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Sequence of Repunit numbers with bicomplex and biquaternion coefficients

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Abstract This study explores a mathematical development of the repunit sequence, consisting of numbers exclusively formed by the digit 1 in a specific numerical base. In this context, we introduce two new classes of sequences: the bicomplex repunit numbers and the biquaternion repunit numbers. These sequences were developed based on the sets of complex numbers, bicomplex numbers, quaternions, and biquaternions. Accordingly, we discuss their respective recurrence relations and associated properties, such as the Binet formula, generating function, and the Taiguri-Vajda, Catalan, Cassini, and d'Ocagne identities. Additionally, we present the partial sums of the terms of these sequences.

1 Introduction

The repunit numbers $\{R_n\}_{n\geq 0}$ are the terms of the sequence $\{0, 1, 11, 111, 1111, 1111, \ldots\}$ where each term satisfies the non-homogeneous recursive formula $R_{n+1} = 10R_n + 1$, with $R_0 = 0$. In [22] the repunit sequence $\{R_n\}_{n\geq 0}$ is defined recursively by a second-order homogeneous recurrence, namely,

$$R_0 = 0\ , R_1 = 1 \text{ and } R_{n+1} = 11R_n - 10R_{n-1}\ ,$$

where R_n denotes the n-th repunit number.

Bicomplex sequences represent a generalization of numerical sequences, being defined in the context of bicomplex numbers. These numbers constitute a natural extension of complex numbers, with additional properties that make them relevant in various fields of mathematics and physics, as explored in the research works [1, 13, 15, 17, 29]. Similarly, biquaternion sequences are defined in the set of biquaternionic numbers, an expansion of quaternions that incorporates complex numbers into their structure. Biquaternionic numbers have applications in linear algebra and geometric transformations, as highlighted in recent studies [2, 20, 27].

In the literature, there are several works that address the representation of Lucas-type sequences through bicomplex, such as [6, 8, 17, 19, 26, 28, 29]. As presented by these researchers, we can recursively define a sequence through bicomplex using Equation (1.1), for all $n \geq 0$:

$$BL_n = L_n + L_{n+1}\mathbf{i_1} + L_{n+2}\mathbf{i_2} + L_{n+3}\mathbf{i_3}\ , \tag{1.1}$$

where L_0, L_1, L_2, and L_3 are the initial terms of the Lucas sequence.

Here we consider the recurrence

$$R^*_{n+1} = 11R^*_n - 10R^*_{n-1}$$

where R^*_n is a bicomplex number or biquaternion number of order n.

In this work, we will present the repunit sequence with bicomplex and biquaternion coefficients. To begin, we will provide a background on complex, bicomplex, quaternion, biquaternion numbers, and repunit numbers, aiming to explore essential definitions and properties. Next, we will introduce two new classes of sequences. Subsequently, we will discuss bicomplex and biquaternion repunit numbers, exploring their recurrence relations, properties, and some identities, such as the Taiguri-Vajda, Catalan, Cassini, and d'Ocagne identities.

2 Background and preliminaries results

In this section we discuss some preliminary results as the complex set, the bicomplex set, the quaternion set, the biquaternion set, and the definition and essential result on repunit numbers.

2.1 Complex, bicomplex, quaternion and biquaternion numbers

Consider the field of complex numbers, denoted by $(\mathbb{C}, +, \cdot)$. The set of complex numbers is defined as $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$, where i is a complex unit. We know that $(\mathbb{C}, +)$ and $(\mathbb{C}, \cdot)$ are abelian groups, and then, the conjugate of a complex number $x = a + bi$ is defined by $\overline{x} = a - bi$. A gaussian number is a complex number $z = a + bi$, where a and b are integers, (see [7, 8, 24]). For all a, b, c, d integer numbers, the following arithmetic operations in the gaussian set hold:

$$(a + bi)^2 = (a^2 - b^2) + 2abi ,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i .$$

The bicomplex numbers, denoted by $(\mathbb{B}, +, \cdot)$, are defined by four base elements $1, i_1, i_2, i_3$ where i_1, i_2, i_3 satisfy the following properties:

$$i_1^2 = i_2^2 = -1 \text{ and } i_3 = i_1 i_2 = i_2 i_1 .$$

A bicomplex number can be expressed in the following form:

$$a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 = (a_0 + a_1 i_1) + (a_2 + a_3 i_1) i_2 .$$

For more details about the bicomplex numbers, one can see in [1, 2, 6, 15, 17].

