in T$. As f is onto, so there exist $a, b, c \in S$ such that $x = f(a)$, $y = f(b)$, and $z = f(c)$. We claim that $aSbSc \subseteq I$. Let $s, s' \in S$. Then $f(asbs'c) = xf(s)yf(s')z \in f(I)$, so there exists $i \in I$ such that $f(asbs'c) = f(i)$. But I is a strong ideal of S, so there exists $i' \in I$ such that $i + i' = 0$. It follows that

$$f(asbs'c + i') = f(asbs'c) + f(i') = f(i) + f(i') = f(i + i') = f(0) = 0.$$

So $asbs'c + i' \in \ker f \subseteq I$ but $i' \in I$ and I is subtractive, so $asbs'c \in I$. Thus $aSbSc \subseteq I$. Since $0 \neq xTyTz = f(aSbSc)$, then $aSbSc \neq 0$. So we have $0 \neq aSbSc \subseteq I$. But I is a weakly-2-absorbing ideal of S, hence $ab \in I$ or $ac \in I$ or $bc \in I$. Thus $xy = f(ab) \in f(I)$ or $xz = f(ac) \in f(I)$ or $yz = f(bc) \in f(I)$. Therefore, $f(I)$ is a weakly-2-absorbing ideal of T. □

The following theorem gives a sufficient condition for a proper ideal of a semiring S to be a weakly 2-absorbing ideal.

Theorem 2.10. *Let S be a semiring and P be a proper ideal of S. Suppose that whenever $0 \neq IJK \subseteq P$ for some ideals I, J, K of S, then $IJ \subseteq P$ or $IK \subseteq P$ or $JK \subseteq P$. Then P is a weakly 2-absorbing ideal of S.*

Proof. Assume that $0 \neq aSbSc \subseteq P$ for $a, b, c \in S$. Take $I = SaS$, $J = SbS$, and $K = ScS$. Then $IJK = (SaS)(SbS)(ScS) \subseteq SaSbScS \subseteq SPS \subseteq P$. Hence $0 \neq aSbSc \subseteq (SaS)(SbS)(ScS) = IJK \subseteq P$. By hypothesis, we have $IJ \subseteq P$ or $IK \subseteq P$ or $JK \subseteq P$. But $ab \in (SaS)(SbS) = IJ$, $ac \in (SaS)(ScS) = IK$, and $bc \in (SbS)(ScS) = JK$. Hence $ab \in P$ or $ac \in P$ or $bc \in P$. Therefore, P is a weakly 2-absorbing ideal of S. □

Definition 2.11. Let S be a semiring and P be a weakly 2-absorbing ideal of S.

(i) If $a, b, c \in S$, then we say that (a, b, c) is a triple-zero of P if $aSbSc = 0$, $ab \notin P$, $ac \notin P$ and $bc \notin P$.

(ii) If $IJK \subseteq P$ for some ideals I, J, K of S, then we say that P is free triple-zero with respect to IJK if (a, b, c) is not a triple-zero of P for every $a \in I, b \in J$, and $c \in K$.

Example 2.12. Let $S = \mathbb{Z}^\circ \times \mathbb{Z}_8$ and $P = \{(0, 0)\}$. Then clearly, by definition, P is a weakly 2-absorbing ideal of S. Since $(0, 2)^3 = (0, 0)$ and $(0, 2)^2 = (0, 4) \notin P$, then $\big((0, 2), (0, 2), (0, 2)\big)$ is a triple-zero of P.

Remark 2.13. If P is a weakly 2-absorbing ideal of a semiring S that is not a 2-absorbing ideal, then P has a triple-zero (a, b, c) for some $a, b, c \in S$.

Lemma 2.14. *Let P be a weakly 2-absorbing subtractive ideal of a semiring S. Assume that $aSbK \subseteq P$ for some $a, b \in S$ and some ideal K of S such that (a, b, c) is not a triple-zero of P for every $c \in K$. If $ab \notin P$, then $aK \subseteq P$ or $bK \subseteq P$.*

Proof. Assume that $ab \notin P$. If $aK \nsubseteq P$ and $bK \nsubseteq P$, there exist $c_1, c_2 \in K$ such that $ac_1 \notin P$ and $bc_2 \notin P$. If $aSbSc_1 \neq 0$, then we have $0 \neq aSbSc_1 \subseteq aSbK \subseteq P$ but P is weakly 2-absorbing, $ab \notin P$, and $ac_1 \notin P$, so $bc_1 \in P$. If $aSbSc_1 = 0$, then since $ab \notin P$, $ac_1 \notin P$, and (a, b, c_1) is not a triple-zero of P, we have $bc_1 \in P$. Similarly, since $aSbSc_2 \subseteq aSbK \subseteq P$, P is weakly 2-absorbing, (a, b, c_2) is not a triple-zero of P, $ab \notin P$, and $bc_2 \notin P$, then we have $ac_2 \in P$. Again, since $aSbS(c_1 + c_2) \subseteq aSbK \subseteq P$, P is weakly 2-absorbing, $(a, b, c_1 + c_2)$ is not a triple-zero of P, and $ab \notin P$, then we have $a(c_1 + c_2) \in P$ or $b(c_1 + c_2) \in P$. If $ac_1 + ac_2 = a(c_1 + c_2) \in P$, then since P is subtractive and $ac_2 \in P$, we obtain $ac_1 \in P$, a contradiction. If $bc_1 + bc_2 = b(c_1 + c_2) \in P$, then since P is subtractive and $bc_1 \in P$, we obtain $bc_2 \in P$, a contradiction. Therefore, we have $aK \subseteq P$ or $bK \subseteq P$. □

Theorem 2.15. *Let P be a weakly 2-absorbing subtractive ideal of a semiring S. Assume that $0 \neq IJK \subseteq P$ for some ideals I, J, K of S such that P is a free triple-zero with respect to IJK. Then $IJ \subseteq P$ or $IK \subseteq P$ or $JK \subseteq P$.*

