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Some results on ς-stochastic fractional differential equations

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his paper investigates the ς-Stochastic Fractional Differential Equations (ς-SFDE), focusing on their existence, uniqueness, and Ulam-Hyers stability. We establish the continuous dependence of solutions on initial data and their regularity in time. Utilizing the Banach Fixed-Point Theorem (BFPT) and properties of stochastic calculus, we derive key results. Additionally, we provide illustrative examples to support our findings.

1 Introduction

Stochastic differential equations (SDEs) may be construed as integral equations or as differential equations, wherein the stochastic integrals are accompanied by the presence of a Brownian motion. Initially, Kiyoshi Ito conducted a study on stochastic differential equations (SDE) with the purpose of constructing diffusions. These diffusions are characterized by being strong Markovian processes and continuous, whose generators are represented by differential operators of second-order. Furthermore, in order to achieve this objective, he has implemented the use of stochastic calculus. Alternatively, the SDE could be viewed as ordinary differential equations disturbed by random noise. This perspective allows for modeling systems subject to random perturbations.

Over the past few decades, fractional differential equations (FDE) have emerged as a significant technique for modeling real-world phenomena. Several studies have been conducted on this approach (see [1, 2, 3, 4, 5]). Here are some recent papers on stochastic fractional differential equations (SFDE) (see [6, 7, 8, 9, 10, 11]) that provide the latest developments on the subject. Recent works on fractional inequalities and stability analysis include [12, 13, 14], which explore various aspects of fractional calculus and its applications.

R. Almeida in [15] has proposed a fractional derivative with respect to a function ς. Most of the works focus on this operator in the deterministic case (see [16, 17, 18, 19, 20, 21, 22, 5]). In the stochastic case, authors in [24], have studied the existence, uniqueness and HUS for ς-Caputo SFDE.

Research has increasingly focused on HUS due to its applications, including when determining exact solutions is difficult (see [25, 26, 27, 28, 29, 30]).

Our results may find use in control theory and signal processing, where stochastic fractional models are better able to represent the dynamics of complex systems with random disturbances and memory effects.

It is to our knowledge that no work has been performed that addresses the existence, uniqueness, and HUS of ς-SFDEs. In this context, this work generalizes [30] to ς-SFDE. The key points of our article can be summarized as follows:

(1) demonstrate the existence and uniqueness of solutions for ς-SFDEs based on the BFPT.

(2) discuss the continuity dependence on the initial data.

(3) study the regularity of the solutions of ς-SFDEs.

(4) investigate the Ulam-Hyers stability of ς-SFDEs.

The article is structured as follows: In Section 2, we present the basic definitions of ς-fractional derivative and some fundamental notations needed to establish our results. In section 3, we investigate the existence, uniqueness, regularity, continuity dependence on the initial data and the Ulam-Hyers stability of the solutions of ς-SFDE. Section 4 is dedicated to illustrating our theories by exhibiting three examples.

2 Preliminaries

Let consider the interval $I=[a,b],\ a<b$ and $\varsigma \in C^1(I)$ such that $\varsigma'(\rho)>0$, for all $\rho \in I$.

Let $(\Omega, F, (\mathbb{F}_\rho)_{\rho\in[a,b]}, \mathbb{P})$ be a complete probability space and B_t be a standard Brownian motion. For $\rho \in [a,b]$, let $\Xi_\rho = L^2(\Omega, \mathbb{F}_\rho, \mathbb{P})$ be the set of all $\mathbb{F}_\rho$ -mesurable and mean square integrable functions $\zeta = (\zeta_1, ..., \zeta_n) : \Omega \to \mathbb{R}^n$ with the norm:

$$\| \zeta \|_{ms} = \sqrt{\mathbb{E} \| \zeta \|^2}.$$

Definition 2.1. The fractional integral of $f \in L^1(I)$ w.r.t. ς is defined by

$$\varsigma I_a^\varepsilon f(\rho) = \frac{1}{\Gamma(\varepsilon)} \int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1} f(l)dl, \varepsilon > 0.$$

Definition 2.2. For $\varepsilon \in (0,1)$, the fractional derivative of a function f w.r.t. ς is defined by

$$\varsigma D_a^\varepsilon f(\rho) = \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{1}{\varsigma'(\rho)} \frac{d}{d\rho} \right) \int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{-\varepsilon} f(l)dl.$$

We consider the following ς-SFDE:

$$\begin{cases} \varsigma D_a^\varepsilon y(\rho) = \sigma_1(\rho, y(\rho)) + \sigma_2(\rho, y(\rho)) \frac{dB_\rho}{d\rho}, \rho \in (a,b], \\ \varsigma I^{1-\varepsilon} y(a) = \varphi, \end{cases} \tag{2.1}$$

where $\varphi \in \Xi_a$ is the initial condition and $\sigma_1, \sigma_2 : I \times \mathbb{R}^n \to \mathbb{R}^n$ are two measurable functions.

We say that the process $y(\rho)$ is a solution of the ς-SFDE (2.1) if it satisfies the following equation:

$$y(\rho) = \frac{(\varsigma(\rho)-\varsigma(a))^{\varepsilon-1}\varphi}{\Gamma(\varepsilon)} + \frac{1}{\Gamma(\varepsilon)} \int_a^\rho \varsigma'(t)(\varsigma(\rho)-\varsigma(t))^{\varepsilon-1} \sigma_1(t, y(t))dt + \frac{1}{\Gamma(\varepsilon)} \int_a^\rho \varsigma'(t)(\varsigma(\rho)-\varsigma(t))^{\varepsilon-1} \sigma_2(t, y(t))dB_t. \tag{2.2}$$

Now, we introduce the following assumptions:

Assumption 2.3. Assume that

$$\| \sigma_1(l, z_1) - \sigma_1(l, z_2) \| + \| \sigma_2(l, z_1) - \sigma_2(l, z_2) \| \leq M \| z_1 - z_2 \|, \forall (l, z_1, z_2) \in I \times \mathbb{R}^n \times \mathbb{R}^n,$$

where $M > 0$.