The set of quaternion (Hamiltonian) numbers, denoted by $\mathbb{H}$, is defined as

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \text{ with } i^2 = j^2 = k^2 = ijk = -1\}.$$

According to [3, 7, 11, 14, 16, 24, 25] , $(\mathbb{H}, +, \cdot)$ forms a vector space with a base $1, i, j, k$, which is composed of unit 1 and its imaginary units i, j and k. The addition of two quaternion numbers is defined by summing their components. So, the addition operation in the quaternion numbers is both commutative and associative. Zero is the null element. Concerning the addition operation, the symmetric element of x is $-x$, which is defined as having all the components of x changed in their signals. This implies that, $(\mathbb{H}, +)$ is an abelian group. The conjugate of a quaternion number $x = a + bi + cj + dk$ is defined by $\overline{x} = a - bi - cj - dk$. When the real component is equal to zero, the quaternion is called pure. Quaternion multiplication follows the usual algebraic multiplication rules from the definition of quaternion numbers; the multiplication table of the quaternion units is given by Table 1:

Table 1. The multiplication table for quaternion units

$\bullet$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

The set of biquaternions $\mathbb{Q}$ extends the set of quaternions by incorporating the complex numbers. They are often referred to as complex quaternions or bicomplex quaternions. A biquaternion is typically written as $q = a + be$ where a and b are quaternions, and $e^2 = -1$ is the imaginary unit from complex numbers. If a and b are given by:

$$a = a_0 + a_1 i + a_2 j + a_3 k$$

$$b = b_0 + b_1 i + b_2 j + b_3 k$$

then a biquaternion $q \in \mathbb{Q}$ can be expressed as:

$$q = a + be = (a_0 + a_1 i + a_2 j + a_3 k) + (b_0 + b_1 i + b_2 j + b_3 k)e$$

So, the biquaternion algebra $\mathbb{Q}$ is a two dimensional vector space over the algebra $\mathbb{H}$ of quaternion numbers with its basis $1,\ e$ satisfying the multiplication rules given by Table 2.

$\bullet$	1	i	j	k	e	ei	ej	ek
1	1	i	j	k	e	ei	ej	ek
i	i	-1	k	$-j$	ei	$-e$	ek	$-ej$
j	j	$-k$	-1	i	ej	$-ek$	$-e$	ei
k	k	j	$-i$	-1	ek	ej	$-ei$	$-e$
e	e	ei	ej	ek	-1	$-i$	$-j$	$-k$
ei	ei	$-e$	$-ek$	ej	i	-1	k	$-j$
ej	ej	ek	$-e$	$-ei$	j	$-k$	-1	i
ek	ek	$-ej$	ei	$-e$	k	j	$-i$	-1

Table 2. The multiplication table for biquaternion units

In this case, any element in $\mathbb{Q}$ can be represented as $q = h_0 + h_1 e$, where $h_0, h_1 \in \mathbb{H}$. From this definition, the real numbers, the complex numbers, and the real quaternions, all can be regarded as the special cases of biquaternions. It should be pointed out that $\mathbb{Q}$ is not a division algebra over $\mathbb{C}$, namely, there exist non zero $a, b \in \mathbb{Q}$ such that $ab = 0$.

