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DEFERRED RIESZ STATISTICAL CONVERGENCE VIA J_p^2-POWER SERIES METHOD FOR DOUBLE SEQUENCES AND ITS APPLICATION TO KOROVKIN TYPE APPROXIMATION THEOREM

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Abstract In this article, we study the statistical $\left(\overline{N}, s_m, t_n\right)$ convergence, strongly Riesz convergence, Riesz statistical convergence concepts for double real number sequences. We also compare the concepts of deferred Riesz convergence, strongly deferred Riesz convergence, deferred Riesz statistical convergence for double sequences as related to the J_p^2-power series method. As an application, a Krovkin's type approximation theorem related to this method is given.

1 Introduction

In this paper, deferred Riesz statistical convergence as well as -deferred Riesz statistical convergence in terms of power series method for real or complex sequences are introduced and studied. Their interconnection with Riesz statistical convergence is explored and illustrative examples in support of our results are presented. Applications of these convergences in the form of a Korovkin-type approximation theorem are established and illustrations demonstrating the superiority of our proven theorem over the classical Korovkin theorem are offered. Finally, the rate of convergence is computed.

Statistical convergence was first given by [22] in 1951. Then, statistical convergence was extended to various fields by many authors (see [2], [10], [15], [16], [23], [24], [25], [26], [28], [31], [32], [34], [35], [36], [39], [40], [44] and [45] etc.)

In 1988, Connor [10] showed that strong Cesàro summability implies statistical convergence, but the converse requires boundedness. Later, strong convergence and statistical convergence were generalized and compared by many authors (see for example [42]).

The idea of convergence by a power series method was first given in Tauberian theorems (see [4], [5] and [30] etc.). Recently, definitions of statistical convergence by the power series method and convergence by a modulus function and the power series method have been given and also some of their properties have been investigated (see [6], [7], [8], [48], [49], [51] and [52] etc.). Yıldız and Demirci [53] also showed that P_p-strong convergence requires P_p-statistical convergence according to the P_p-power series method. For double sequences, Coşar et al. in [11] discussed some other properties and also, statistical convergence and strong convergence were discussed by Mursaleen et al. [42] with respect to a modulus function and the J_p^2-power series method. The first statistical Krovkin-type approximation theorem was given by Gadjiev and Orhan in [25]. Of course, related Krovkin type approximation theorems have also been discussed by various authors (see [9], [17], [18], [19], [20], [21], [50], [54] and [55], etc.).

In 2003, Móricz [38] and Mursaleen-Edely [39] gave a definition of statistical convergence for double sequences, independently. In the same year, Móricz [37] gave the double Cesàro statistical $(C,1,1)$ definition and showed that double statistical convergence requires double statistical $(C,1,1)$ convergence. Karakaya-Chishti in [29], Mursaleen and co-workers in [41] gave definition of the Riesz statistical convergence and in [41] they gave the Krovkin type approximation theorem.

Agnew [1] gave the definition of Deferred Cesàro convergence. After this study, Jena et al. introduced the statistical Deferred Cesaro convergence and statistical Deferred Cesàro summability and gave the approximation theorem in [27]. Dağadur and Sezgek gave theorems related to cluster points with respect to double deferred Cesàro convergence in [14]. Deferred Nörlund and deferred Riesz means were studied in [47], [46] and also Dağadur and Çatal dealt with the deferred Nörlund and deferred Riesz means convergence of Mellin-Fourier series in [13].

Cai [9] also gave the concepts of deferred Riesz statistical convergence and strongly deferred Riesz summability and studied the implications between them. He also defined deferred Riesz convergence according to the power series method and gave a Krovkin type theorem associated with it.

In Section 2, the definitions of statistical $\left(\overline{N},s_m,t_n\right)$ convergence, strongly Riesz convergence, Riesz statistical convergence for double sequences were given and it was shown that Riesz statistical convergence requires the other two convergences with boundedness condition. In Section 3, the definitions of deferred Riesz convergence, strongly deferred Riesz convergence, deferred Riesz statistical convergence for double sequences are given and strongly deferred Riesz convergence and deferred Riesz convergence are compared. It is also shown that deferred Riesz statistical convergence for double sequences, with the condition of boundedness, requires deferred Riesz convergence and statistical convergence of the deferred Riesz mean. In Section 4, the definitions of Riesz statistical convergence according to the J_p^2-power series method, and deferred Riesz statistical convergence according to the J_p^2-power series method are given. Examples are given showing that the concepts of J_p^2-power series method, Riesz statistical convergence according to the J_p^2-power series method, and deferred Riesz statistical convergence according to the J_p^2-power series method are incomparable. In the last section, a Korovkin type theorem is given as an application.

2 Basic definitions

In this section, the concept of double deferred Riesz statistical convergence and strong double deferred Riesz summability means for double sequences of real numbers will be presented and some inclusion relations between them will be established.

Definition 2.1. [43, p.10] Let $\{x_n\}$ be a complex sequence. Let $\{s_n\}$ be a sequence of non-negative real numbers such that $s_0>0$,

$$R_n := \sum_{k=0}^{n} s_k,$$

and

$$\mathcal{R}_n\left(x\right) := \frac{1}{R_n}\sum_{k=0}^{n} s_k x_k.$$

If

$$\lim_{n\to\infty} \mathcal{R}_n\left(x\right) = L$$

then $\{x_n\}$ is said to be Riesz (weighted mean) summable to L and is written as

$$x_j \to L, \left(\overline{N},s_n\right).$$

The Riesz mean is regular $\Leftrightarrow R_n\to\infty$.