Assumption 2.4. Assume that the functions $\sigma_1(.,0), \sigma_2(.,0)$ satisfying

$$\| \sigma_2(.,0) \|_\infty = \operatorname*{ess\,sup}_{l\in I} \| \sigma_2(l,0) \| < \infty$$

and

$$\int_a^b \| \sigma_1(l,0) \|^2 dl < \infty.$$

3 Main results

Let the Banach space $\mathbb{H}^2(I)$ which contains all process f which are $\mathbb{F}_t$-adapted and measurable which satisfies $ess\sup_{\rho\in I}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}f(\rho)\|_{ms}<\infty$ with the norm $\|\cdot\|_{\mathbb{H}^2}$ given by:

$$\|f\|_{\mathbb{H}^2}=ess\sup_{\rho\in I}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}f(\rho)\|_{ms},$$

and consider an operator G on $\mathbb{H}^2(I)$ defined as follows:

$$\begin{aligned}G_\varphi y(\rho)=&\frac{(\varsigma(\rho)-\varsigma(a))^{\varepsilon-1}\varphi}{\Gamma(\varepsilon)}+\frac{1}{\Gamma(\varepsilon)}\int_a^\rho\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}\sigma_1(l,y(l))dl\\&+\frac{1}{\Gamma(\varepsilon)}\int_a^\rho\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}\sigma_2(l,y(l))dB_l,\forall y\in\mathbb{H}^2(I),\varphi\in\Xi_a.\end{aligned}\quad(3.1)$$

3.1 Existence, uniqueness and continuity dependence on the initial data

In this section, we establish the results concerning the existence, uniqueness, and continuity dependence on the initial data of the solutions of the ς-SFDE defined by equation (2.1).

Lemma 3.1. *For every $\varphi\in\Xi_a$, G_φ is well-defined.*

Proof. Let $y\in\mathbb{H}^2(I)$, then

$$\begin{aligned}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}G_\varphi y(\rho)\|_{ms}^2\leqslant&\frac{3\|\varphi\|_{ms}^2}{\Gamma(\varepsilon)^2}\\&+\frac{3}{\Gamma(\varepsilon)^2}\mathbb{E}\left(\|\int_a^\rho\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}\sigma_1(l,y(l))dl\|^2\right)\\&+\frac{3}{\Gamma(\varepsilon)^2}\mathbb{E}\left(\|\int_a^\rho\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}\sigma_2(l,y(l))dB_l\|^2\right).\end{aligned}\quad(3.2)$$

Using the Cauchy-Schwartz inequality and Fubini's theorem, we obtain:

$$\begin{aligned}&\mathbb{E}\left(\|\int_a^\rho\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}\sigma_1(l,y(l))dl\|^2\right)\\&=\mathbb{E}\Big(\|\int_a^\rho\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}\\&\qquad(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(\varsigma(l)-\varsigma(a))^{\varepsilon-1}\sigma_1(l,y(l))dl\|^2\Big)\\&\leqslant\left(\int_a^\rho(\varsigma'(l))^2(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl\right)\\&\qquad\mathbb{E}\Big(\int_a^\rho\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}\sigma_1(l,y(l))\|^2dl\Big).\end{aligned}\quad(3.3)$$

With the change of variables $u=\frac{(\varsigma(l)-\varsigma(a))}{(\varsigma(\rho)-\varsigma(a))}$, for $\rho>a$ and the Beta function

$$B(r_1,r_2)=\int_0^1u^{r_1-1}(1-u)^{r_2-1}du,r_1,r_2>0,$$

we obtain:

$$\begin{aligned}&\int_a^\rho(\varsigma'(l))^2(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl\\&\leqslant K(\varsigma(\rho)-\varsigma(a))^{2\varepsilon-1}\left(\int_0^1u^{2\varepsilon-2}(1-u)^{2\varepsilon-2}du\right)\\&\leqslant K(\varsigma(b)-\varsigma(a))^{2\varepsilon-1}B(\alpha,\beta),\end{aligned}\quad(3.4)$$

where
$$K = \sup_{t \in I} \varsigma'(t) \text{ and } \alpha = \beta = 2\varepsilon - 1.$$

Therefore using Assumption 2.3, we have:
$$\|(\varsigma(l) - \varsigma(a))^{1-\varepsilon}\sigma_1(l, y(l))\|^2 \leqslant 2M^2\|(\varsigma(l) - \varsigma(a))^{1-\varepsilon}y(l)\|^2 + 2(\varsigma(l) - \varsigma(a))^{2-2\varepsilon}\|\sigma_1(l, 0)\|^2.$$

Then,
$$\mathbb{E}\left(\int_a^\rho \|(\varsigma(l) - \varsigma(a))^{1-\varepsilon}\sigma_1(l, y(l))\|^2 dl\right) \leqslant 2M^2(b-a)\|y\|^2_{\mathbb{H}^2} + 2(\varsigma(b) - \varsigma(a))^{2-2\varepsilon}\int_a^b \|\sigma_1(l, 0)\|^2 dl. \quad (3.5)$$

Thus, by (3.3), (3.4) and (3.5), we obtain:
$$\mathbb{E}\left(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{\varepsilon-1}(\varsigma(\rho) - \varsigma(a))^{1-\varepsilon}\sigma_1(l, y(l))dl\|^2\right) \leqslant 2K(\varsigma(b) - \varsigma(a))^{2\varepsilon-1}B(\alpha, \beta)\Big(M^2(b-a)\|y\|^2_{\mathbb{H}^2} + (\varsigma(b) - \varsigma(a))^{2-2\varepsilon}\int_a^b \|\sigma_1(l, 0)\|^2 dl\Big). \quad (3.6)$$

On the other hand, using Ito's isometry formula and applying Assumption 2.3, we get:
$$\begin{aligned}
&\mathbb{E}\Big(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{\varepsilon-1}(\varsigma(\rho) - \varsigma(a))^{1-\varepsilon}\sigma_2(l, y(l))dB_l\|^2\Big) \\
&= \mathbb{E}\Big(\int_a^\rho (\varsigma'(l))^2(\varsigma(\rho) - \varsigma(l))^{2\varepsilon-2}(\varsigma(\rho) - \varsigma(a))^{2-2\varepsilon}(\varsigma(l) - \varsigma(a))^{2\varepsilon-2} \\
&\qquad \| (\varsigma(l) - \varsigma(a))^{1-\varepsilon}\sigma_2(l, y(l))\|^2 dl\Big) \\
&\leqslant 2M^2\|y\|^2_{\mathbb{H}^2}\Big(\int_a^\rho (\varsigma'(l))^2(\varsigma(\rho) - \varsigma(a))^{2-2\varepsilon}(\varsigma(l) - \varsigma(a))^{2\varepsilon-2}(\varsigma(\rho) - \varsigma(l))^{2\varepsilon-2}dl\Big) \\
&\quad + 2\|\sigma_2(., 0)\|^2_\infty\Big(\int_a^\rho (\varsigma'(l))^2(\varsigma(\rho) - \varsigma(a))^{2-2\varepsilon}(\varsigma(\rho) - \varsigma(l))^{2\varepsilon-2}dl\Big). \quad (3.7)
\end{aligned}$$