2.2 The repunit numbers

The recurrence of the repunit sequence, in its homogeneous form, is given by: $R_{n+1} = 11R_n - 10R_{n-1}$, with initial conditions $R_0 = 0$, $R_1 = 1$, and for $n = 1, 2, \ldots$. Such sequence is the sequence A002275 of the on-line encyclopedia of integer sequences (see OEIS [23]). This is a sequence of the type $H_{n+2} = pH_{n+1} + qH_n$, with initial terms $H_0 = a$ and $H_1 = b$. This sequence was introduced, in 1965, by Horadam [9], and it generalizes many sequences with characteristic equation of recurrence relation of form $x^2 - px - q = 0$, a recurrence equation of order 2, (see also [10, 18]). For more details, see the references [21, 22]. Some works explore the connections of this sequence with the classic Lucas sequence, a sequence Horadam-type, among which we can highlight [12]. Consider $p = 11$ and $q = -10$ fixed, and the recurrence

$$R_{n+1} = 11R_n - 10R_{n-1}\,, \tag{2.1}$$

where R_n denotes the n-th repunit number. The explicit formula for the $n-$th repunit number is given by

$$R_n = \frac{10^n - 1}{9}\,, \tag{2.2}$$

(see [22] and references therein).

In addition, our next auxiliary result presents a generating function for the repunit numbers and can be accessed at [4, Proposition 1].

Lemma 2.1. *The generating function for the repunit numbers $\{R_n\}_{n\geq 0}$ denoted by $G_{R_n}(x)$, is given by*

$$G_{R_n}(x) = \frac{x}{1 - 11x + 10x^2}. \tag{2.3}$$

Next, in Figure 1, the development of the generating function for repunit numbers is presented, carried out using the Maxima software.

Figure 1. Development of the generating function for repunit numbers (Prepared by the authors).

3 Bicomplex repunit sequence

We consider the field of bicomplex numbers, denoted by $(\mathbb{B}, +, \cdot)$. In this field, let $a,\ b,\ c,\ d, e,$ and f be real numbers. We define $X = (a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{i_3})$ and $Y = (e + f\mathbf{i_1} + g\mathbf{i_2} + h\mathbf{i_3})$. When working with bicomplex numbers, we will use the following facts:

$$X^2 = (a^2 - b^2 - c^2 + d^2) + 2(ab - cd)\mathbf{i_1} + 2(ac - bd)\mathbf{i_2} + 2(ad + bc)\mathbf{i_3}\ ,$$
$$XY = (ad - be - cf) + (ae + bd)\mathbf{i_1} + (af + cd)\mathbf{i_2} + (bf + ce)\mathbf{i_3}\ .$$

By substituting $a = e = f = g = h = 1,\ b = 10,\ c = 100$, and $d = 1000$ into the preceding relations, we obtain

Lemma 3.1. *Let* $X = 1+10\mathbf{i_1}+100\mathbf{i_2}+1000\mathbf{i_3}$ *and* $Y = 1+\mathbf{i_1}+\mathbf{i_2}+\mathbf{i_3}$ *be bicomplex numbers. Then, we have:*

(a) $X^2 = 989901 - 199980\mathbf{i_1} - 19800\mathbf{i_2} + 2000\mathbf{i_3}$,

(b) $XY = 891 - 1089\mathbf{i_1} - 909\mathbf{i_2} + 1111\mathbf{i_3}$.

Definition 3.2. For all integers $n \geq 0$, the set of bicomplex repunit sequence is defined as $\{BR_n\}_{n\geq 0}$ and the $n+2$-th term is given by

$$BR_{n+2} = R_n + R_{n+1}\mathbf{i_1} + R_n\mathbf{i_2} + R_{n+3}\mathbf{i_3}\ , \tag{3.1}$$

where R_j is the j-th repunit number.

The bicomplex $\{BR_n\}_{n\geq 0}$ is the bicomplex repunit sequence that satisfies the Equation (3.1), and the first two terms are $BR_0 = \mathbf{i_1} + 11\mathbf{i_2} + 111\mathbf{i_3}$ and $BR_1 = 1 + 11\mathbf{i_1} + 111\mathbf{i_2} + 1111\mathbf{i_3}$.

As a consequence of the recurrence relation given by Equation (2.1) we have:

Proposition 3.3. *The sequence* $\{BR_n\}_{n\geq 0}$ *of bicomplex repunit numbers satisfies the following second order recursive relation:*

$$BR_{n+2} = 11BR_{n+1} - 10BR_n\ , \tag{3.2}$$

with initial values $BR_0 = \mathbf{i_1} + 11\mathbf{i_2} + 111\mathbf{i_3}$ *and* $BR_1 = 1 + 11\mathbf{i_1} + 111\mathbf{i_2} + 1111\mathbf{i_3}$.