Definition 2.2. [3] Let (s_n) and (t_n) be two sequences of positive real numbers such that

$$S_m := \sum_{j=0}^{m} s_j \to \infty \text{ and } T_n := \sum_{k=0}^{n} t_k \to \infty \text{ for } m,\, n\to\infty$$

where $s_0 > 0$, $t_0 > 0$ and $s_n \neq 0$, $t_n \neq 0$. Let

$$\mathcal{R}_{nm}(x) := \frac{1}{S_m T_n} \sum_{j=0}^{m} \sum_{k=0}^{n} s_j t_k x_{jk} \tag{2.1}$$

If $R_{nm}(x) \to L$ for $m, n \to \infty$ in the Pringstime sense, that is, if $p - \lim\limits_{nm} R_{nm}(x) = L$, then

$$x_{jk} \to L, \left(\overline{N}, s_m, t_n\right)$$

is written and the double sequence (x_{jk}) is said to be Riesz convergent to L. Of course, since S_n, $T_n \to \infty$ (for $n \to \infty$), the Riesz mean $\left(\overline{N}, s_m, t_n\right)$ is also regular.

If $s_n = t_n = 1$ is taken $(C, 1, 1)$ Cesaro mean is obtained.

Definition 2.3. [12] If

$$\lim_{n,m\to\infty} \frac{1}{mn} \left|\{(j,k) : j \leq m, k \leq n \text{ ve } |x_{jk} - L| \geq \varepsilon\}\right| = 0$$

for ech $\varepsilon > 0$, then the double sequence $\{x_{jk}\}$ is said to be statistically convergent to L and $S^2 - \lim x_{jk} = L$ is written.

Definition 2.4. If $S^2 - \lim\limits_{n,m\to\infty} R_{nm}(x) = L$, i.e;

$$\lim_{n,m\to\infty} \frac{1}{S_m T_n} \left|\{(j,k) : j \leq S_m,\, k \leq T_n \text{ ve } |\mathcal{R}_{nm}(x) - L| \geq \varepsilon\}\right| = 0,$$

then we write $S^2_{\left(\overline{N}, s_j, t_k\right)} - \lim\limits_{j,k\to\infty} x_{jk} = L$ and (x_{jk}) is statistically convergent $\left(\overline{N}, s_m, t_n\right)$ to some number L.

Definition 2.5. If

$$\lim_{n,m\to\infty} \frac{1}{S_m T_n} \sum_{j=0}^{m} \sum_{k=0}^{n} s_j t_k |x_{jk} - L| = 0$$

then

$$x_{jk} \to L,\ \left|\left(\overline{N}, s, t\right)\right| \text{ or } \left|\left(\overline{N}, s, t\right)\right| - \lim x_{jk} = L$$

is written and the double sequence (x_{jk}) is said to be strongly Riesz convergent to L.

Definition 2.6. If

$$\lim_{n,m\to\infty} \frac{1}{S_m T_n} \left|\{(j,k) : j \leq S_m, k \leq T_n \text{ ve } s_j t_k |x_{jk} - L| \geq \varepsilon\}\right| = 0$$

for ech $\varepsilon > 0$, then the double sequence $\{x_{jk}\}$ is said to be Riesz statistically convergent to L and $\mathcal{R}S^2 - \lim x_{jk} = L$ is written.

If $s_j = t_k = 1$ is taken, then it reduces to the statistical convergence of the sequence double $\{x_{jk}\}$ to L.

Theorem 2.7. *Let* $s_j t_k |x_{jk} - L| \leq M$ *for* $j, k \in \mathbb{N}^2$. *If* $\mathcal{R}S^2 - \lim x_{jk} = L$, *then* $\left(\overline{N}, s_m, t_n\right) - \lim x_{jk} = L$ *and* $S^2_{\left(\overline{N}, s_j, t_k\right)} - \lim\limits_{j,k\to\infty} x_{jk} = L$.

Proof. Let

$$A_{mn} := \{(j,k) : j \leq S_m, k \leq T_n \text{ and } s_j t_k |x_{jk} - L| \geq \varepsilon\}$$

for each $\varepsilon > 0$. Since $\mathcal{R}S^2 - \underline{\lim} x_{jk} = L$,

$$\lim_{n,m\to\infty} \frac{1}{S_m T_n} |A_{mn}| = 0.$$

Let's say

$$B_{mn} := \{(j,k) : j \leq S_m, k \leq T_n \text{ and } s_j t_k |x_{jk} - L| < \varepsilon\}.$$

By the definition of $\mathcal{R}_{nm}(x)$ (2.1), there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} |\mathcal{R}_{mn}(x) - L| &= \left| \frac{1}{S_m T_n} \left(\sum_{\substack{j=0 \\ s_j t_k |x_{jk} - L| \geq \varepsilon}}^{m} \sum_{k=0}^{n} s_j t_k + \sum_{\substack{j=0 \\ s_j t_k |x_{jk} - L| < \varepsilon}}^{m} \sum_{k=0}^{n} s_j t_k \right) x_{jk} - L \right| \\ &\leq \frac{1}{S_m T_n} \left(\sum_{\substack{j=0 \\ s_j t_k |x_{jk} - L| \geq \varepsilon}}^{m} \sum_{k=0}^{n} s_j t_k + \sum_{\substack{j=0 \\ s_j t_k |x_{jk} - L| < \varepsilon}}^{m} \sum_{k=0}^{n} s_j t_k \right) |x_{jk} - L| \\ &\leq 2M \frac{1}{S_m T_n} |A_{mn}| + \varepsilon < 2\varepsilon, \text{(because } S_m, T_n \to \infty \text{ for } m,n \to \infty) \end{aligned}$$

for $m,n > n_0(\varepsilon)$. Hence $\lim_{m,n\to\infty} \mathcal{R}_{mn}(x) = L$. Thus $\left(\overline{N}, s_m, t_n\right) - \underline{\lim}\, x_{jk} = L$. Also from here we get

$$\frac{1}{S_m T_n} |\{(j,k) : j \leq S_m, k \leq T_n, \text{ and } |\mathcal{R}_{mn}(x) - L| \geq 2\varepsilon\}| \leq \frac{n_0}{S_M} + \frac{n_0}{T_N}, M,N > n_0.$$

Hence for all $\varepsilon > 0$,

$$\lim_{M,N\to\infty} \frac{1}{S_M T_N} |\{(j,k) : j \leq S_M, k \leq T_N, \text{ and } |\mathcal{R}_{mn}(x) - L| \geq 2\varepsilon\}| = 0,$$

i.e. $S^2_{\left(\overline{N}, s_j, t_k\right)} - \lim_{j,k\to\infty} x_{jk} = L.$ □

3 Double Deferred Riesz statistical convergence

Let us consider the sequences $a = \{a_n\}$, $b = \{b_n\}$, $c = \{c_n\}$ and $d = \{d_n\}$ consisting of non-negative integers with the conditions $a(m) < b(m)$, $c(n) < d(n)$, $b(m) \to \infty$ and $d(n) \to \infty$ as (for $n,m \to \infty$).