Since,
$$\begin{aligned}
&\int_a^\rho (\varsigma'(l))^2(\varsigma(\rho) - \varsigma(a))^{2-2\varepsilon}(\varsigma(\rho) - \varsigma(l))^{2\varepsilon-2}dl \\
&\leqslant K(\varsigma(b) - \varsigma(a))^{2-2\varepsilon}\max_{\rho \in I}\int_a^\rho \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{2\varepsilon-2}dl \\
&\leqslant \frac{K}{2\varepsilon - 1}\left(\varsigma(b) - \varsigma(a)\right), \quad (3.8)
\end{aligned}$$

then, using (3.4) and (3.8), we get
$$\mathbb{E}\left(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{\varepsilon-1}(\varsigma(\rho) - \varsigma(a))^{1-\varepsilon}\sigma_2(l, y(l))dB_l\|^2\right) \leqslant 2KM^2\|y\|^2_{\mathbb{H}^2}(\varsigma(b) - \varsigma(a))^{2\varepsilon-1}B(\alpha, \beta) + \frac{2K}{2\varepsilon - 1}(\varsigma(b) - \varsigma(a))\|\sigma_2(., 0)\|^2_\infty. \quad (3.9)$$

Therefore, by (3.1),(3.6) and (3.9), G_φ is well defined. □

Theorem 3.2. *The ς-SFDE (2.1) has a unique solution.*

Proof. For $\tau > 0$, let the norm $\|\cdot\|_\tau$ on $\mathbb{H}^2(I)$ by:

$$\|\xi\|_\tau = \sup_{\rho\in I}\sqrt{\frac{\mathbb{E}\left(\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}\xi(\rho)\|^2\right)}{e^{\tau(\varsigma(\rho)-\varsigma(a))}}}, \quad \forall \xi \in \mathbb{H}^2(I).$$

We have $(\mathbb{H}^2(I), \|\cdot\|_\tau)$ is a Banach space because $\|\cdot\|_{\mathbb{H}^2}$ and $\|\cdot\|_\tau$ are equivalent.

We will show that G_φ is contractive for some $\tau > 0$.

Let $y_1, y_2 \in \mathbb{H}^2(I)$. Using (3.1), we get for all $\rho \in I$:

$$\begin{aligned}
&\mathbb{E}\left(\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}G_\varphi y_1(\rho)-(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}G_\varphi y_2(\rho)\|^2\right)\\
&\leqslant \frac{2}{\Gamma(\varepsilon)^2}\mathbb{E}\left(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\sigma_1(l,y_1(l))-\sigma_1(l,y_2(l)))dl\|^2\right)\\
&+\frac{2}{\Gamma(\varepsilon)^2}\mathbb{E}\left(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\sigma_2(l,y_1(l))-\sigma_2(l,y_2(l)))dB_l\|^2\right).
\end{aligned} \tag{3.10}$$

By applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\mathbb{E}\left(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\sigma_1(l,y_1(l))-\sigma_1(l,y_2(l)))dl\|^2\right)\\
&\quad\leqslant KM^2(\rho-a)\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}\\
&\qquad\qquad\mathbb{E}\left(\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(y_1(l)-y_2(l))\|^2\right)dl.
\end{aligned} \tag{3.11}$$

Moreover, using Ito's isometry, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\sigma_2(l,y_1(l))-\sigma_2(l,y_2(l)))dB_l\|^2\right)\\
&\quad=\mathbb{E}\Big(\int_a^\rho (\varsigma'(l))^2(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}\\
&\qquad\qquad\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(\sigma_2(l,y_1(l))-\sigma_2(l,y_2(l)))\|^2dl\Big)\\
&\quad\leqslant KM^2\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}\\
&\qquad\qquad\mathbb{E}\Big(\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(y_1(l)-y_2(l))\|^2\Big)dl.
\end{aligned} \tag{3.12}$$

Then,

$$\begin{aligned}
&\mathbb{E}\left(\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}G_\varphi y_1(\rho)-(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}G_\varphi y_2(\rho)\|^2\right)\\
&\quad\leqslant\frac{2KM^2}{\Gamma(\varepsilon)^2}[b-a+1]\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}\\
&\qquad\cdot\frac{\mathbb{E}\left(\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(y_1(l)-y_2(l))\|^2\right)}{e^{\tau(\varsigma(l)-\varsigma(a))}}e^{\tau(\varsigma(l)-\varsigma(a))}dl\\
&\quad\leqslant\frac{2KM^2}{\Gamma(\varepsilon)^2}[b-a+1]\|y_1-y_2\|_\tau^2 Z(\rho),
\end{aligned} \tag{3.13}$$

where

$$Z(\rho)=\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}e^{\tau(\varsigma(l)-\varsigma(a))}dl.$$

Thanks to the change of variable $v=\varsigma(l)-\varsigma(a)$, we can derive that

$$Z(\rho)=\vartheta^{2-2\varepsilon}\int_0^\vartheta v^{2\varepsilon-2}(\vartheta-v)^{2\varepsilon-2}e^{v\tau}dv,$$

with $\vartheta = \varsigma(\rho) - \varsigma(a)$.