Proof. Assume that $IJ \nsubseteq P$. We prove that $IK \subseteq P$ or $JK \subseteq P$. Assume, by contradiction, that $IK \nsubseteq P$ and $JK \nsubseteq P$. So there exist $x_1 \in I$ and $x_2 \in J$ such that $x_1K \nsubseteq P$ and $x_2K \nsubseteq P$. But $x_1Sx_2K \subseteq IJK \subseteq P$ and since $x_1K \nsubseteq P$ and $x_2K \nsubseteq P$, then by Lemma 2.14, we have $x_1x_2 \in P$. Now since $IJ \nsubseteq P$, there exist $y_1 \in I$ and $y_2 \in J$ such that $y_1y_2 \notin P$. Now, since $y_1Sy_2K \subseteq IJK \subseteq P$ and $y_1y_2 \notin P$, then again by Lemma 2.14, we have $y_1K \subseteq P$ or $y_2K \subseteq P$. We have the following cases:

Case 1: ($y_1K \subseteq P$ and $y_2K \nsubseteq P$). Since $x_1Sy_2K \subseteq IJK \subseteq P$, $x_1K \nsubseteq P$ and $y_2K \nsubseteq P$, then by Lemma 2.14, $x_1y_2 \in P$. Since $y_1K \subseteq P$, $x_1K \nsubseteq P$ and P is subtractive, then $(x_1 + y_1)K \nsubseteq P$. But since $(x_1 + y_1)Sy_2K \subseteq IJK \subseteq P$, $(x_1 + y_1)K \nsubseteq P$ and $y_2K \nsubseteq P$, we obtain $(x_1 + y_1)y_2 \in P$ by Lemma 2.14. But $(x_1 + y_1)y_2 = x_1y_2 + y_1y_2 \in P$, $x_1y_2 \in P$ and P is subtractive, so $y_1y_2 \in P$, a contradiction.

Case 2: ($y_2K \subseteq P$ and $y_1K \nsubseteq P$). Since $y_1Sx_2K \subseteq IJK \subseteq P$, $y_1K \nsubseteq P$, and $x_2K \nsubseteq P$, then by Lemma 2.14, $y_1x_2 \in P$. Also, since $y_2K \subseteq P$, $x_2K \nsubseteq P$, and P is subtractive, then $(x_2 + y_2)K \nsubseteq P$. So we have $y_1S(x_2 + y_2)K \subseteq P$, $y_1K \nsubseteq P$, and $(x_2 + y_2)K \nsubseteq P$, hence by Lemma 2.14, $y_1(x_2 + y_2) \in P$ but P is subtractive and $y_1x_2 \in P$, so $y_1y_2 \in P$, a contradiction.

Case 3: ($y_1K \subseteq P$ and $y_2K \subseteq P$). For $i = 1, 2$, since $y_iK \subseteq P$, $x_iK \nsubseteq P$ and P is subtractive, then $(x_i + y_i)K \nsubseteq P$. But $x_1S(x_2 + y_2)K \subseteq P$, $x_1K \nsubseteq P$ and $(x_2 + y_2)K \nsubseteq P$, so $x_1(x_2 + y_2) \in P$ by Lemma 2.14. Since $x_1x_2 \in P$, $x_1x_2 + x_1y_2 \in P$ and P is subtractive, we have $x_1y_2 \in P$. Also, since $(x_1 + y_1)Sx_2K \subseteq P$, $x_2K \nsubseteq P$ and $(x_1 + y_1)K \nsubseteq P$, then by Lemma 2.14, we have $(x_1 + y_1)x_2 \in P$, so $x_1x_2 + y_1x_2 \in P$ but $x_1x_2 \in P$ and P is subtractive, hence $y_1x_2 \in P$. Finally, since $(x_1 + y_1)S(x_2 + y_2)K \subseteq P$, $(x_1 + y_1)K \nsubseteq P$ and $(x_2 + y_2)K \nsubseteq P$, we have $(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2 \in P$ Lemma 2.14. But since P is subtractive and $x_1x_2, x_1y_2, y_1x_2 \in P$, so $y_1y_2 \in P$, a contradiction. Therefore, $IK \subseteq P$ or $JK \subseteq P$. □

Lemma 2.16. *Let P be a weakly 2-absorbing subtractive ideal of a semiring S and let (a, b, c) be a triple-zero of P for some $a, b, c \in S$. Then:*

(1) $aSbP = PbSc = aPc = 0$.

(2) $P^2c = aP^2 = PbP = 0$.

Proof. First, note that since (a, b, c) is a triple-zero of P, then $aSbSc = 0$ and so

$$asbtc = 0 \text{ for all } s, t \in S.....(*).$$

(1) Suppose that $aSbP \neq 0$. Then there exist $s \in S$ and $x \in P$ such that $asbx \neq 0$. Now by $(*)$, $asbc = 0$. Hence $asb(x + c) = asbx + asbc = asbx \neq 0$. It follows that $0 \neq aSbS(x + c) \subseteq P$. Since P is weakly 2-absorbing and $ab \notin P$, we have $a(x + c) \in P$ or $b(x + c) \in P$. But P is subtractive, so $ac \in P$ or $bc \in P$, which is a contradiction since (a, b, c) is a triple-zero of P. Next, suppose that $PbSc \neq 0$. Then there exist $s \in S$ and $x \in P$ such that $xbsc \neq 0$. Now $(a + x)bsc = absc + xbsc = xbsc \neq 0$. Hence $0 \neq (a + x)SbSc \subseteq P$. Since P is weakly 2-absorbing and $bc \notin P$, we have $(a + x)b \in P$ or $(a + x)c \in P$. But P is subtractive, so $ab \in P$ or $ac \in P$, which is a contradiction since (a, b, c) is a triple-zero of P. Finally, suppose that $aPc \neq 0$. Then there exists $x \in P$ such that $axc \neq 0$. Now $a(b + x)c = abc + axc = axc \neq 0$. Hence $0 \neq aS(b + x)Sc \subseteq P$. Since P is weakly 2-absorbing and $ac \notin P$, we have $a(b + x) \in P$ or $(b + x)c \in P$. But P is subtractive, so $ab \in P$ or $bc \in P$, which is a contradiction since (a, b, c) is a triple-zero of P.