Definition 3.1. [56] Let $x = \{x_{jk}\}$ be a double sequence. If

$$\lim_{n,m\to\infty} \frac{|\{(j,k) : a(m)+1 \leq j \leq b(m), c(n)+1 \leq k \leq d(n) : |x_{jk} - L| \geq \varepsilon\}|}{(b(m) - a(m))(d(n) - c(n))} = 0$$

for ech $\varepsilon > 0$, then the double sequence $\{x_{jk}\}$ is said to be deferred statistically convergent to L and is denoted by $x_{jk} \to L \left(DS^{cd}_{ab}\right)$.

The situation is the same when $a(m) \leq j \leq b(m), c(n) \leq k \leq d(n)$ is used instead of $a(m)+1 \leq j \leq b(m), c(n)+1 \leq k \leq d(n)$ in Definition 3.1 (see [12]).

If $b(m) = m$, $d(n) = n$, $a(m) = c(n) = 0$ is taken in this definition, then double deferred statistically convergent is reduced to double statistically convergent.

Definition 3.2. Let $x = \{x_{jk}\}$ be a double sequence and

$$\left(D\mathcal{R}^{cd}_{ab} x\right)_{mn} = \frac{1}{S^*_m T^*_n} \sum_{j=a(m)+1}^{b(m)} \sum_{k=c(n)+1}^{d(n)} s_j t_k x_{jk} \tag{3.1}$$

where

$$S^*_m := \sum_{j=a(m)+1}^{b(m)} s_j \to \infty, T^*_n := \sum_{k=c(n)+1}^{d(n)} t_k \to \infty \text{ for } m,n \to \infty. \tag{3.2}$$

If $\lim_{n,m\to\infty} \left(D\mathcal{R}^{cd}_{ab} x\right)_{mn} = L$, then the double sequence $\{x_{jk}\}$ is said to be deferred Riesz summable in L and is denoted by $x_{jk} \to L \left(D\mathcal{R}^{cd}_{ab}\right)$ or $D\mathcal{R}^{cd}_{ab} - \lim x_{jk} = L$.

Of course, since S_n^*, $T_n^* \to \infty$ (for $n \to \infty$), for every choice of sequences $a = \{a_n\}$, $b = \{b_n\}$, $c = \{c_n\}$ and $d = \{d_n\}$ that satisfies the above condition, DR_{ab}^{cd} method is regular. If $s_j = t_k = 1$ is taken, then it reduces to the deferred convergence of the sequence double $\{x_{jk}\}$ to L.

Definition 3.3. Let $x = \{x_{jk}\}$ be a double sequence. If

$$\lim_{m,n\to\infty} \frac{1}{S_m^* T_n^*} \sum_{j=a(m)+1}^{b(m)} \sum_{k=c(n)+1}^{d(n)} s_j t_k \left|x_{jk} - L\right| = 0, \tag{3.3}$$

then the double sequence $\{x_{jk}\}$ is said to be strongly deferred Riesz summable on L and is denoted by $x_{jk} \to L\left(\overline{D\mathcal{R}_{ab}^{cd}}\right)$ or $\overline{D\mathcal{R}_{ab}^{cd}} - \lim x_{jk} = L$.

If $s_j = t_k = 1$ is taken, then it reduces to the strongly deferred summable of the double sequence $\{x_{jk}\}$ to L.

Definition 3.4. Let DR_{ab}^{cd} be a regular Riesz summability method. If

$$\lim_{n,m\to\infty} \frac{\left|\{(j,k) : j \le S_m^*, k \le T_n^* \text{ and } s_j t_k \left|x_{jk} - L\right| \ge \varepsilon\}\right|}{S_m^* T_n^*} = 0$$

for each $\varepsilon > 0$, then the sequence $\{x_{jk}\}$ is said to be deferred Riesz statistically convergent to L and

$$D\mathcal{R}_{ab}^{cd} S^2 - \lim x_{jk} = L$$

is written.

The following theorem compares strong double deferred Riesz convergence with statistical double deferred Riesz convergence.

Theorem 3.5. *Let $\{x_{jk}\}$ be double sequences. If $x_{jk} \to L\left(\overline{D\mathcal{R}_{ab}^{cd}}\right)$, then $x_{jk} \to L\left(D\mathcal{R}_{ab}^{cd} S^2\right)$. But the converse is not true.*

Proof. Assume that $x_{jk} \to L\left(\overline{D\mathcal{R}_{ab}^{cd}}\right)$ and let

$$U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon) := \{(j,k) : j \le S_m^*, \le k \le T_n^* \text{ and } s_j t_k \left|x_{jk} - L\right| \ge \varepsilon\}$$

Now, we have

$$\frac{1}{S_m^* T_n^*} \sum_{j=a(m)+1}^{b(m)} \sum_{k=c(n)+1}^{d(n)} s_j t_k \left|x_{jk} - L\right|$$

$$\begin{aligned}
&= \tfrac{1}{S_m^* T_n^*} \sum_{\substack{j=a(m)+1 \\ (j,k)\in U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)} \sum_{k=c(n)+1}^{d(n)} s_j t_k \left|x_{jk} - L\right| + \tfrac{1}{S_m^* T_n^*} \sum_{\substack{j=a(m)+1 \\ (j,k)\notin U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)} \sum_{k=c(n)+1}^{d(n)} s_j t_k \left|x_{jk} - L\right| \\
&\ge \tfrac{1}{S_m^* T_n^*} \sum_{\substack{j=a(m)+1 \\ (j,k)\in U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)} \sum_{k=c(n)+1}^{d(n)} s_j t_k \left|x_{jk} - L\right| \ge \tfrac{1}{S_m^* T_n^*} \sum_{\substack{j=a(m)+1 \\ (j,k)\in U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)} \sum_{k=c(n)+1}^{d(n)} \varepsilon \\
&\ge \varepsilon \tfrac{1}{S_m^* T_n^*} \left|U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)\right|.
\end{aligned}$$

Taking the limit for $j, k \to \infty$, we obtain $x_{jk} \to L\left(D\mathcal{R}_{ab}^{cd} S^2\right)$. That is, $\left(\overline{D\mathcal{R}_{ab}^{cd}}\right) \subset \left(D\mathcal{R}_{ab}^{cd} S^2\right)$. □

Now let us give a counter-example to show that the converse of the theorem is not true.