Thus, by using Lemma 3.2 in [27], with the positive constant $\Lambda_{2\varepsilon-1}$, we get

$$Z(\rho) \leqslant \frac{e^{\tau\vartheta}}{\tau^{2\varepsilon-1}}\left(\Lambda_{2\varepsilon-1} + \frac{\Gamma(2\varepsilon-1)}{2^{2\varepsilon-2}}\right).$$

Therefore, we have

$$\frac{\mathbb{E}\left(\|(\vartheta^{1-\varepsilon}G_\varphi y_1(\rho) - \vartheta^{1-\varepsilon}G_\varphi y_2(\rho)\|^2\right)}{e^{\tau\vartheta}} \leqslant \frac{2KM^2}{\tau^{2\varepsilon-1}\Gamma(\varepsilon)^2}[b-a+1]\left(\Lambda_{2\varepsilon-1} + \tfrac{\Gamma(2\varepsilon-1)}{2^{2\varepsilon-2}}\right)\|y_1 - y_2\|_\tau^2.$$

Hence,

$$\|G_\varphi y_1 - G_\varphi y_2\|_\tau \leqslant C_\tau\|y_1 - y_2\|_\tau,$$

where

$$C_\tau = \sqrt{\frac{2KM^2}{\tau^{2\varepsilon-1}\Gamma(\varepsilon)^2}[b-a+1]\left(\Lambda_{2\varepsilon-1} + \frac{\Gamma(2\varepsilon-1)}{2^{2\varepsilon-2}}\right)}.$$

We choose $\tau > 0$ such that $C_\tau < 1$, therefore G_φ has a unique fixed point. Hence the ς-SPDE (2.1) has a unique solution. □

Theorem 3.3. *Let $\pi_1, \pi_2 \in \Xi_a$ such that $\pi_1 \neq \pi_2$. Then,*

$$\lim_{\pi_1\to\pi_2}\sup_{\rho\in I}\|(\varsigma(\rho) - \varsigma(a))^{1-\varepsilon}(\Theta_1(\rho) - \Theta_2(\rho))\|_{ms} = 0,$$

where $\Theta_1(\rho)$ and $\Theta_2(\rho)$ are the solutions of (2.1) with initial conditions π_1 and π_2 respectively.

Proof. From (2.1), we get

$$\begin{aligned}\Theta_1(\rho) - \Theta_2(\rho) = {} & \frac{(\varsigma(\rho) - \varsigma(a))^{\varepsilon-1}(\pi_1 - \pi_2)}{\Gamma(\varepsilon)} \\ & + \frac{1}{\Gamma(\varepsilon)}\int_a^\rho \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{\varepsilon-1}\left[\sigma_1(l,\Theta_1(l)) - \sigma_1(l,\Theta_2(l))\right]dl \\ & + \frac{1}{\Gamma(\varepsilon)}\int_a^\rho \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{\varepsilon-1}\left[\sigma_2(l,\Theta_1(l)) - \sigma_2(l,\Theta_2(l))\right]dB_l. \end{aligned} \quad (3.14)$$

Then,

$$\begin{aligned}\|(\varsigma(\rho) - \varsigma(a))^{1-\varepsilon}(\Theta_1(\rho) - \Theta_2(\rho))\|_{ms}^2 \leqslant {} & \frac{3\|\pi_1 - \pi_2\|_{ms}^2}{\Gamma(\varepsilon)^2} \\ & + \frac{3}{\Gamma(\varepsilon)^2}\mathbb{E}\left(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{\varepsilon-1}(\varsigma(\rho) - \varsigma(a))^{1-\varepsilon}\left[\sigma_1(l,\Theta_1(l)) - \sigma_1(l,\Theta_2(l))\right]dl\|^2\right) \\ & + \frac{3}{\Gamma(\varepsilon)^2}\mathbb{E}\left(\|\int_a^\rho \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{\varepsilon-1}(\varsigma(\rho) - \varsigma(a))^{1-\varepsilon}\left[\sigma_2(l,\Theta_1(l)) - \sigma_2(l,\Theta_2(l))\right]dB_l\|^2\right) \\ & = \frac{3}{\Gamma(\varepsilon)^2}\left(\mathbb{E}\left(\|\pi_1 - \pi_2\|^2\right) + I_1(\rho) + I_2(\rho)\right). \end{aligned} \quad (3.15)$$

Using Assumption 2.3 and Cauchy-Schwarz inequality, we get:

$$\begin{aligned} I_1(\rho) \leqslant KM^2(\rho - a)\int_a^\rho & \varsigma'(l)(\varsigma(\rho) - \varsigma(l))^{2\varepsilon-2}(\varsigma(\rho) - \varsigma(a))^{2-2\varepsilon}(\varsigma(l) - \varsigma(a))^{2\varepsilon-2} \\ & \mathbb{E}\left(\|(\varsigma(l) - \varsigma(a))^{1-\varepsilon}(\Theta_1(l) - \Theta_2(l))\|^2\right)dl. \end{aligned} \quad (3.16)$$

Using Ito's isometry, we get:

$$I_2(\rho) \le KM^2 \int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2} \cdot \mathbb{E}\left(\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(\Theta_1(l)-\Theta_2(l))\|^2\right) dl. \quad (3.17)$$

Then,

$$\begin{aligned}\frac{\Gamma(\varepsilon)^2}{3}&\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\Theta_1(\rho)-\Theta_2(\rho))\|^2 \\ &\le \mathbb{E}\left(\|\pi_1-\pi_2\|^2\right) + M^2K(b-a+1)\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2} \\ &(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}\mathbb{E}\left(\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(\Theta_1(l)-\Theta_2(l))\|^2\right) dl. \quad (3.18)\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\frac{\Gamma(\varepsilon)^2}{3}&\mathbb{E}\left(\frac{\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\Theta_1(\rho)-\Theta_2(\rho))\|^2}{e^{r(\varsigma(\rho)-\varsigma(a))}}\right) \\ &\le \mathbb{E}\left(\|\pi_1-\pi_2\|^2\right) + \frac{KM^2(b-a+1)}{e^{r(\varsigma(\rho)-\varsigma(a))}}\|\Theta_1-\Theta_2\|_\tau^2 \int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2} \\ &\qquad (\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}e^{r(\varsigma(l)-\varsigma(a))}dl \\ &\le \mathbb{E}\left(\|\pi_1-\pi_2\|^2\right) + \frac{KM^2(b-a+1)}{\tau^{2\varepsilon-1}}\left(\Lambda_{2\varepsilon-1}+\frac{\Gamma(2\varepsilon-1)}{2^{2\varepsilon-1}}\right)\|\Theta_1-\Theta_2\|_\tau^2. \quad (3.19)\end{aligned}$$