(2) Suppose that $P^2c \neq 0$. Then there exist $x, y \in P$ such that $xyc \neq 0$. By (1), we have $xbc = 0$ and $ayc = 0$. Now $(a + x)(b + y)c = abc + ayc + xbc + xyc = xyc \neq 0$. Hence $0 \neq (a + x)S(b + y)Sc \subseteq P$. Since P is weakly 2-absorbing, we have $(a + x)c \in P$ or $(b + y)c \in P$ or $(a + x)(b + y) \in P$. But P is subtractive, so $ac \in P$ or $bc \in P$ or $ab \in P$, which is a contradiction since (a, b, c) is a triple-zero of P. Next, suppose that $aP^2 \neq 0$. Then there exist $x, y \in P$ such that $axy \neq 0$. By (1), we have $aby = 0$ and $axc = 0$. Now $a(b + x)(c + y) = abc + aby + axc + axy = axy \neq 0$. Hence $0 \neq aS(b + x)S(c + y) \subseteq P$. Since P is weakly 2-absorbing, we have $a(b + x) \in P$ or $a(c + y) \in P$ or $(b + x)(c + y) \in P$. But P is subtractive, so $ab \in P$ or $ac \in P$ or $bc \in P$, which is a contradiction since (a, b, c) is a triple-zero of P. Finally, suppose $PbP \neq 0$. Then there exist $x, y \in P$ such that $xby \neq 0$. By (1), $aby = 0$ and $xbc = 0$. Now $(a + x)b(c + y) = abc + aby + xbc + xby = xby \neq 0$. Hence $0 \neq (a + x)SbS(c + y) \subseteq P$. Since P is weakly 2-absorbing, we have $(a + x)b \in P$ or $b(c + y) \in P$ or $(a + x)(c + y) \in P$. But P is subtractive, so $ab \in P$ or $bc \in P$ or $ac \in P$, which is a contradiction since (a, b, c) is a triple-zero of P. □

Theorem 2.17. *Let P be a weakly 2-absorbing subtractive ideal of a semiring S that is not a 2-absorbing ideal, then $P^3 = 0$.*

Proof. Assume that P is a weakly 2-absorbing subtractive ideal of S that is not a 2-absorbing ideal. Then by Remark 2.13, P has a triple-zero (a, b, c) for some $a, b, c \in S$. If $P^3 \neq 0$, there exist $x, y, z \in P$ such that $xyz \neq 0$. Then by Lemma 2.16, we have $(a + x)(b + y)(c + z) = xyz \neq 0$. Hence $0 \neq (a + x)S(b + y)S(c + z) \subseteq P$. Since P is weakly 2-absorbing, either $(a + x)(b + y) \in P$ or $(a + x)(c + z) \in P$ or $(b + y)(c + z) \in P$, and since P is subtractive, then either $ab \in P$ or $ac \in P$ or $bc \in P$ which is a contradiction since (a, b, c) is a triple-zero of P. Thus $P^3 = 0$. □

Corollary 2.18. *Let S be a semiring such that every proper ideal is a weakly 2-absorbing ideal. If P is a proper subtractive ideal of S such that $P = P^2$, then either $P = 0$ or P is 2-absorbing.*

Proof. Let P be a proper subtractive ideal of S such that $P = P^2$. If $P \neq 0$, then $P^3 = PP^2 = PP = P^2 = P \neq 0$ but P is a weakly 2-absorbing subtractive ideal, so by Theorem 2.17, P is 2-absorbing. □

For a semiring S, let $\mathcal{P}(S)$ denote the intersection of all prime ideals of S.

Corollary 2.19. *If P is a weakly 2-absorbing subtractive ideal of a semiring S that is not a 2-absorbing ideal, then $P \subseteq \mathcal{P}(S)$.*

Proof. Follows from Theorem 2.17 and the fact that $\mathcal{P}(S)$ is a semiprime ideal. □

The next example shows that a proper ideal I of a semiring S with $I^3 = 0$ need not be weakly 2-absorbing.

Example 2.20. Let $S = Ide(\mathbb{Z}_{16})$, the semiring of all ideals of the ring $\mathbb{Z}_{16}$ and let $I = \{(0), (8)\}$. Then I is an ideal of S and $I^3 = 0$ but I is not a weakly 2-absorbing ideal of S since $\{(0)\} \neq (2)S(2)S(2) = S(2)(2)(2) = S(8) \subseteq I$ but $(2)(2) = (4) \notin I$.

Recall that if S is a semiring and $x \in S$, then $(0 :_S x)$ denotes the set $\{s \in S \mid sx = 0\}$.

Proposition 2.21. *Let S be a semiring and $x \in S$. If Sx is subtractive and $(0 :_S x) \subseteq Sx$, then Sx is weakly 2-absorbing if and only if it is 2-absorbing.*

Proof. Suppose Sx is a weakly 2-absorbing subtractive left ideal of S that is not 2-absorbing, then Sx has a triple-zero (a, b, c) for some $a, b, c \in S$. Hence $aSbSc = 0$ with $ab \notin Sx$, $ac \notin Sx$, and $bc \notin Sx$. Now $aSbS(c + x) = aSbSc + aSbSx = aSbSx \neq 0$ since $ab \notin Sx$ and $(0 :_S x) \subseteq Sx$. Now, since $0 \neq aSbS(c + x) \subseteq Sx$, $ab \notin Sx$, and Sx is weakly 2-absorbing, we have $a(c + x) \in Sx$ or $b(c + x) \in Sx$. But Sx is subtractive and $ax, bx \in Sx$, so we have $ac \in Sx$ or $bc \in Sx$, which is a contradiction. Thus Sx is 2-absorbing. □

We end this section by defining the notion of a strongly weakly 2-absorbing ideal of a semiring S and studying some properties of this notion.