Example 3.6. Let's define the sequences $\{x_{jk}\}$ as follows:

$$x_{jk}=\begin{cases} j^2k^2 & ,b(m)-\sqrt{b(m)}\le j\le b(m)\,,\,d(n)-\sqrt{d(n)}\le k\le d(n)\\ 0 & \text{, otherwise}\end{cases}$$

where

$$\begin{array}{ll} a(m)+1<b(m)-\sqrt{b(m)}<b(m) & ,b(m)\to\infty\\ c(n)+1<d(n)-\sqrt{d(n)}<d(n) & ,d(n)\to\infty\end{array}$$

and let's take $s_j=t_j=1$ for each j. for an arbitrary $\varepsilon>0$, we have

$$\frac{1}{S_m^*T_n^*}\left|\{(j,k):a(m)\le j\le b(m),c(n)\le k\le d(n)\text{ and }s_jt_k\left|x_{jk}-L\right|\ge\varepsilon\}\right|$$

$$=\frac{|\{a(m)+1\le j\le b(m),c(n)+1\le k\le d(n):\ |x_{jk}-L|\ge\varepsilon\}|}{(b(m)-a(m))\ (d(n)-c(n))}$$

$$=\frac{\sqrt{b(m)}}{(b(m)-a(m))}\frac{\sqrt{d(n)}}{(d(n)-c(n))}\to 0,\text{ (for }n,m\to\infty).$$

This ensures that $x_{jk}\to 0\left(D\mathcal{R}_{ab}^{cd}S^2\right)$. On the other hand, we see that

$$\frac{1}{S_m^*T_n^*}\sum_{j=a(m)+1}^{b(m)}\sum_{k=c(n)+1}^{d(n)}s_jt_k\left|x_{jk}-0\right|$$

$$=\frac{1}{(b(m)-a(m))(d(n)-c(n))}\sum_{j=a(m)+1}^{b(m)}\sum_{k=c(n)+1}^{d(n)}x_{jk}$$

$$\ge\frac{\sqrt{b(m)}\left(b(m)-\sqrt{b(m)}\right)^2}{(b(m)-a(m))}\frac{\sqrt{d(n)}\left(d(n)-\sqrt{d(n)}\right)^2}{(d(n)-c(n))}\to\infty\text{ as }n,m\to\infty.$$

Hence we have $x_{jk}\nrightarrow 0\left(\overline{D\mathcal{R}_{ab}^{cd}}\right)$. That is, double deferred Riesz statistical convergence does not require strongly deferred Riesz convergence.

Theorem 3.7. *Let $s_jt_k\left|x_{jk}-L\right|\le M$ for each $j,k\in\mathbb{N}^2$. If $x_{jk}\to L\left(D\mathcal{R}_{ab}^{cd}S^2\right)$, then $x_{jk}\to L\left(\overline{D\mathcal{R}_{ab}^{cd}}\right)$.*

Proof. Assume that $x_{jk}\to L\left(D\mathcal{R}_{ab}^{cd}S^2\right)$, $s_jt_k\left|x_{jk}-L\right|\le M$ for $j,k\in\mathbb{N}^2$ and let

$$U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon):=\{(j,k):j\le S_m^*,\,k\le T_n^*\text{ and }s_jt_k\left|x_{jk}-L\right|\ge\varepsilon\}$$

again. Then, we have

$$\frac{1}{S_m^*T_n^*}\sum_{j=a(m)+1}^{b(m)}\sum_{k=c(n)+1}^{d(n)}s_jt_k\left|x_{jk}-L\right|$$

$$=\frac{1}{S_m^*T_n^*}\sum_{\substack{j=a(m)+1\\(j,k)\in U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)}\sum_{k=c(n)+1}^{d(n)}s_jt_k\left|x_{jk}-L\right|+\frac{1}{S_m^*T_n^*}\sum_{\substack{j=a(m)+1\\(j,k)\notin U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)}\sum_{k=c(n)+1}^{d(n)}s_jt_k\left|x_{jk}-L\right|$$

$$\le\frac{M}{S_m^*T_n^*}\left|U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)\right|+\varepsilon\to\varepsilon,\text{ (for }n\to\infty).$$

Hence we obtain $x_{jk}\to L\left(\overline{D\mathcal{R}_{ab}^{cd}}\right)$. □

Now we show that if the double sequence (x_{jk}) is deferred Riesz statistically convergent, then (x_{jk}) is deferred Riesz convergent and the sequence (x_{jk}) is $\left(D\mathcal{R}_{ab}^{cd}\right)$-convergent to L.