Note that the recurrence (3.2) has the same characteristic equation given by

$$r^2 - 11r + 10 = 0\ , \tag{3.3}$$

whose distinct roots are $r_1 = 10$ and $r_2 = 1$. Then, for $n \geq 0$, the expression

$$x_n = c_1(x_1)^n + c_2(x_2)^n ,$$

with c_1, c_2 bicomplex numbers that are solutions of Equation (3.3). Let us determine the bicomplex constants c_1 and c_2, considering that $BR_0 = \mathbf{i_1} + 11\mathbf{i_2} + 111\mathbf{i_3}$ and $BR_1 = 1 + 11\mathbf{i_1} + 111\mathbf{i_2} + 1111\mathbf{i_3}$, and we obtain the linear system,

$$\begin{cases} \mathbf{i_1} + 11\mathbf{i_2} + 111\mathbf{i_3} = c_1 + c_2 \\ 1 + 11\mathbf{i_1} + 111\mathbf{i_2} + 1111\mathbf{i_3} = 10c_1 + c_2 \end{cases} .$$

Solving the linear system we find $c_1 = \frac{1+10\mathbf{i_1}+100\mathbf{i_2}+1000\mathbf{i_3}}{9}$ and $c_2 = -\frac{1+\mathbf{i_1}+\mathbf{i_2}+\mathbf{i_3}}{9}$. So we have just shown the Binet formula for the bicomplex repunit sequence $\{BR_n\}_{n\geq 0}$.

Proposition 3.4 (Binet's formula). *For all non-negative integers n, we have*

$$BR_n = \frac{10^n - 1}{9} + \frac{10^{n+1} - 1}{9}\mathbf{i_1} + \frac{10^{n+2} - 1}{9}\mathbf{i_2} + \frac{10^{n+3} - 1}{9}\mathbf{i_3} , \tag{3.4}$$

where BR_n is the n-th bicomplex repunit number.

Proof. We have that a general solution to Equation (3.2) is of the form $BR_n = c_1(10)^n + c_2(1)^n$. So

$$\begin{aligned} BR_n =& c_1(10)^n + c_2(1)^n \\ =& \frac{1 + 10\mathbf{i_1} + 100\mathbf{i_2} + 1000\mathbf{i_3}}{9}(10)^n - \frac{1 + \mathbf{i_1} + \mathbf{i_2} + \mathbf{i_3}}{9}(1)^n \\ =& \frac{10^n - 1}{9} + \frac{10^{n+1} - 1}{9}\mathbf{i_1} + \frac{10^{n+2} - 1}{9}\mathbf{i_2} + \frac{10^{n+3} - 1}{9}\mathbf{i_3} , \end{aligned}$$

which completes the proof □

Now consider the sequence of the partial sums, denoted by $BS_n = BR_0 + BR_1 + BR_2 + \cdots + BR_n$, where $n \geq 0$. Here, $\{BR_n\}_{n\geq 0}$ represents the bicomplex repunit sequence.

Proposition 3.5. *Let $\{BR_n\}_{n\geq 0}$ be the bicomplex repunit sequence. For all non negative integers n, we have*

$$BS_n = \frac{10^{n+1} - 10 - 9n}{81} + \frac{10^{n+2} - 19 - 9n}{81}\mathbf{i_1} + \frac{10^{n+3} - 190 - 9n}{81}\mathbf{i_2} + \frac{10^{n+4} - 1981 - 9n}{81}\mathbf{i_3} .$$

Proof. We have that

$$\begin{aligned} BS_n =& BR_0 + BR_1 + BR_2 + \cdots + BR_n \\ =& (R_1 + R_2 + \cdots + R_n) + (R_1 + R_2 + \cdots + R_{n+1})\mathbf{i_1} \\ &+ (R_1 + R_1 + R_2 + R_3 + \cdots + R_{n+2})\mathbf{i_2} + (R_3 + \cdots + R_{n+2})\mathbf{i_3} . \end{aligned}$$

Now, it follows from Proposition 6.1 in [21] that

$$\begin{aligned} BS_n =& \frac{10(10^n - 1) - 9n}{81} + \frac{10(10^{n+1} - 1) - 9(n+1)}{81}\mathbf{i_1} + \frac{10^{n+3} - 109 - 9n}{81}\mathbf{i_2} \\ &+ \frac{10^{n+4} - 1981 - 9n}{81}\mathbf{i_3} , \end{aligned}$$

then, we obtain the result . □

Using the Binet formula, we obtain the Tagiuri-Vajda identity for the bicomplex repunit below.