Definition 2.22. Let P be a proper ideal of a semiring S. Then P is a strongly weakly 2-absorbing ideal of S if whenever $0 \neq IJK \subseteq P$ for some ideals I, J, K of S, then $IJ \subseteq P$ or $IK \subseteq P$ or $JK \subseteq P$.

Remark 2.23. By Theorem 2.10, every strongly weakly 2-absorbing ideal of a semiring S is weakly 2-absorbing.

Theorem 2.24. *Let P be a proper subtractive ideal of a semiring S such that $P^3 \neq 0$. The following statements are equivalent:*

(1) P is a strongly weakly 2-absorbing ideal of S.

(2) P is a weakly 2-absorbing ideal of S.

(3) P is a 2-absorbing ideal of S.

Proof. $(1) \Rightarrow (2)$. Follows from Remark 2.23.
$(2) \Rightarrow (3)$. Follows from Theorem 2.17 and the fact that $P^3 \neq 0$.
$(3) \Rightarrow (1)$. Follows from [2, Corollary 2.17]. □

Proposition 2.25. *Let P be a proper ideal of a semiring S. The following statements are equivalent:*

(1) P is a strongly weakly 2-absorbing ideal of S.

(2) Whenever $0 \neq IJK \subseteq P$ *with* $P \subseteq I$ *for some ideals* I, J, K *of* S*, then* $IJ \subseteq P$ *or* $IK \subseteq P$ *or* $JK \subseteq P$*.*

Proof. $(1) \Rightarrow (2)$. This follows from Definition 2.22.
$(2) \Rightarrow (1)$. Assume that $0 \neq IJK \subseteq P$ for some ideals I, J, K of S. Then $0 \neq IJK \subseteq (I+P)JK = IJK + PJK \subseteq P$. So by (2) and since $P \subseteq I + P$, we have $(I+P)J \subseteq P$ or $(I+P)K \subseteq P$ or $JK \subseteq P$. Hence $IJ \subseteq P$ or $IK \subseteq P$ or $JK \subseteq P$. Thus P is a strongly weakly 2-absorbing ideal of S. □

Proposition 2.26. *Let* S *be a semiring. Then every ideal of* S *is a strongly weakly 2-absorbing ideal if and only if for any ideals* I, J, K *of* S *with* $IJK \neq 0$, $IJ = IJK$ *or* $IK = IJK$ *or* $JK = IJK$*.*

Proof. Let I, J, K be ideals of S such that $IJK \neq 0$. We have two cases: Case 1: $IJK \neq S$. Then IJK is strongly weakly 2-absorbing. Since $0 \neq IJK \subseteq IJK$, then $IJ \subseteq IJK$ or $IK \subseteq IJK$ or $JK \subseteq IJK$ and so $IJ = IJK$ or $IK = IJK$ or $JK = IJK$. Case 2: $IJK = S$. Then $I = J = K = S$ and so $IJ = IK = JK = S^2 = S^3 = IJK$. Conversely, suppose that P is a proper ideal of S and suppose $0 \neq IJK \subseteq P$ for some ideals I, J, K of S. By hypothesis, $IJ = IJK \subseteq P$ or $IK = IJK \subseteq P$ or $JK = IJK \subseteq P$. Thus P is a strongly weakly 2-absorbing ideal of S. □

Corollary 2.27. *Let* S *be a semiring such that every ideal is a strongly weakly 2-absorbing ideal, then for any ideal* P *of* S*, either* $P^3 = P^2$ *or* $P^3 = 0$*.*

3 Weakly 2-absorbing ideals in some semiring costructions

The following theorem determines when $I \ltimes M$ is a weakly 2-absorbing ideal in the idealization $S \ltimes M$.

Theorem 3.1. *Let* S *be a semiring,* M *an* S*-*S*-bisemimodule, and* I *a proper ideal of* S*. The following statements are equivalent:*

(1) $I \ltimes M$ *is a weakly 2-absorbing ideal of* $S \ltimes M$

(2) I *is a weakly 2-absorbing ideal of* S *and for any triple-zero* (a, b, c) *of* I*,* $aSbSM = MSbSc = aMc = 0$*.*