Theorem 3.8. *Let $s_jt_k\left|x_{jk}-L\right|\le M$ for $j,k\in\mathbb{N}^2$. If $D\mathcal{R}_{ab}^{cd}S^2-\lim x_{jk}=L$, then $D\mathcal{R}_{ab}^{cd}-\lim x_{jk}=L$ and $S^2-\lim\left(D\mathcal{R}_{ab}^{cd}x\right)_{mn}=L$.*

Proof. Let $D\mathcal{R}_{ab}^{cd}S^2-\lim x_{jk}=L$. Then, we have

$$\lim_{n,m\to\infty}\frac{\left|U_{nm}^{\mathcal{R}_{ab}^{cd}}\left(\varepsilon\right)\right|}{S_m^*T_n^*}=0$$

where

$$U_{nm}^{\mathcal{R}_{ab}^{cd}}\left(\varepsilon\right):=\left\{(j,k):j\le S_m^*,\le k\le T_n^*\text{ and }s_jt_k\left|x_{jk}-L\right|\ge\varepsilon\right\}$$

for each $\varepsilon>0$. Since $s_jt_k\left|x_{jk}-L\right|\le M$ for $j,k\in\mathbb{N}^2$ and $D\mathcal{R}_{ab}^{cd}S^2-\lim x_{jk}=L$, it can be seen that $|L|\le M$. Hence, there exists $n_0\left(\varepsilon\right)\in\mathbb{N}$ such that

$$\begin{aligned}
\left|\left(D\mathcal{R}_{ab}^{cd}x\right)_{mn}-L\right| &=\left|\frac{1}{S_m^*T_n^*}\left(\sum_{\substack{j=a(m)+1\\(j,k)\in U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)}\sum_{k=c(n)+1}^{d(n)}s_jt_k+\sum_{\substack{j=a(m)+1\\(j,k)\notin U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)}\sum_{k=c(n)+1}^{d(n)}s_jt_k\right)x_{jk}-L\right|\\
&\le\frac{1}{S_m^*T_n^*}\left(\sum_{\substack{j=a(m)+1\\(j,k)\in U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)}\sum_{k=c(n)+1}^{d(n)}s_jt_k+\sum_{\substack{j=a(m)+1\\(j,k)\notin U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)}\sum_{k=c(n)+1}^{d(n)}s_jt_k\right)\left|x_{jk}-L\right|\\
&\le\frac{1}{S_m^*T_n^*}\sum_{\substack{j=a(m)+1\\(j,k)\in U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)}\sum_{k=c(n)+1}^{d(n)}M+\frac{1}{S_m^*T_n^*}\sum_{\substack{j=a(m)+1\\(j,k)\notin U_{nm}^{\mathcal{R}_{ab}^{cd}}(\varepsilon)}}^{b(m)}\sum_{k=c(n)+1}^{d(n)}\varepsilon\\
&\le M\frac{1}{S_m^*T_n^*}\left|U_{nm}^{\mathcal{R}_{ab}^{cd}}\left(\varepsilon\right)\right|+\varepsilon<2\varepsilon,\text{ (because }S_m^*,\ T_n^*\to\infty\text{ for }m,n\to\infty)
\end{aligned}$$

for $m,n>n_0\left(\varepsilon\right)$. Hence $D\mathcal{R}_{ab}^{cd}-\lim x_{jk}=L$. Additionally,

$$\frac{1}{S_M^*T_N^*}\left|V_{nm}^{\mathcal{R}_{ab}^{cd}}\left(\varepsilon\right)\right|\le\frac{n_0}{S_M}+\frac{n_0}{T_N},\ M,N>n_0$$

where

$$V_{nm}^{\mathcal{R}_{ab}^{cd}}\left(\varepsilon\right):=\left\{(j,k):j\le S_M^*,\le k\le T_N^*\text{ and }\left|\left(D\mathcal{R}_{ab}^{cd}x\right)_{mn}-L\right|\ge2\varepsilon\right\}$$

for all $\varepsilon>0$. Therefore

$$\lim_{M,N\to\infty}\frac{1}{S_M^*T_N^*}\left|\left\{(j,k):j\le S_M^*,k\le T_N^*,\text{ and }\left|\left(D\mathcal{R}_{ab}^{cd}x\right)_{mn}-L\right|\ge2\varepsilon\right\}\right|=0,$$

for all $\varepsilon>0$, i.e. $S^2-\lim_{-}\left(D\mathcal{R}_{ab}^{cd}x\right)_{mn}\to L$. □

4 Deferred Riesz statistical convergence via power series method

Let (p_{jk}) be a non-negative real double sequence with $p_{00}>0$ such that the corresponding power series

$$p\left(s,t\right)=\sum_{j,k=0}^{\infty}p_{jk}t^js^k$$

has radius of convergence $R=1$. Let

$$\sigma_x\left(s,t\right):=\frac{1}{p\left(s,t\right)}\sum_{j,k=0}^{\infty}p_{jk}s^jt^kx_{jk}\text{ for }0<s,t<1. \tag{4.1}$$

(see [4]).

Definition 4.1. [4] Let $\{x_{jk}\}$ be a sequence of real numbers. If

$$\lim_{s,t\to 1^-} \sigma_x(s,t) = \lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{j,k=0}^{\infty} p_{jk} s^j t^k x_{jk} = L,$$

for all $(s,t) \in (0,1)\times(0,1)$, then $\{x_{jk}\}$ is convergent to L with respect to J_p^2-power series method and $J_p^2 - \lim\limits_{s,t\to 1^-} x_{jk} = L$ is written. If

$$J_p^2 - \lim x_{jk} = L \text{ and } \sup_{t,s\in(0,1)} |\sigma_x(t,s)| < \infty,$$

then double sequence $x = (x_{jk})$ is called boundedly summable to $x = (x_{jk})$ by J_p^2-power series method and we write bJ_p^2- $\lim\limits_{s,t\to 1^-} x_{jk} = L$.

Under the conditions, for $0 < s,t < 1$

$$J_p^2 \text{ is regular } \Leftrightarrow \begin{array}{l} \lim\limits_{s,t\to 1^-} \frac{p_\lambda(s)}{p(s,t)} = 0 \\ \text{and } \lim\limits_{s,t\to 1^-} \frac{p_\mu(t)}{p(s,t)} = 0 \end{array} \text{ for any fixed } \lambda,\mu\in\mathbb{N} \tag{4.2}$$

where

$$p_\lambda(s) = \sum_j p_{j\lambda} s^j \text{ and } p_\mu(t) = \sum_k p_{\mu k} t^k$$

(see [4]).