Theorem 3.6. *[Tagiuri-Vajda's identity] Let m, n, k be any natural number, we have*

$$BR_{m+n}BR_{m+k} - BR_m BR_{m+n+k} = 10^m(891 - 1089\mathbf{i_1} - 909\mathbf{i_2} + 1111\mathbf{i_3})R_k R_n ,$$

where BR_n is the n-th bicomplex repunit number, and R_n is the n-th repunit number.

Proof. According to Proposition 3.4, and considering $X = (1 + 10\mathbf{i_1} + 100\mathbf{i_2} + 1000\mathbf{i_3})$ and $Y = (1 + \mathbf{i_1} + \mathbf{i_2} + \mathbf{i_3})$, we obtain that

$$\begin{aligned} BR_{m+n}BR_{m+k} &= \left(\frac{X10^{m+n} - Y}{9}\right)\left(\frac{X10^{m+k} - Y}{9}\right) \\ &= \frac{X^2 10^{2m+n+k} - XY10^{m+n} - XY10^{m+k} + Y^2}{81} , \end{aligned}$$

and, on the other hand,

$$\begin{aligned} BR_m BR_{m+n+k} &= \left(\frac{X10^{m} - Y}{9}\right)\left(\frac{X10^{m+n+k} - Y}{9}\right) \\ &= \frac{X^2 10^{2m+n+k} - XY10^{m} - XY10^{m+n+k} + Y^2}{81} . \end{aligned}$$

Hence, Binet's formula for the repunits numbers (Equation (2.2)), and once again, Proposition 3.4 lead to

$$\begin{aligned} BR_{m+n}BR_{m+k} - BR_m BR_{m+n+k} &= XY\frac{10^{m+n+k} - 10^{m+k} - 10^{m+n} + 10^m}{81} \\ &= XY\frac{(10^{m+k} - 10^m)(10^n - 1)}{81} \\ &= XY10^m\left(\frac{10^k - 1}{9}\right)\left(\frac{10^n - 1}{9}\right) . \end{aligned}$$

To establish this result, we use Lemma 3.1. □

The following identities arise as a direct consequence of the Tagiuri-Vajda Identity, as established in Theorem 3.6. The subsequent results will present the detailed derivation.

Proposition 3.7 (Catalan's identity). *Let m, n be any natural number. For $m \geq n$ we have*

$$(BR_m)^2 - BR_{m-n}BR_{m+n} = 10^{m-n}(891 - 1089\mathbf{i_1} - 909\mathbf{i_2} + 1111\mathbf{i_3}) \cdot (BR_n)^2 ,$$

where BR_n is the n-th bicomplex repunit number.

Proposition 3.8 (Cassini's identity). *For all $m \geq 1$, we have*

$$(BR_m)^2 - BR_{m-1}BR_{m+1} = 10^{m-1}(891 - 1089\mathbf{i_1} - 909\mathbf{i_2} + 1111\mathbf{i_3}) ,$$

where BR_n is the n-th bicomplex repunit number.

Proposition 3.9 (d'Ocagne's identity). *Let m, n be any natural number. For $m \geq n$ we have*

$$BR_{m+1}BR_n - BR_m BR_{n+1} = 10^m(891 - 1089\mathbf{i_1} - 909\mathbf{i_2} + 1111\mathbf{i_3})BR_{m-n} ,$$

where BR_n is the n-th bicomplex repunit number.

Next, we will present the generating function for the bicomplex repunit sequence.