Proof. $(1) \Rightarrow (2)$. Assume that $I \ltimes M$ is a weakly 2-absorbing ideal of $S \ltimes M$. Let $a, b, c \in S$ and suppose $0 \neq aSbSc \subseteq I$. Then $\{(0,0)\} \neq (a,0)S \ltimes M(b,0)S \ltimes M(c,0) \subseteq I \ltimes M$ but $I \ltimes M$ is a weakly 2-absorbing ideal, so we have $(a,0)(b,0) \in I \ltimes M$ or $(a,0)(c,0) \in I \ltimes M$ or $(b,0)(c,0) \in I \ltimes M$. This implies that $(ab,0) \in I \ltimes M$ or $(ac,0) \in I \ltimes M$ or $(bc,0) \in I \ltimes M$. Hence $ab \in I$ or $ac \in I$ or $bc \in I$. So I is a weakly 2-absorbing ideal of S. Now, let (a,b,c) be a triple-zero of I. Then $aSbSc = 0$, $ab \notin I$, $ac \notin I$, and $bc \notin I$. If $aSbSM \neq 0$, then there exist $s_1, s_2 \in S$ and $m \in M$ such that $as_1bs_2m \neq 0$. Now $(0,0) \neq (as_1bs_2c, as_1bs_2m) = (a,0)(s_1,0)(b,0)(s_2,0)(c,m) \in (a,0)S \ltimes M(b,0)S \ltimes M(c,m) \subseteq aSbSc \ltimes M = 0 \ltimes M \subseteq I \ltimes M$. But $(a,0)(b,0) = (ab,0) \notin I \ltimes M$, $(a,0)(c,m) = (ac,am) \notin I \ltimes M$, and $(b,0)(c,m) = (bc,bm) \notin I \ltimes M$, a contradiction since $I \ltimes M$ is a weakly 2-absorbing ideal. If $MSbSc \neq 0$, then there exist $s_1, s_2 \in S$ and $n \in M$ such that $ns_1bs_2c \neq 0$. As above, we have $(0,0) \neq (as_1bs_2c, ns_1bs_2c) = (a,n)(s_1,0)(b,0)(s_2,0)(c,0) \in (a,n)S \ltimes M(b,0)S \ltimes M(c,0) \subseteq aSbSc \ltimes M = 0 \ltimes M \subseteq I \ltimes M$. But $(a,n)(b,0) = (ab,nb) \notin I \ltimes M$, $(a,n)(c,0) = (ac,nc) \notin I \ltimes M$, and $(b,0)(c,0) = (bc,0) \notin I \ltimes M$, a contradiction since $I \ltimes M$ is a weakly 2-absorbing ideal. If $aMc \neq 0$, then there exists $k \in M$ such that $akc \neq 0$. Now, $(0,0) \neq (abc, akc) = (a,0)(1,0)(b,k)(1,0)(c,0) \in (a,0)S \ltimes M(b,k)S \ltimes M(c,0) \subseteq aSbSc \ltimes M = 0 \ltimes M \subseteq I \ltimes M$. But $(a,0)(b,k) = (ab,ak) \notin I \ltimes M$, $(a,0)(c,0) = (ac,0) \notin I \ltimes M$, and $(b,k)(c,0) = (bc,kc) \notin I \ltimes M$, a contradiction since $I \ltimes M$ is a weakly 2-absorbing ideal. Thus $aSbSM = MSbSc = aMc = 0$.

$(2) \Rightarrow (1)$. Let I be a weakly 2-absorbing ideal of S and suppose $\{(0,0)\} \neq (a,m)S \ltimes M(b,n)S \ltimes M(c,k) \subseteq I \ltimes M$. Then $aSbSc \subseteq I$. If $aSbSc \neq 0$, then since I is a weakly 2-absorbing ideal of S and $0 \neq aSbSc \subseteq I$, we have $ab \in I$ or $ac \in I$ or $bc \in I$, which implies $(a,m)(b,n) = (ab, an+mb) \in I \ltimes M$ or $(a,m)(c,k) = (ac, ak+mc) \in I \ltimes M$ or $(b,n)(c,k) = (bc, bk+nc) \in I \ltimes M$. So suppose that $aSbSc = 0$. If $ab \notin I$, $ac \notin I$, and $bc \notin I$, then (a,b,c) is a triple-zero of I and so by assumption, $aSbSM = MSbSc = aMc = 0$. Now $\{(0,0)\} \neq (a,m)S \ltimes M(b,n)S \ltimes M(c,k) \subseteq aSbSc \ltimes (aSbSM + aMc + MSbSc) = \{(0,0)\}$, a contradiction. So $ab \in I$ or $ac \in I$ or $bc \in I$ and hence $(a,m)(b,n) \in I \ltimes M$ or $(a,m)(c,k) \in I \ltimes M$ or $(b,n)(c,k) \in I \ltimes M$.

□

Corollary 3.2. *Let (S,M) be a local semiring such that $M^3 = 0$ and let $E = MS$. Then $I \ltimes E$ is a weakly 2-absorbing ideal of $S \ltimes E$ for every proper ideal I of S.*

Proof. Let (S,M) be a local semiring such that $M^3 = 0$, $E = MS$, and I be a proper ideal of S. Then by Example 2.2, I is a weakly 2-absorbing ideal of S. Let (a,b,c) be a triple zero of I. Then $aSbSc = 0$, $ab \notin I$, $ac \notin I$, and $bc \notin I$. If a or b or c is a unit in S, then as in Example 2.2, $bc \in aSbSc = 0$ or $ac \in aSbSc = 0$ or $ab \in aSbSc = 0$, respectively. But then $bc \in I$ or $ac \in I$ or $ab \in I$, a contradiction. So a, b, and c are nonunits of S and hence $a,b,c \in M$. Thus $aSbSE = aSbSMS \subseteq M^3S = 0$, $ESbSc = MSSbSc \subseteq M^3 = 0$, and $aEc = aMSc \subseteq M^3 = 0$. Therefore, by Theorem 3.1, $I \ltimes E$ is a weakly 2-absorbing ideal of $S \ltimes E$.

□

An application of Theorem 3.1 is the following counterexample of a nonzero weakly 2-absorbing ideal that is not a 2-absorbing ideal.

Example 3.3. Let $S = Ide(\mathbb{Z}_8)$, the semiring of all ideals of the ring $\mathbb{Z}_8$, that is, $S = \{(0), (1) = \mathbb{Z}_8, (2), (4)\}$ and $M = \{(0), (4)\}$. Let $I = \{(0)\}$. Then clearly, by definition, I is a weakly 2-absorbing ideal of S. Now, the only triple-zero of I is $((2),(2),(2))$ and since $(2)(4) = (0)$, then we have $(2)S(2)SM = MS(2)S(2) = (2)M(2) = 0$. Thus by Theorem 3.1, $I \ltimes M$ is a weakly 2-absorbing ideal of $S \ltimes M$. But $I \ltimes M$ is not a 2-absorbing ideal of $S \ltimes M$ since $((2),(0))((2),(0))((2),(0)) = ((0),(0)) \in I \ltimes M$ but $((4),(0)) \notin I \ltimes M$.