Thus, we have

$$\frac{\Gamma(\varepsilon)^2}{3}\|\Theta_1-\Theta_2\|_\tau^2 \le \mathbb{E}\left(\|\pi_1-\pi_2\|^2\right) + \chi(\tau)\|\Theta_1-\Theta_2\|_\tau^2,$$

where

$$\chi(\tau) = \frac{KM^2(b-a+1)}{\tau^{2\varepsilon-1}}\left(\Lambda_{2\varepsilon-1}+\frac{\Gamma(2\varepsilon-1)}{2^{2\varepsilon-2}}\right).$$

Therefore,

$$\left(\frac{\Gamma(\varepsilon)^2}{3}-\chi(\tau)\right)\|\Theta_1-\Theta_2\|_\tau^2 \le \|\pi_1-\pi_2\|_{ms}^2.$$

Hence,

$$\lim_{\pi_1\to\pi_2}\sup_{\rho\in I}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\Theta_1(\rho)-\Theta_2(\rho))\|_{ms} = 0.$$

□

3.2 Regularity

Theorem 3.4. *Suppose that Assumptions 2.3 and Assumptions 2.4 hold. Then, the solutions of the ς-SFDE (2.1) satisfy:*

(i) for any $c \in]a,b]$,

$$\lim_{\rho\to c}\mathbb{E}\left\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}y(\rho)-(\varsigma(c)-\varsigma(a))^{1-\varepsilon}y(c)\right\|^2 = 0,$$

and

(ii)

$$\lim_{\rho\to a}\mathbb{E}\left\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}y(\rho)-\frac{\varphi}{\Gamma(\varepsilon)}\right\|^2 = 0.$$

Proof. Let $a < \mu_1 < \mu_2 \leqslant b$. We have

$$\begin{aligned}
&(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}y(\mu_2)-(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}y(\mu_1)\\
&\quad=\frac{1}{\Gamma(\varepsilon)}\int_a^{\mu_1}\varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\sigma_1(l,y(l))dl\\
&\quad-\frac{1}{\Gamma(\varepsilon)}\int_a^{\mu_1}\varsigma'(l)(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\sigma_1(l,y(l))dl\\
&\quad+\frac{1}{\Gamma(\varepsilon)}\int_a^{\mu_1}\varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\sigma_2(l,y(l))dB_l\\
&\quad-\frac{1}{\Gamma(\varepsilon)}\int_a^{\mu_1}\varsigma'(l)(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\sigma_2(l,y(l))dB_l\\
&\quad+\frac{1}{\Gamma(\varepsilon)}\int_{\mu_1}^{\mu_2}\varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\sigma_1(l,y(l))dl\\
&\quad+\frac{1}{\Gamma(\varepsilon)}\int_{\mu_1}^{\mu_2}\varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\sigma_2(l,y(l))dB_l\\
&\quad=\frac{1}{\Gamma(\varepsilon)}\Big(T_1+T_2+T_3+T_4\Big). \qquad (3.20)
\end{aligned}$$

Taking the expectation of both sides, we obtain

$$\frac{\Gamma(\varepsilon)^2}{4}\mathbb{E}\Big\|(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}y(\mu_2)-(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}y(\mu_1)\Big\|^2\leqslant\mathbb{E}\|T_1\|^2+\mathbb{E}\|T_2\|^2+\mathbb{E}\|T_3\|^2+\mathbb{E}\|T_4\|^2.$$

By using some classical stochastic calculus tools, we derive the following inequalities:

$$\begin{aligned}
\mathbb{E}\|T_1\|^2&=\mathbb{E}\Big\|\int_a^{\mu_1}\varsigma'(l)\Big[(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\\
&\quad-(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\Big]\sigma_1(l,y(l))dl\Big\|^2\\
&\leqslant\int_a^{\mu_1}\varsigma'(l)^2\Big[(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\\
&\quad-(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\Big]^2(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl\\
&\quad\mathbb{E}\Big(\int_a^{\mu_1}(\varsigma(l)-\varsigma(a))^{2-2\varepsilon}\parallel\sigma_1(l,y(l))\parallel^2dl\Big)\\
&\leqslant c_1K\int_a^{\mu_1}\varsigma'(l)\Big[(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\\
&\quad-(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\Big]^2(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl\\
&=z_1-z_2, \qquad (3.21)
\end{aligned}$$

where

$$c_1=\mathbb{E}\Big(\int_a^b(\varsigma(l)-\varsigma(a))^{2-2\varepsilon}\parallel\sigma_1(l,y(l))\parallel^2dl\Big).$$

We have

$$\begin{aligned}
z_1&=c_1K\int_a^{\mu_1}\varsigma'(l)(\varsigma(\mu_1)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\mu_1)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl\\
&=c_1K(\varsigma(\mu_1)-\varsigma(a))^{2\varepsilon-1}B(2\varepsilon-1,2\varepsilon-1). \qquad (3.22)
\end{aligned}$$

Using the change of variables $u=\frac{(\varsigma(l)-\varsigma(a))}{(\varsigma(\mu_2)-\varsigma(a))}$, we obtain,

$$z_2 = c_1 K \int_a^{\mu_1} \varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl$$
$$= c_1 K(\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\int_0^{\frac{(\varsigma(\mu_1)-\varsigma(a))}{(\varsigma(\mu_2)-\varsigma(a))}} u^{2\varepsilon-2}(1-u)^{2\varepsilon-2}du. \quad (3.23)$$

Then by (3.22) and (3.23), we get

$$\mathbb{E}\|T_1\|^2 \leqslant c_1 K[(\varsigma(\mu_1)-\varsigma(a))^{2\varepsilon-1}B(2\varepsilon-1, 2\varepsilon-1)$$
$$-(\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\int_0^{\frac{(\varsigma(\mu_1)-\varsigma(a))}{(\varsigma(\mu_2)-\varsigma(a))}} u^{2\varepsilon-2}(1-u)^{2\varepsilon-2}du]$$
$$\leqslant c_1 K\Phi_1(\mu_1,\mu_2). \quad (3.24)$$