Proposition 3.10. *The generating function for the bicomplex repunit numbers is given by*

$$G_{BR_n}(x) = \frac{x}{1 - 11x + 10x^2} + \frac{1}{1 - 11x + 10x^2}\mathbf{i_1} + \frac{11 - 10x}{1 - 11x + 10x^2}\mathbf{i_2} + \frac{111 - 110x}{1 - 11x + 10x^2}\mathbf{i_3}$$

Proof. Consider the generating function for the sequence of bicomplex repunit numbers given by $G_{BR_n}(x) = \sum_{n=0}^{\infty} B_n x^n$. Combining expressions $-11xG_{BR_n}(x)$ e $10x^2 G_{BR_n}(x)$, we obtain

$$\begin{aligned} G_{BR_n}(x) =& \frac{GR_0 + (BR_1 - 11BR_0)x}{1 - 11x + 10x^2} \\ =& \frac{x}{1 - 11x + 10x^2} + \frac{1}{1 - 11x + 10x^2}\mathbf{i_1} + \frac{11 - 10x}{1 - 11x + 10x^2}\mathbf{i_2} + \frac{111 - 110x}{1 - 11x + 10x^2}\mathbf{i_3} , \end{aligned}$$

which concludes the proof. □

4 Biquaternion repunit sequence

Similar to the previous section, we now introduce biquaternion repunit numbers and explore some of their properties. We present the Taiguri-Vajda identity and, as a consequence, the classical identities of Catalan, Cassini, and d'Ocagne. Furthermore, we investigate partial sums of the terms sequence . Complementing the work in [4], we prove two results about quaternion repunit numbers, which are crucial for the proofs of the main results in this section.

Consider the following definition of the biquaternion repunit sequence.

Definition 4.1. For all integers $n \geq 0$, the set of biquaternion repunit numbers is denoted by $\{QR_n\}_{n\geq 0}$ and the n-th term of this set is defined by

$$QR_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k + (R_{n+4} + R_{n+5}i + R_{n+6}j + R_{n+7}k)e \,, \tag{4.1}$$

with initial conditions

$$QR_0 = R_0 + R_1 i + R_2 j + R_3 k + (R_4 + R_5 i + R_6 j + R_7 k)e$$

and

$$QR_1 = R_1 + R_2 i + R_3 j + R_4 k + (R_5 + R_6 i + R_7 j + R_8 k)e \,,$$

where R_n is the n-th repunit number.

According [4] the quaternion repunit is $HR_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k$, so we can rewrite the Equation (4.1) in the form

$$QR_n = HR_n + HR_{n+4}e \,. \tag{4.2}$$

The following proposition establishes the same recurrence relation for the biquaternion repunit numbers.

Proposition 4.2. *For all $n \geq 0$, the biquaternion repunit sequence $\{QR_n\}_{n\geq 0}$ satisfies the recurrence relation*

$$QR_{n+2} = 11QR_{n+1} - 10QR_n \tag{4.3}$$

Proof. Making use from the Equation (4.2), we have

$$\begin{aligned} &11QR_{n+1} - 10QR_n \\ =&11(HR_{n+1} + HR_{n+5}e) - 10(HR_n + HR_{n+4}e) \\ =&HR_{n+2} + HR_{n+6}e \;=\; QR_{n+2} \,, \end{aligned}$$

since $HR_{n+2} = 11HR_{n+1} - 10HR_n$, see [4, Proposition 8]. □

The next auxiliary result presents the Binet formula for quaternions repunit numbers.

Lemma 4.3. *[4, Proposition 12] For all non-negative integers n, we have*

$$HR_n = \frac{10^n - 1}{9} + \frac{10^{n+1} - 1}{9}i + \frac{10^{n+2} - 1}{9}j + \frac{10^{n+3} - 1}{9}k \,, \tag{4.4}$$

where HR_n is the n-th term of the quaternion repunit sequence.

The next result provides the Binet formula for biquaternions repunit numbers.