Let $\mathcal{C} \subset \mathbb{N}^2$, where $\mathbb{N}^2$ denotes set of non-negative integers. If the limit

$$\delta_{J_p^2}(\mathcal{C}) = \lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{\substack{j,k=0 \\ (j,k)\in\mathcal{C}}}^{\infty} p_{jk} s^j t^k$$

exists, then $\delta_{J_p^2}(\mathcal{C})$ is called the double density of $\mathcal{C}$ with respect to J_p^2. Let $\{x_{jk}\}$ be a double sequence of real numbers. If $\delta_{J_p^2}(A(\varepsilon)) = 0$, where $A(\varepsilon) = \left\{(j,k)\in\mathbb{N}^2 : |x_{jk} - L| \geq \varepsilon\right\}$ for all $\varepsilon > 0$, then, $\{x_{jk}\}$ is said to be statistically convergent to L according to J_p^2-power series method and is denoted by $x_{jk} \to L \left(st_{J_p^2}\right)$ or $st_{J_p^2} - \lim x_{jk} = L$ [42].

We define deferred Riesz statistical convergence using J_p^2-power series method for sequences of real numbers.

Definition 4.2. A sequence $\{x_{jk}\}$ is Riesz statistically convergent via J_p^2-power series method to L if

$$\lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{(j,k)\in A_{nm}(\varepsilon)} p_{jk} s^j t^k = 0, \tag{4.3}$$

where

$$A_{nm}(\varepsilon) = \{(j,k) : j \leq S_m,\, k \leq T_n \text{ and } s_j t_k |x_{jk} - L| \geq \varepsilon\}$$

for all $\varepsilon > 0$. Then we write it as $\mathcal{R}S_{J_p^2} - \lim x_{jk} = L$.

Definition 4.3. A sequence $\{x_{jk}\}$ is deferred Riesz statistically convergent via J_p^2-power series method to L if

$$\lim_{s,t\to 1^-} \frac{1}{p(s,t)} \sum_{(j,k)\in A_{nm}(\varepsilon)} p_{jk} s^j t^k = 0, \tag{4.4}$$

where

$$B_{nm}(\varepsilon) = \{(j,k) : j \leq S_m^*,\, k \leq T_n^* \text{ and } s_j t_k |x_{jk} - L| \geq \varepsilon\}$$

for all $\varepsilon > 0$. Then we write it as $D\mathcal{R}S_{J_p^2} - \lim x_{jk} = L$.

Example 4.4. Let

$$x_{jk}=\begin{cases} 0 & , j,k \text{ is square} \\ jk & , \text{otherwise.} \end{cases} \tag{4.5}$$

$a_m=2m$, $b_m=4m$, $c_n=2n$, $d_n=4n$, $L=0$ and $s_j=t_k=1$, for all $j,k\in\mathbb{N}$. Also let

$$p_{jk}=\begin{cases} 1 & , j,k \text{ is square} \\ 0 & , \text{otherwise.} \end{cases}$$

Then, since

$$\sum_{(j,k)\in A_{nm}(\varepsilon)} p_{jk}s^jt^k=0$$

where

$$A_{nm}(\varepsilon)=\{(j,k): 2m<j\leq 4m,\, 2n<k\leq 4n \text{ and } j,k \text{ is not square}\},$$

$$\lim_{s,t\to 1^-}\frac{1}{p(s,t)}\sum_{(j,k)\in A_{nm}(\varepsilon)} p_{jk}s^jt^k=0,$$

i.e. $D\mathcal{R}S_{J_p^2}-\lim x_{jk}=0$. At the same time, $\mathcal{R}S_{J_p^2}-\lim x_{jk}=0$. From (4.5), we have

$$\lim_{n,m\to\infty}\frac{1}{mn}\left|\{(j,k): j\leq m, k\leq m \text{ and } |x_{jk}|\geq\varepsilon\}\right|\neq 0,$$

for ech $\varepsilon>0$. Hence $\mathcal{R}S^2-\lim x_{jk}\neq 0$.

Example 4.5. Define a sequence $\{x_{jk}\}$ as:

$$x_{jk}=\begin{cases} \sqrt{jk} & , j \text{ and } k \text{ is square} \\ 0 & , \text{otherwise.} \end{cases}$$

Let $s_j=t_k=1$, for all $j,k\in\mathbb{N}$. Then

$$\begin{aligned}&\lim_{n,m\to\infty}\frac{1}{S_mT_n}\left|\{(j,k): j\leq S_m^*,\, k\leq T_n^* \text{ and } s_jt_k\left|x_{jk}-0\right|\geq\varepsilon\}\right| \\ &=\lim_{n,m\to\infty}\frac{1}{mn}\left|\{(j,k): j\leq m,\, k\leq n \text{ and } \left|x_{jk}-0\right|\geq\varepsilon\}\right|=\lim_{n,m\to\infty}\frac{\sqrt{m}\sqrt{n}}{mn}=0\end{aligned}$$

Hence, $\{x_{jk}\}$ is Riesz statistically convergent to 0.

$$p_{jk}=\begin{cases} 1 & , j,k \text{ is square} \\ 0 & , \text{otherwise.} \end{cases}$$

and consider $a_m=2m$, $b_m=4m$, $c_n=2n$, $d_n=4n$ in (4.3). Then, $D\mathcal{R}S_{J_p^2}-\lim x_{jk}\neq 0$. At the same time, $\mathcal{R}S_{J_p^2}-\lim x_{jk}\neq 0$. because

$$\lim_{s,t\to 1^-}\frac{1}{p(s,t)}\sum_{(j,k)\in A_{nm}(\varepsilon)} p_{jk}s^jt^k\neq 0,$$

where $A_{nm}(\varepsilon)=\{(j,k): 2m<j\leq 4m,\, 2n<k\leq 4n \text{ and } j,k \text{ is square}\}$. Thus, $D\mathcal{R}S_{J_p^2}-\lim x_{jk}\neq 0$, but since

$$\lim_{n,m\to\infty}\frac{1}{S_mT_n}\left|\{(j,k): j\leq S_m,\, k\leq T_n \text{ and } s_jt_k\left|x_{jk}-0\right|\geq\varepsilon\}\right|=0,$$

we have $\mathcal{R}S^2-\lim x_{jk}=0$.