Concerning the T_2, we have

$$\mathbb{E}\|T_2\|^2 = \mathbb{E}\Big\|\int_a^{\mu_1}\varsigma'(l)\Big[(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}$$
$$-(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\Big]\sigma_2(l,y(l))dB_l\Big\|^2. \quad (3.25)$$

By Ito's isometry, we get

$$\mathbb{E}\|T_2\|^2 = \mathbb{E}\Big(\int_a^{\mu_1}\varsigma'(l)^2\Big[(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}$$
$$-(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\Big]^2(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}$$
$$\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}\sigma_2(l,y(l))\|^2 dl\Big)$$
$$\leqslant 2M^2K\|y\|^2_{\mathbb{H}^2}\int_a^{\mu_1}\varsigma'(l)\Big[(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}$$
$$-(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\Big]^2(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl$$
$$+2K\|\sigma_2(.,0)\|^2_\infty\int_a^{\mu_1}\varsigma'(l)\Big[(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}$$
$$-(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\Big]^2 dl. \quad (3.26)$$

Thus,

$$\mathbb{E}\|T_2\|^2 \leqslant \Big(2M^2K\|y\|^2_{\mathbb{H}^2} + 2K(\varsigma(\mu_1)-\varsigma(a))^{2-2\varepsilon}\|\sigma_2(.,0)\|^2_\infty\Big)\Phi_2(\mu_1,\mu_2), \quad (3.27)$$

where

$$\Phi_2(\mu_1,\mu_2) = \int_a^{\mu_1}\varsigma'(l)\Big[(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}-(\varsigma(\mu_1)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}\Big]^2$$
$$(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl. \quad (3.28)$$

For T_3, we apply Cauchy-Schwartz inequality, Assumptions 2.3 and Assumptions 2.4 yielding that

$$\begin{aligned}\mathbb{E}\|T_3\|^2 &= \mathbb{E}\Big\| \int_{\mu_1}^{\mu_2} \varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\sigma_1(l,y(l))dl\Big\|^2 \\ &\leqslant \int_{\mu_1}^{\mu_2} \varsigma'(l)^2(\varsigma(\mu_2)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl \\ &\qquad \mathbb{E}\Big((\varsigma(l)-\varsigma(a))^{2-2\varepsilon} \parallel \sigma_1(l,y(l)) \parallel^2 dl\Big) \\ &\leqslant c_1K(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}\int_{\mu_1}^{\mu_2} \varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{2\varepsilon-2}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl. \end{aligned} \tag{3.29}$$

Using the change of variables $u = \frac{(\varsigma(l)-\varsigma(a))}{(\varsigma(\mu_2)-\varsigma(a))}$, we obtain,

$$\begin{aligned}&(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}\int_{\mu_1}^{\mu_2} \varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{2\varepsilon-2}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl \\ &\qquad = (\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\int_{\frac{(\varsigma(\mu_1)-\varsigma(a))}{(\varsigma(\mu_2)-\varsigma(a))}}^{1} u^{2\varepsilon-2}(1-u)^{2\varepsilon-2}du. \end{aligned} \tag{3.30}$$

Therefore,

$$\begin{aligned}\mathbb{E}\|T_3\|^2 &\leqslant c_1K(\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\int_{\frac{(\varsigma(\mu_1)-\varsigma(a))}{(\varsigma(\mu_2)-\varsigma(a))}}^{1} u^{2\varepsilon-2}(1-u)^{2\varepsilon-2}du \\ &\leqslant c_1K(\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\Phi_3(\mu_1,\mu_2). \end{aligned} \tag{3.31}$$

Applying Ito's isometry, we get

$$\begin{aligned}\mathbb{E}\|T_4\|^2 &= \mathbb{E}\Big\| \int_{\mu_1}^{\mu_2} \varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{\varepsilon-1}(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}\sigma_2(l,y(l))dB_l\Big\|^2 \\ &= \mathbb{E}\Big(\int_{\mu_1}^{\mu_2} \varsigma'(l)^2(\varsigma(\mu_2)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}\|\sigma_2(l,y(l))\|^2dl\Big) \\ &= \mathbb{E}\Big(\int_{\mu_1}^{\mu_2} \varsigma'(l)^2(\varsigma(\mu_2)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2} \\ &\qquad \|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}\sigma_2(l,y(l))\|^2dl\Big) \\ &\leqslant 2M^2K\|y\|^2_{\mathbb{H}^2}\Big(\int_{\mu_1}^{\mu_2} \varsigma'(l)(\varsigma(\mu_2)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}dl\Big) \\ &\quad + 2\|\sigma_2(.,0)\|^2_\infty\Big(\int_{\mu_1}^{\mu_2} \varsigma'(l)^2(\varsigma(\mu_2)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}dl\Big). \end{aligned} \tag{3.32}$$

Then,

$$\begin{aligned}\mathbb{E}\|T_4\|^2 &\leqslant 2M^2K\|y\|^2_{\mathbb{H}^2}(\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\int_{\frac{(\varsigma(\mu_1)-\varsigma(a))}{(\varsigma(\mu_2)-\varsigma(a))}}^{1} u^{2\varepsilon-2}(1-u)^{2\varepsilon-2}du \\ &\quad + \frac{2K}{2\varepsilon-1}\|\sigma_2(.,0)\|^2_\infty(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}(\varsigma(\mu_2)-\varsigma(\mu_1))^{2\varepsilon-1} \\ &\leqslant 2M^2K\|y\|^2_{\mathbb{H}^2}(\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\Phi_4(\mu_1,\mu_2) \\ &\quad + \frac{2K}{2\varepsilon-1}\|\sigma_2(.,0)\|^2_\infty(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}(\varsigma(\mu_2)-\varsigma(\mu_1))^{2\varepsilon-1}. \end{aligned} \tag{3.33}$$

Therefore,

$$\mathbb{E}\Big\|(\varsigma(\mu_2)-\varsigma(a))^{1-\varepsilon}y(\mu_2)-(\varsigma(\mu_1)-\varsigma(a))^{1-\varepsilon}y(\mu_1)\Big\|^2$$
$$\leqslant \frac{4}{\Gamma(\varepsilon)}\Bigg[c_1K\Phi_1(\mu_1,\mu_2)+\Big(2M^2K\|y\|^2_{\mathbb{H}^2}+2K(\varsigma(\mu_1)-\varsigma(a))^{2-2\varepsilon}\|\sigma_2(.,0)\|^2_\infty\Big)\Phi_2(\mu_1,\mu_2)$$
$$+c_1K(\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\Phi_3(\mu_1,\mu_2)+2M^2K\|y\|^2_{\mathbb{H}^2}(\varsigma(\mu_2)-\varsigma(a))^{2\varepsilon-1}\Phi_4(\mu_1,\mu_2)$$
$$+\frac{2K}{2\varepsilon-1}\|\sigma_2(.,0)\|^2_\infty(\varsigma(\mu_2)-\varsigma(a))^{2-2\varepsilon}(\varsigma(\mu_2)-\varsigma(\mu_1))^{2\varepsilon-1}\Bigg]. \quad (3.34)$$

Hence, (i) holds from the last inequality.