Proposition 4.4 (*Binet's Formula*). *For all non-negative integers n, we have*

$$\begin{aligned} QR_n &= \frac{10^n-1}{9} + \frac{10^{n+1}-1}{9}i + \frac{10^{n+2}-1}{9}j + \frac{10^{n+3}-1}{9}k \\ &\quad + \frac{10^{n+4}-1}{9}e + \frac{10^{n+5}-1}{9}ie + \frac{10^{n+6}-1}{9}je + \frac{10^{n+7}-1}{9}ke \, . \end{aligned} \tag{4.5}$$

Proof. By combining Lemma 4.3 with Equation (4.2), we obtain the result. □

4.1 Some sums

In this section, we present the results of our investigation of partial sums of the terms of the biquaternion repunit numbers with a variable integer number of terms. We consider the sequence of partial sums, defined as the sum of terms of the biquaternion repunit sequence, for a given integer value of n

$$\sum_{k=0}^{n} QR_k = QR_0 + QR_1 + QR_2 + QR_3 + \ldots + QR_n \ ,$$

for $n \geq 0$, and $\{QR_n\}_{n\geq 0}$ is the biquaternion repunit sequence.

According to Equation (4.2), the partial sums of the biquaternion repunit sequence can be expressed in terms of the partial sums of the quaternion repunits. In this sense, before exhibiting the partial sums of $\{QR_n\}_{n\geq 0}$, we need to demonstrate the following result.

Proposition 4.5. *Let $\{HR_n\}_{n\geq 0}$ be the quaternion repunit sequence, we have the following identities:*

(a) $$\sum_{k=0}^{n} HR_k = \frac{10R_n - n}{9} + \frac{10R_{n+1}-(n+1)}{9}i + \frac{10R_{n+2}-(n+11)}{9}j + \frac{10R_{n+3}-(n+111)}{9}k$$

(b)

$$\begin{aligned} \sum_{k=0}^{n} HR_{2k} =& \frac{10^2R_{2n} - 11n}{99} + \frac{R_{2n+3}-11(n+1)-1}{99}i + \frac{10^2R_{2n+2}-11(n+1)}{99}j \\ &+ \frac{R_{2n+5}-11(n+11)-1}{99}k \end{aligned}$$

(c)

$$\begin{aligned} \sum_{k=0}^{n} HR_{2k+1} =& \frac{R_{2n+3}-11(n+1)-1}{99} + \frac{10^2R_{2n+2}-11(n+1)}{99}i + \frac{R_{2n+5}-11(n+11)-1}{99}j \\ &+ \frac{10^2R_{2n+4}-11(n+101)}{99}k \ , \end{aligned}$$

where R_n is the n-th repunit number.

Proof.

(a) It suffices to consult Proposition 13 in [4].

(b) Note that

$$\begin{aligned} \sum_{k=0}^{n} HR_{2k} =& HR_0 + HR_2 + HR_4 + \ldots + HR_{2n} \\ =& (R_0 + R_1 i + R_2 j + R_3 k) + \ldots + (R_{2n} + R_{2n+1} i + R_{2n+2} j + R_{2n+3} k) \\ =& (R_0 + R_2 + \ldots + R_{2n}) + (R_1 + \ldots + R_{2n+1})i + (R_2 + \ldots + R_{2n+2})j \\ &+ (R_1 + R_3 + R_5 + \ldots + R_{2n+3} - R_1)k \\ =& \sum_{k=0}^{n} R_{2k} + \left(\sum_{k=0}^{n} R_{2k+1}\right) i + \left(\sum_{k=0}^{n+1} R_{2k}\right) j + \left(\sum_{k=0}^{n+1} R_{2k+1} - R_1\right) k \ . \end{aligned}$$

The result is a consequence of Proposition 11 in [5].

(c) The proof exhibits similarities to that of item (b) and then we omit the proof. □

It follows from the previous proposition that:

Proposition 4.6. *For all integers $n \geq 0$, we have the following identities:*

(a) $$\sum_{k=0}^{n}(-1)^k HR_k = -\frac{10^{2n+2}-1}{99} - \frac{10^{2n+3}-10}{99}i - \frac{10^{2n+4}-100}{99}j - \frac{10^{2n+5}-1000}{99}k\,,$$
if n is odd, and

(b) $$\sum_{k=0}^{n}(-1)^k HR_k = \frac{10^{2n+3}-10}{99} + \frac{10^{2n+4}-1}{99}i + \frac{10^{2n+5}+89}{99}j + \frac{10^{2n+6}+989}{99}k\,,$$
if n is even, where $\{R_n\}_{n\geq 0}$ is the repunit sequence, and $\{HR_n\}_{n\geq 0}$ is the quaternion repunit sequence.