The examples given above show that the Riesz statistical convergence via the J_p^2-power series method $\left(D\mathcal{R}S_{J_p^2}\right)$ and the Riesz statistical convergence $\left(\mathcal{R}S^2\right)$ are incomparable.

5 Korovkin type Theorem with respect to $\mathcal{DRS}_{J_p^2}$

Korovkin's theorem, as presented by Korovkin [33], holds considerable importance in mathematics as it provides important insights into the uniform approximation of continuous functions on compact metric spaces. The theorem states that it is possible to uniformly approximate these functions through a sequence of positive linear operators.

Let $C(\mathcal{K})$ be the vector space of all real-valued continuous functions f on a compact subset $\mathcal{K}$ of real numbers. Then $C(\mathcal{K})$ is a Banach space with norm

$$\|f\| = \sup_{x\in\mathcal{K}} |f(x)|, \, f \in C(\mathcal{K}).$$

Let $\mathcal{L} : C(\mathcal{K}) \to C(\mathcal{K})$ be a linear operator. Then, we say $\mathcal{L}$ is a positive linear operator if $\mathcal{L}(f,x) \geq 0$, $\forall\, f(x) \geq 0$, where $x \in \mathcal{K}$.

Theorem 5.1. *[33] Let $(\mathcal{L}_n)$ be a sequence of positive linear operators such that $\mathcal{L}_n : C(\mathcal{K}) \to C(\mathcal{K})$. If*

$$\lim_n \|L_n(f_i) - f_i\| = 0 \text{ for } i = 0,1,2 \tag{5.1}$$

where

$$f_0(y) = 1,\ f_1(y) = y,\ f_2(y) = y^2,$$

then

$$\lim_n \|L_n(f) - f\| = 0 \text{ for } \forall f \in C(\mathcal{K}) \tag{5.2}$$

Let $\mathcal{K} \subset \mathbb{R}^2$ be a compact subset and

$$C(\mathcal{K}) := \{f \mid\ f : \mathcal{K} \to \mathbb{R} \text{ continuous}\}.$$

The space of continuous functions $C(\mathcal{K})$ is a Banach space with norm

$$\|f\| = \sup_{(x,y)\in\mathcal{K}} |f(x,y)|, \, f \in C(\mathcal{K}).$$

Let $\mathcal{L}$:$C(\mathcal{K}) \to C(\mathcal{K})$ be a linear operator. If $f \geq 0$ implies $\mathcal{L}f \geq 0$, then $\mathcal{L}$ is called a positive linear operator. In the literature, the value of $\mathcal{L}f$ for the point $(x,y) \in \mathcal{K}$ is denoted by $\mathcal{L}(f;x,y)$.

Theorem 5.2. *Let $\{\mathcal{L}_{j,k}\}$ be a sequence of positive linear operators such that $\mathcal{L}_{j,k} : C(\mathcal{K}) \to C(\mathcal{K})$. If*

$$\mathcal{DRS}_{J_p^2} - \lim_{j,k} \|\mathcal{L}_{j,k}(f_r) - f_r\| = 0, \text{ for } (r = 0,1,2,3), \tag{5.3}$$

where

$$f_0(x,y) = 1,\ f_1(x,y) = x,\ f_2(x,y) = y \text{ and } f_3(x,y) = x^2 + y^2,$$

then

$$\mathcal{DRS}_{J_p^2} - \lim_{j,k} \|\mathcal{L}_{j,k}(f) - f\| = 0,\ \forall f \in C(\mathcal{K}). \tag{5.4}$$

Proof. Let $f \in C(\mathcal{K})$. Then, since $\mathcal{K} \subset \mathbb{R}^2$ is compact and f is continuous in $\mathcal{K}$, $|f(x,y)| \leq \|f\|$ for each $(x,y) \in \mathcal{K}$. Also, again from the definition of continuity of f in $\mathcal{K}$,

$$|f(t,s) - f(x,y)| \leq \|f\| \text{ for } \forall (x,y) \in \mathcal{K} \tag{5.5}$$

and for every $\varepsilon > 0$ there exists $\exists\delta > 0$ such that

$$|f(t,s) - f(x,y)| < \varepsilon, \text{ with } |t-x| < \delta \text{ and } |s-y| < \delta. \tag{5.6}$$

From (5.5) and (5.6), we have

$$|f(t,s) - f(x,y)| \leq \varepsilon + \frac{2\|f\|}{\delta^2}\left((t-x)^2 + (s-y)^2\right). \tag{5.7}$$

Since $\mathcal{L}_{j,k}$ is a positive linear operator, we get

$$|\mathcal{L}_{j,k}(f;x,y) - f(x,y)|$$

$$\begin{aligned}
&= |\mathcal{L}_{j,k}(f(t,s) - f(x,y);x,y) - f(x,y)(\mathcal{L}_{j,k}(f_0;x,y) - f_0(x,y))| \\
&\le \mathcal{L}_{j,k}(|f(t,s) - f(x,y)|;x,y) + \|f\|\,|\mathcal{L}_{j,k}(f_0;x,y) - f_0(x,y)| \\
&\le \mathcal{L}_{j,k}\left(\varepsilon + \tfrac{2\|f\|}{\delta^2}\left((t-x)^2 + (s-y)^2\right);x,y\right) + \|f\|\,|\mathcal{L}_{j,k}(f_0;x,y) - f_0(x,y)| \\
&\le \left(\varepsilon + \|f\| + \tfrac{2\|f\|}{\delta^2}\left((\max|x|)^2 + (\max|y|)^2\right)\right)|\mathcal{L}_{j,k}(f_0;x,y) - f_0(x,y)| \\
&+ \tfrac{4\|f\|}{\delta^2}(\max|x|)\,|\mathcal{L}_{j,k}(f_1;x,y) - f_1(x,y)| + \tfrac{4\|f\|}{\delta^2}(\max|y|)\,|\mathcal{L}_{j,k}(f_2;x,y) - f_2(x,y)| \\
&+ \tfrac{2\|f\|}{\delta^2}|\mathcal{L}_{j,k}(f_3;x,y) - f_3(x,y)| + \varepsilon \\
&\le \mathcal{B}\{|\mathcal{L}_{j,k}(f_0;x,y) - f_0(x)| + |\mathcal{L}_{j,k}(f_1;x,y) - f_1(x,y)| \\
&+ |\mathcal{L}_{j,k}(f_2;x,y) - f_2(x,y)| + |\mathcal{L}_{j,k}(f_3;x,y) - f_3(x,y)|\} + \varepsilon
\end{aligned} \tag{5.8}$$

where

$$\mathcal{B} = \max\left\{\varepsilon + \|f\| + \frac{2\|f\|}{\delta^2}\left((\max|x|)^2 + (\max|y|)^2\right), \frac{4\|f\|(\max|x|)}{\delta^2}, \frac{2\|f\|}{\delta^2}\right\}.$$