The proof of (ii) is similar to (i). □

3.3 Ulam-Hyers stability

In this section, we discuss the Ulam-Hyers stability w.r.t. θ for the ς-SFDE (2.1) on I.

Definition 3.5. The ς-SFDE (2.1) is Ulam-Hyers stable w.r.t. θ, if there is $L>0$ satisfying: for every $\theta>0$ and all solution $\varpi\in\mathbb{H}^2(I)$ of the following inequality:

$$\mathbb{E}\Big\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\varpi(\rho)-\Big[(\varsigma(\rho)-\varsigma(a))^{\varepsilon-1}\frac{\varphi}{\Gamma(\varepsilon)}$$
$$-\Big(\int_a^\rho\frac{\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}}{\Gamma(\varepsilon)}(\sigma_1(l,\varpi(l))dl+\sigma_2(l,\varpi(l))dB_l)\Big)\Big]\Big\|^2\leqslant\theta,\quad\forall\rho\in I,\quad (3.35)$$

there is $\xi\in\mathbb{H}^2(I)$ a solution of (2.1), with initial condition ${}^{\varsigma}I^{1-\varepsilon}\xi(a)=\varphi$, such that:

$$\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\varpi(\rho)-\xi(\rho))\|^2\leqslant L\theta,\quad\forall\rho\in I.$$

Theorem 3.6. *Assume that Assumption 2.3 and Assumption 2.4 hold. Then, the ς-SFDE (2.1) is Ulam-Hyers stable on I.*

Proof. Let $\theta>0$ and $\varpi\in\mathbb{H}^2(I)$ a solution of (3.35).

Let $\xi(\rho)$ the solution of (2.1) with ${}^{\varsigma}I^{1-\varepsilon}\xi(a)=\varphi$, then

$$\xi(\rho)=(\varsigma(\rho)-\varsigma(a))^{\varepsilon-1}\frac{\varphi}{\Gamma(\varepsilon)}+\int_a^\rho\frac{\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}}{\Gamma(\varepsilon)}(\sigma_1(l,\xi(l))dl+\sigma_2(l,\xi(l))dB_l). \quad (3.36)$$

Thus,

$$\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\varpi(\rho)-\xi(\rho))\|^2$$
$$\leqslant 2\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\varpi(\rho)-(\varsigma(\rho)-\varsigma(a))^{\varepsilon-1}\frac{\varphi}{\Gamma(\varepsilon)}$$
$$-\int_a^\rho\frac{\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}}{\Gamma(\varepsilon)}\sigma_1(l,\varpi(l))dl+\sigma_2(l,\varpi(l))dB_l)\|^2$$
$$+2\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}\int_a^\rho\frac{\varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}}{\Gamma(\varepsilon)}[\sigma_1(l,\varpi(l))-\sigma_1(l,\xi(l)))dl$$
$$+(\sigma_2(l),\varpi(l))-\sigma_2(l),\xi(l)))dB_l]\rangle\|^2. \quad (3.37)$$

Then, applying Assumption 2.3, Assumption 2.4, Ito's isometry and Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
&\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\varpi(\rho)-\xi(\rho))\|^2\\
&\leqslant 2\theta+\frac{4}{\Gamma(\varepsilon)^2}\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}\sigma_1(l,\varpi(l))-\sigma_1(l,\xi(l)))dl\|^2\\
&\quad+\frac{4}{\Gamma(\varepsilon)^2}\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{\varepsilon-1}\sigma_2(l,\varpi(l))-\sigma_2(l,\xi(l)))dB_l\|^2\\
&\leqslant 2\theta+\frac{4}{\Gamma(\varepsilon)^2}KM^2(\rho-a)\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}\\
&\qquad\mathbb{E}\left(\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(\varpi(l)-\xi(l))\|^2\right)dl\\
&\quad+\frac{4}{\Gamma(\varepsilon)^2}KM^2\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}\\
&\qquad\mathbb{E}\Big(\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(\varpi(l)-\xi(l))\|^2\Big)dl\\
&\leqslant 2\theta+\frac{4}{\Gamma(\varepsilon)^2}KM^2(\rho-a+1)\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}\\
&\qquad\mathbb{E}\left(\|(\varsigma(l)-\varsigma(a))^{1-\varepsilon}(\varpi(l)-\xi(l))\|^2\right)dl. \qquad (3.38)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\varpi(\rho)-\xi(\rho))\|^2}{e^{\tau(\varsigma(\rho)-\varsigma(a))}} &\leqslant 2\theta+\frac{4M^2K(\rho-a+1)}{\Gamma(\varepsilon)^2e^{\tau(\varsigma(\rho)-\varsigma(a))}}\|\varpi-\xi\|_\tau^2\\
&\int_a^\rho \varsigma'(l)(\varsigma(\rho)-\varsigma(l))^{2\varepsilon-2}(\varsigma(\rho)-\varsigma(a))^{2-2\varepsilon}(\varsigma(l)-\varsigma(a))^{2\varepsilon-2}e^{\tau(\varsigma(l)-\varsigma(a))}dl\\
&\leqslant 2\theta+\frac{4M^2K(\rho-a+1)}{\Gamma(\varepsilon)^2\tau^{2\varepsilon-1}}\left(\Lambda_{2\varepsilon-1}+\frac{\Gamma(2\varepsilon-1)}{2^{2\varepsilon-2}}\right)\|\varpi-\xi\|_\tau^2. \qquad (3.39)
\end{aligned}
$$