Theorem 3.4. *Let $S = S_1 \times S_2$ be a decomposable semiring and I be a proper subtractive ideal of S_1. The following statements are equivalent:*

(1) I is a 2-absorbing ideal of S_1.

(2) $I \times S_2$ is a 2-absorbing ideal of S.

(3) $I \times S_2$ is a weakly 2-absorbing ideal of S.

Proof. (1) ⇒ (2). Let $(a,b)S(c,d)S(e,f) \subseteq I \times S_2$, then $aS_1cS_1e \subseteq I$. By (1), we have $ac \in I$ or $ae \in I$ or $ce \in I$. So $(a,b)(c,d) \in I \times S_2$ or $(a,b)(e,f) \in I \times S_2$ or $(c,d)(e,f) \in I \times S_2$. Hence $I \times S_2$ is a 2-absorbing ideal of S.
(2) ⇒ (3). Clear.
(3) ⇒ (1). Since I is subtractive, then so is $I \times S_2$. Now since $I \times S_2$ is a proper ideal of S, $I \times S_2 \nsubseteq \mathcal{P}(S)$. So by (3) and Corollary 2.19, $I \times S_2$ is a 2-absorbing ideal of S. Now to prove (1), let $aS_1bS_1c \subseteq I$, then $(a,0)S(b,0)S(c,0) \subseteq I \times S_2$. By (3), we have $(a,0)(b,0) \in I \times S_2$ or $(a,0)(c,0) \in I \times S_2$ or $(b,0)(c,0) \in I \times S_2$. Hence $ab \in I$ or $ac \in I$ or $bc \in I$. Thus I is a 2-absorbing ideal of S_1.

□

Theorem 3.5. *Let $S = S_1 \times S_2$ be a product of semirings. Let I_1 be a nonzero proper subtractive ideal of S_1 and I_2 be a nonzero ideal of S_2. The following statements are equivalent:*

(1) $I_1 \times I_2$ is a weakly 2-absorbing ideal of S.

(2) I_1 is a 2-absorbing ideal of S_1 and $I_2 = S_2$ or I_1 is a prime ideal of S_1 and I_2 is a prime ideal of S_2.

(3) $I_1 \times I_2$ is a 2-absorbing ideal of S.

Proof. (1) $\Rightarrow$ (2). Assume that $I_1 \times I_2$ is a weakly 2-absorbing ideal of S. If $I_2 = S_2$, then by Theorem 3.4, we have I_1 is a 2-absorbing ideal of S_1. Now assume that $I_2 \neq S_2$. We prove that I_i is a prime ideal of S_i for $i = 1, 2$. Suppose $aS_1b \subseteq I_1$ for some $a, b \in S_1$ and take $0 \neq i_2 \in I_2$. Then $\{(0,0)\} \neq (a,1)S(b,1)S(1,i_2) \subseteq aS_1bS_1 \times I_2 \subseteq I_1 \times I_2$. But $(a,1)(b,1) = (ab,1) \notin I_1 \times I_2$, so either $(a,1)(1,i_2) = (a,i_2) \in I_1 \times I_2$ or $(b,1)(1,i_2) = (b,i_2) \in I_1 \times I_2$, and hence either $a \in I_1$ or $b \in I_1$. Thus I_1 is a prime ideal of S_1. Next, suppose $cS_2d \subseteq I_2$ for some $c, d \in S_2$ and take $0 \neq i_1 \in I_1$. Then $\{(0,0)\} \neq (i_1,1)S(1,c)S(1,d) \subseteq I_1 \times S_2cS_2d \subseteq I_1 \times I_2$. But $(1,c)(1,d) = (1,cd) \notin I_1 \times I_2$, so either $(i_1,1)(1,c) = (i_1,c) \in I_1 \times I_2$ or $(i_1,1)(1,d) = (i_1,d) \in I_1 \times I_2$, and hence either $c \in I_2$ or $d \in I_2$. Thus I_2 is a prime ideal of S_2.
(2) $\Rightarrow$ (3). If I_1 is a 2-absorbing ideal of S_1 and $I_2 = S_2$, then by Theorem 3.4, $I_1 \times I_2 = I_1 \times S_2$ is a 2-absorbing ideal of S. Now assume that I_k is a prime ideal of S_k for $k = 1, 2$. Let $(a_1,b_1)S(a_2,b_2)S(a_3,b_3) \subseteq I_1 \times I_2$ for some $(a_j,b_j) \in S$, $j = 1, 2, 3$. Then $a_1S_1a_2S_1a_3 \subseteq I_1$ and $b_1S_2b_2S_2b_3 \subseteq I_2$. But I_k is a prime ideal of S_k for $k = 1, 2$. So $a_i \in I_1$ for some $i \in \{1,2,3\}$ and $b_j \in I_2$ for some $j \in \{1,2,3\}$. Without loss of generality, assume $a_1 \in I_1$ and $b_3 \in I_2$. Hence $(a_1,b_1)(a_3,b_3) \in I_1 \times I_2$. Thus $I_1 \times I_2$ is a 2-absorbing ideal of S.
(3) $\Rightarrow$ (1). Clear. □

Corollary 3.6. *Let $S = S_1 \times S_2$ be a product of semirings. Let I_1 be a nonzero proper subtractive ideal of S_1 and I_2 be an ideal of S_2. If $I_1 \times I_2$ is a weakly 2-absorbing ideal of S that is not a 2-absorbing ideal, then $I_2 = \{0\}$.*

The following example shows that the condition I_2 is a nonzero ideal of S_2 in Theorem 3.5 is essential.

Example 3.7. Let $S_1 = S \ltimes M$ and $I_1 = I \ltimes M$, where $S = Ide(\mathbb{Z}_8)$, $M = \{(0),(4)\}$, and $I = \{(0)\}$. Let S_2 be a semifield. From Example 3.3, I_1 is a weakly 2-absorbing ideal of S_1 that is not a 2-absorbing ideal, and since S_2 is a semifield, then $I_1 \times \{0\}$ is a weakly 2-absorbing ideal of $S_1 \times S_2$ that is not a 2-absorbing ideal.