For any given $\mu > 0$, there exists $\varepsilon > 0$ such that $\varepsilon < \mu$. Let's define sets

$$\mathcal{K}_{n,m} := \{(j,k) : j < S_m^*,\ k \le T_n^* \text{ and } p_j q_k\,|\mathcal{L}_{j,k}(f;x,y) - f(x,y)| \ge \mu\}$$

and for $r = 0,1,2,3$,

$$\mathcal{K}_{n,m,r} := \left\{(j,k) : j < S_m^*,\ k \le T_n^* \text{ and } p_j q_k\,|\mathcal{L}_{j,k}(f_r;x,y) - f_r(x,y)| \ge \frac{\mu-\varepsilon}{4\mathcal{B}}\right\}.$$

From (5.8), we have

$$\mathcal{K}_{n,m} \subset \bigcup_{r=0}^{3} \mathcal{K}_{n,m,r} \tag{5.9}$$

and hence we get

$$\frac{1}{p(s,t)} \sum_{j,k \in \mathcal{K}_{n,m}} p_{jk} t^j s^k \le \sum_{r=0}^{3} \frac{1}{p(s,t)} \sum_{(j,k) \in \mathcal{K}_{n,m,r}} p_{jk} t^j s^k. \tag{5.10}$$

From here, using (5.3), we obtain

$$D\mathcal{R}S_{J_p^2} - \lim_{m,n} \|\mathcal{L}_{m,n}(f) - f\| = 0, \forall f \in C(\mathcal{K}).$$

Therefore, the proof is complete. □

Corollary 5.3. *Let $\{\mathcal{L}_{j,k}\}$ be a sequence of positive linear operators such that $\mathcal{L}_{j,k} : C(\mathcal{K}) \to C(\mathcal{K})$. If*

$$\mathcal{R}S_{J_p^2} - \lim_{j,k} \|\mathcal{L}_{j,k}(f_r) - f_r\| = 0, \text{for } (r = 0,1,2,3),$$

where

$$f_0(x,y) = 1,\ f_1(x,y) = x,\ f_2(x,y) = y \text{ and } f_3(x,y) = x^2 + y^2,$$

then

$$\mathcal{R}S_{J_p^2} - \lim_{j,k} \|\mathcal{L}_{j,k}(f) - f\| = 0, \forall f \in C(\mathcal{K}).$$

Remark 5.4. 3.1 Let us consider the Bernstein operators

$$B_{m,n}(f;x,y) = \sum_{j=0}^{m} \sum_{k=0}^{n} f\left(\frac{j}{m}, \frac{k}{n}\right) \binom{m}{j}\binom{n}{k} x^j (1-x)^{m-j} y^k (1-y)^{n-k},$$

where $(x,y) \in D = [0,1] \times [0,1]$ and $f \in C(\mathcal{K})$. Let us take the sequence of positive linear operators defined on $C(\mathcal{K})$ as follows:

$$p_{mn} = \begin{cases} 1 & , m,n \text{ square} \\ 0 & , \text{otherwise} \end{cases}$$

and

$$\mathcal{L}_{m,n}(f;x,y) = (1+p_{mn})\, B_{m,n}(f;x,y)\,. \tag{5.11}$$

Since

$$\begin{aligned} B_{m,n}(f_0;x,y) &= 1 \\ B_{m,n}(f_1;x,y) &= x \\ B_{m,n}(f_2;x,y) &= y \\ B_{m,n}(f_3;x,y) &= x^2+y^2+\tfrac{x-x^2}{m}+\tfrac{y-y^2}{n} \end{aligned}$$

from, we have

$$\begin{aligned} \mathcal{L}_{m,n}(f_0;x,y) &= (1+p_{mn})\,, \\ \mathcal{L}_{m,n}(f_1;x,y) &= (1+p_{mn})\,x, \\ \mathcal{L}_{m,n}(f_2;x,y) &= (1+p_{mn})\,y, \\ \mathcal{L}_{m,n}(f_3;x,y) &= (1+p_{mn})\left(x^2+y^2+\tfrac{x-x^2}{m}+\tfrac{y-y^2}{n}\right). \end{aligned} \tag{5.12}$$

Since $DRS_{J_p^2} - \lim p_{mn} = 0$, we get

$$D\mathcal{R}S_{J_p^2} - \underline{\lim}\, \|\mathcal{L}_{m,n}(f_i) - f_i\| = 0 \text{ for } \forall i = 0,1,2,3$$

Hence, from Theorem 5.2,

$$D\mathcal{R}S_{J_p^2} - \underline{\lim}\, \|\mathcal{L}_{m,n}(f) - f\| = 0 \text{ for } \forall f \in C(\mathcal{K})\,.$$

Since the sequence $\{p_{mn}\}$ does not approach zero, $\mathcal{L}_{m,n}$ cannot satisfy Theorem 3.1. This demonstrates that Theorem 5.2 is a non-trivial generalization of Theorem 5.1.

6 Conclusion

In this paper, we have defined the statistical $\left(\overline{N}, s_m, t_n\right)$ convergence, strongly Riesz convergence, Riesz statistical convergence for double real number sequences. We compared the concepts of deferred Riesz convergence, strongly deferred Riesz convergence, deferred Riesz statistical convergence for double sequences as related to the J_p^2−power series method. We applied this new method of summability to prove a Krovkin's type approximation theorem.

Declarations

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