We consider $\tau>0$ such that $B(\tau)=\frac{4M^2K(\rho-a+1)}{\Gamma(\varepsilon)^2\tau^{2\varepsilon-1}}\left(\Lambda_{2\varepsilon-1}+\frac{\Gamma(2\varepsilon-1)}{2^{2\varepsilon-2}}\right)<1$. Note that $B(\tau)\to 0$ as $\tau\to\infty$, so such τ exists. Then,

$$\|\varpi-\xi\|_\tau^2\leq\frac{2\theta}{1-B(\tau)}.$$

Hence,

$$\mathbb{E}\|(\varsigma(\rho)-\varsigma(a))^{1-\varepsilon}(\varpi(\rho)-\xi(\rho))\|^2\leq\frac{2e^{\tau(\varsigma(b)-\varsigma(a))}}{1-B(\tau)}\theta.$$

Then, the ς-SFDE (2.1) is Ulam-Hyers stable on I. □

4 Examples

Our theories are illustrated by three examples in this section.

Example 4.1. Let consider the inequality (3.35) for each $\theta>0$ and $\rho\in I$ where $I=[0,5]$, $\varsigma(\rho)=\rho$, $\sigma_1(\rho,\varpi(\rho))=\frac{\cos(\varpi(\rho))}{\sqrt{\rho^4+1}}$, $\sigma_2(\rho,\varpi(\rho))=\frac{\cos(\varpi(\rho))}{\rho^4+3}$.

Let $(\rho,\varpi_1,\varpi_2)\in[0,5]\times\mathbb{R}\times\mathbb{R}$, then

$$|\sigma_1(\rho,\varpi_1)-\sigma_1(\rho,\varpi_2)|\leq|\varpi_1-\varpi_2|, \qquad (4.1)$$

$$|\sigma_2(\rho,\varpi_1)-\sigma_2(\rho,\varpi_2)|\leq\frac{1}{3}|\varpi_1-\varpi_2|, \qquad (4.2)$$

$$||\sigma_2(\cdot,0)||_\infty=\operatorname*{ess\,sup}_{\rho\in[0,5]}|\sigma_2(\rho,0)|\leq\frac{1}{3}, \qquad (4.3)$$

and

$$\int_0^5||\sigma_1(l,0)||^2dl\leq 5.$$

Hence, Assumption 2.3 and Assumption 2.4 are satisfied. Therefore, by Theorem 3.6 the ς-SFDE is Ulam-Hyers stable on $[0,5]$.

Example 4.2. Let consider the inequality (3.35) for every $\theta > 0$ and $\rho \in I$ where $I = [1,3]$, $\varsigma(\rho) = \rho^2$, $\sigma_1(\rho, \varpi(\rho)) = \frac{\cos(\varpi(\rho))}{\sqrt{\rho^2+1}}$, $\sigma_2(\rho, \varpi(\rho)) = \frac{\varpi(\rho)}{\rho^6+5}$.

Let $(\rho, \varpi_1, \varpi_2) \in [1,3] \times \mathbb{R} \times \mathbb{R}$, then

$$|\sigma_1(\rho, \varpi_1) - \sigma_1(\rho, \varpi_2)| \leq |\varpi_1 - \varpi_2|, \tag{4.4}$$

$$|\sigma_2(\rho, \varpi_1) - \sigma_2(\rho, \varpi_2)| \leq \frac{1}{5}|\varpi_1 - \varpi_2|, \tag{4.5}$$

$$||\sigma_2(\cdot, 0)||_\infty = \operatorname*{ess\,sup}_{\rho \in [1,3]} |\sigma_2(\rho, 0)| \leq \frac{1}{5}, \tag{4.6}$$

and

$$\int_1^3 ||\sigma_1(l, 0)||^2 dl \leq 2.$$

Hence, Assumption 2.3 and Assumption 2.4 are satisfied. Therefore, by Theorem 3.6 the ς-SFDE is Ulam-Hyers stable on $[1,3]$.

Example 4.3. Consider a two-dimensional ς-SFDE system for every $\theta > 0$ with $I = [0,2]$, $\varsigma(\rho) = \rho$, and the coefficients defined as:

$$\sigma_1(\rho, y) = \begin{pmatrix} \frac{\sin(y_1)}{3+\rho^2} \\ \frac{\cos(y_2)}{4+\rho^2} \end{pmatrix}, \quad \sigma_2(\rho, y) = \begin{pmatrix} \frac{y_1}{5} \\ \frac{\sin(y_2)}{5+2\rho^2} \end{pmatrix},$$

where $y = (y_1, y_2)^T \in \mathbb{R}^2$. For any $y, z \in \mathbb{R}^2$ and $\rho \in [0,2]$, we have:

$$\|\sigma_1(\rho, y) - \sigma_1(\rho, z)\| \leq \|y - z\|, \tag{4.7}$$

$$\|\sigma_2(\rho, y) - \sigma_2(\rho, z)\| \leq \|y - z\|, \tag{4.8}$$

and

$$\|\sigma_2(\cdot, 0)\|_\infty = 0, \quad \int_0^2 \|\sigma_1(l, 0)\|^2 dl \leq 2.$$

Thus, Assumptions 2.3 and 2.4 are satisfied, and by Theorem 3.6, this two-dimensional ς-SFDE system is Ulam-Hyers stable on $[0,2]$.

5 Conclusion

This paper focuses on investigating the existence, uniqueness, continuity dependence on the initial data, regularity of the solutions and the Ulam-Hyers stability of ς-SFDEs, utilizing BFPT, Ito's isometry formula and classical inequalities such as the Cauchy-Schwartz inequality.

Previous efforts on stochastic fractional differential equations are generalized and extended by our results. Our method offers a unified framework for studying ς-fractional derivatives with regard to arbitrary functions, which gives us more freedom in simulating real-world occurrences than previous research. Although more general than those found in [24], [25], and [26], the stability results achieved here are compatible with those of those studies.

In our forthcoming endeavors, we aim to extend our investigations to encompass pantograph ς-SFDEs, which integrate both fractional and delay components.We also plan to explore applications of our results to systems involving Caputo-Erdlyi-Kober and Hilfer fractional derivatives as referenced in [28], [29], [30], [31], and [32].

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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