Theorem 3.8. *Let $S = S_1 \times S_2$ be a product of semirings and let I be a nonzero proper subtractive ideal of S_1. The following statements are equivalent:*

(1) $I \times \{0\}$ is a weakly 2-absorbing ideal of S.

(2) I is a weakly prime ideal of S_1 and $\{0\}$ is a prime ideal of S_2.

Proof. (1) $\Rightarrow$ (2). Assume that $I \times \{0\}$ is a weakly 2-absorbing ideal of S. First, suppose $0 \neq aS_1b \subseteq I$ for some $a, b \in S_1$. Then $\{(0,0)\} \neq (a,1)S(b,1)S(1,0) \subseteq aS_1bS_1 \times \{0\} \subseteq I \times \{0\}$. But since $I \times \{0\}$ is a weakly 2-absorbing ideal of S and $(a,1)(b,1) = (ab,1) \notin I \times \{0\}$, so either $(a,1)(1,0) \in I \times \{0\}$ or $(b,1)(1,0) \in I \times \{0\}$ and hence $(a,0) \in I \times \{0\}$ or $(b,0) \in I \times \{0\}$. So either $a \in I$ or $b \in I$. Hence I is a weakly prime ideal of S_1. Next, suppose $cS_2d = \{0\}$ for some $c, d \in S_2$. Since $I \neq 0$, there is $0 \neq i \in I$. So $\{(0,0)\} \neq \{i\} \times cS_2d \subseteq (i,1)S(1,c)S(1,d) \subseteq I \times \{0\}$. But $I \times \{0\}$ is a weakly 2-absorbing ideal of S and $(1,c)(1,d) = (1,cd) \notin I \times \{0\}$, so $(i,1)(1,c) \in I \times \{0\}$ or $(i,1)(1,d) \in I \times \{0\}$ and hence $(i,c) \in I \times \{0\}$ or $(i,d) \in I \times \{0\}$. So either $c = 0$ or $d = 0$. Thus $\{0\}$ is a prime ideal of S_2.
(2) $\Rightarrow$ (1). Assume that $\{(0,0)\} \neq (a_1,b_1)S(a_2,b_2)S(a_3,b_3) \subseteq I \times \{0\}$ for some $(a_j,b_j) \in S$, $j = 1, 2, 3$. Then $\{(0,0)\} \neq a_1S_1a_2S_1a_3 \times b_1S_2b_2S_2b_3 \subseteq I \times \{0\}$. So we have $\{0\} \neq a_1S_1a_2S_1a_3 \subseteq I$ and $b_1S_2b_2S_2b_3 = \{0\}$. Since I is a weakly prime ideal of S_1 and $\{0\} \neq a_1S_1a_2S_1a_3 \subseteq I$, then by Lemma 2.4, $a_i \in I$ for some $i \in \{1,2,3\}$. Also, since $\{0\}$ is a prime ideal of S_2 and $b_1S_2b_2S_2b_3 = \{0\}$, then $b_j = 0$ for some $j \in \{1,2,3\}$.Without loss of generality, assume $a_1 \in I$ and $b_2 = 0$. Then $(a_1,b_1)(a_2,b_2) = (a_1a_2,0) \in I \times \{0\}$. Therefore, $I \times \{0\}$ is a weakly 2-absorbing ideal of S. □

Theorem 3.9. *Let $S = S_1 \times S_2$ be a product of semirings, I_1 a nonzero proper subtractive ideal of S_1, and I_2 a proper ideal of S_2. The following statements are equivalent:*

(1) $I_1 \times I_2$ is a weakly 2-absorbing ideal of S that is not a 2-absorbing ideal.

(2) I_1 is a weakly prime ideal of S_1 that is not a prime ideal and $I_2 = \{0\}$ is a prime ideal of S_2.

Proof. (1) $\Rightarrow$ (2). Let $I_1 \times I_2$ be a weakly 2-absorbing ideal of S that is not a 2-absorbing ideal. Then by Corollary 3.6, we have $I_2 = \{0\}$. So $I_1 \times \{0\}$ is a weakly 2-absorbing ideal of S. So by Theorem 3.8, I_1 is a weakly prime ideal of S_1 and $I_2 = \{0\}$ is a prime ideal of S_2. Now we show that I_1 is not a prime ideal of S_1. Suppose not, that is, I_1 is a prime ideal of S_1. Then I_1 is a 2-absorbing ideal of S_1 and so $I_1 \times I_2 = I_1 \times \{0\}$ is a 2-absorbing ideal of S, a contradiction. Hence I_1 is not a prime ideal of S_1.
(2) $\Rightarrow$ (1). Assume that I_1 is a weakly prime ideal of S_1 that is not a prime ideal and $I_2 = \{0\}$ is a prime ideal of S_2. Then by Theorem 3.8, $I_1 \times I_2 = I_1 \times \{0\}$ is a weakly 2-absorbing ideal of S. Next, we want to show that $I_1 \times I_2 = I_1 \times \{0\}$ is not a 2-absorbing ideal of S. Now since I_1 is not a prime ideal of S_1, there exist $a, b \in S_1$ such that $aS_1b \subseteq I_1$ but $a \notin I_1$ and $b \notin I_1$. So $(a, 1)S(b, 1)S(1, 0) = aS_1bS_1 \times \{0\} \subseteq I_1 \times \{0\}$ but $(a, 1)(b, 1) = (ab, 1) \notin I_1 \times \{0\}$, $(a, 1)(1, 0) = (a, 0) \notin I_1 \times \{0\}$, and $(b, 1)(1, 0) = (b, 0) \notin I_1 \times \{0\}$. Therefore, $I_1 \times \{0\}$ is not a 2-absorbing ideal of S. □

Example 3.10. Let $S_1 = \mathbb{Z}_4 \ltimes \mathbb{Z}_2$ and $S_2 = \mathbb{Z}^\circ$. Take $I_1 = \{0\} \ltimes \mathbb{Z}_2$ and $I_2 = \{0\}$. Then clearly $I_2 = \{0\}$ is a prime ideal of S_2. Since $\{0\}$ is a weakly prime ideal of $\mathbb{Z}_4$ and for $ab = 0$ with $a \neq 0$ and $b \neq 0$, we have $a = b = 2$, and so $a\mathbb{Z}_2 = b\mathbb{Z}_2 = 0$, then by [5, Theorem 17], I_1 is a weakly prime ideal of S_1 but I_1 is not a prime ideal of S_1 since $(2, 0)(2, 0) = (0, 0) \in I_1$ but $(2, 0) \notin I_1$. Hence, by Theorem 3.9, $I_1 \times I_2$ is a weakly 2-absorbing ideal of $S_1 \times S_2$ that is not a 2-absorbing ideal.

Theorem 3.11. *Let $S = S_1 \times S_2 \times S_3$ be a product of semirings and let I_i be a subtractive ideal of S_i for $i = 1, 2, 3$ with I_1 is proper. If $I = I_1 \times I_2 \times I_3 \neq \{(0, 0, 0)\}$, then the following statements are equivalent:*

(1) I is a weakly 2-absorbing ideal of S.

(2) I is a 2-absorbing ideal of S.

(3) $I = I_1 \times S_2 \times S_3$ and I_1 is a 2-absorbing ideal of S_1 or $I = I_1 \times I_2 \times S_3$ such that I_1 is a prime ideal of S_1 and I_2 is a prime ideal of S_2 or $I = I_1 \times S_2 \times I_3$ such that I_1 is a prime ideal of S_1 and I_3 is a prime ideal of S_3.

Proof. (1) $\Rightarrow$ (2). Assume that I is a weakly 2-absorbing ideal of S and $I \neq \{(0, 0, 0)\}$. Take $(a_1, a_2, a_3) \in I \setminus \{(0, 0, 0)\}$. We have $(0, 0, 0) \neq (a_1, a_2, a_3) \in (a_1, 1, 1)S(1, a_2, 1)S(1, 1, a_3) \subseteq I$. So $(a_1, 1, 1)(1, a_2, 1) \in I$ or $(a_1, 1, 1)(1, 1, a_3) \in I$ or $(1, a_2, 1)(1, 1, a_3) \in I$. Hence $(a_1, a_2, 1) \in I$ or $(a_1, 1, a_3) \in I$ or $(1, a_2, a_3) \in I$. If $(a_1, a_2, 1) \in I$, then $I_3 = S_3$. Similarly, if $(a_1, 1, a_3) \in I$ or $(1, a_2, a_3) \in I$, then $I_2 = S_2$ or $I_1 = S_1$, respectively. So $I = I_1 \times I_2 \times S_3$ or $I = I_1 \times S_2 \times I_3$ or $I = S_1 \times I_2 \times I_3$. Hence $I \nsubseteq \mathcal{P}(S)$. Since I is a weakly 2-absorbing subtractive ideal of S and $I \nsubseteq \mathcal{P}(S)$, so by Corollary 2.19, I is a 2-absorbing ideal of S.
(2) $\Rightarrow$ (3). Since I is a 2-absorbing ideal of S, I_1 is a 2-absorbing ideal of S_1. And since I_1 is a proper ideal of S_1, then by the proof of the implication $[(1) \Rightarrow (2)]$ in this theorem, either $I_2 = S_2$ or $I_3 = S_3$. Suppose that $I_2 \neq S_2$ and $I_3 = S_3$. We prove that I_1 is a prime ideal of S_1 and I_2 is a prime ideal of S_2. Suppose $a_1S_1b_1 \subseteq I_1$ for some $a_1, b_1 \in S_1$ and suppose $a_2S_2b_2 \subseteq I_2$ for some $a_2, b_2 \in S_2$. Then $(0, 0, 0) \neq (a_1b_1, a_2b_2, 1) \in (a_1, 1, 1)S(1, a_2b_2, 1)S(b_1, 1, 1) \subseteq I$. Since $(a_1, 1, 1)(b_1, 1, 1) \notin I$, then we have $(a_1, 1, 1)(1, a_2b_2, 1) = (a_1, a_2b_2, 1) \in I$ or $(1, a_2b_2, 1)(b_1, 1, 1) = (b_1, a_2b_2, 1) \in I$ and so $a_1 \in I_1$ or $b_1 \in I_1$. Hence I_1 is a prime ideal of S_1. Also, since $(0, 0, 0) \neq (a_1b_1, a_2b_2, 1) \in (a_1b_1, 1, 1)S(1, a_2, 1)S(1, b_2, 1) \subseteq I$ and $(1, a_2, 1)(1, b_2, 1) = (1, a_2b_2, 1) \notin I$, then either $(a_1b_1, 1, 1)(1, a_2, 1) = (a_1b_1, a_2, 1) \in I$ or $(a_1b_1, 1, 1)(1, b_2, 1) = (a_1b_1, b_2, 1) \in I$ and so either $a_2 \in I_2$ or $b_2 \in I_2$. Hence I_2 is a prime ideal of S_2. Finally, suppose that $I_2 = S_2$ and $I_3 \neq S_3$. By an argument similar to the case when I of the form $I_1 \times I_2 \times S_3$ with $I_2 \neq S_2$, we have I_1 is a prime ideal of S_1 and I_3 is a prime ideal of S_3.
(3) $\Rightarrow$ (1). If I is one of the three forms given in (3), then it is easy to check that I is a 2-absorbing ideal of S and thus I is a weakly 2-absorbing ideal of S. □

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