Proof.

(a) If n is odd, so the last term is negative. Then

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k HR_k =& HR_0 - HR_1 + HR_2 - HR_3 + \cdots + HR_{2n} - HR_{2n+1}\\ =&(HR_0 + HR_2 + \cdots + HR_{2n}) - (HR_1 + HR_3 + \cdots + HR_{2n+1})\\ =&\sum_{k=0}^{n} HR_{2k} - \sum_{k=0}^{n} HR_{2k+1}\,.\end{aligned}$$

According to Proposition 4.5, and items (b) and (c), it follows that:

$$-\frac{10^{2n+2}-1}{99} - \frac{10^{2n+3}-10}{99}i - \frac{10^{2n+4}-100}{99}j - \frac{10^{2n+5}-1000}{99}k\,.$$

(b) In case that the last term is positive, we have

$$\begin{aligned}\sum_{k=0}^{2(n+1)}(-1)^k HR_k =& HR_0 - HR_1 + HR_2 - HR_3 + \cdots + HR_{2n} - HR_{2n+1}\\ =&\sum_{k=0}^{n+1} HR_{2k} - \sum_{k=0}^{n} HR_{2k+1}\,.\end{aligned}$$

As in item (a), the result is established by applying Proposition 4.5, items (b) and (c). □

Now, we present some partial sums of terms of the biquaternion repunit numbers, which constitute the main results of this section.

Proposition 4.7. *Let $\{QR_n\}_{n\geq 0}$ be the biquaternion repunit sequence, we have the following identities:*

(a)

$$\begin{aligned}&\sum_{k=0}^{n} QR_k\\ =&\frac{10R_n - n}{9} + \frac{10R_{n+1}-(n+1)}{9}i + \frac{10R_{n+2}-(n+11)}{9}j + \frac{10R_{n+3}-(n+111)}{9}k\\ &+\frac{10R_{n+4}-(n+1111)}{9}e + \frac{10R_{n+5}-(n+11111)}{9}ie + \frac{10R_{n+6}-(n+111111)}{9}je\\ &+\frac{10R_{n+7}-(n+1111111)}{9}ke\,.\end{aligned}$$

(b)

$$\begin{aligned}&\sum_{k=0}^{n} QR_{2k}\\ =&\frac{10^2R_{2n}-11n}{99}+\frac{R_{2n+3}-11(n+1)-1}{99}i+\frac{10^2R_{2n+2}-11(n+1)}{99}j\\ &+\frac{R_{2n+5}-11(n+11)-1}{99}k+\frac{10^2R_{2n+4}-11(n+101)}{99}e+\frac{R_{2n+7}-11(n+1011)-1}{99}ie\\ &+\frac{10^2R_{2n+6}-11(n+10101)}{99}je+\frac{R_{2n+9}-11(n+101011)-1}{99}ke\end{aligned}$$

(c)

$$\begin{aligned}&\sum_{k=0}^{n} QR_{2k+1}\\ =&\frac{R_{2n+3}-11(n+1)-1}{99}+\frac{10^2R_{2n+2}-11(n+1)}{99}i+\frac{R_{2n+5}-11(n+11)-1}{99}j\\ &+\frac{10^2R_{2n+4}-11(n+101)}{99}k+\frac{R_{2n+7}-11(n+1011)-1}{99}e+\frac{10^2R_{2n+6}-11(n+10101)}{99}ie\\ &+\frac{R_{2n+9}-11(n+101011)-1}{99}je+\frac{10^2R_{2n+8}-11(n+1010101)}{99}ke\ ,\end{aligned}$$

where R_n is the n-th repunit number.

Proof. *(a)* Follows from the definition of partial sums of terms of the biquaternion repunit sequence that we have

$$\begin{aligned}\sum_{k=0}^{n} QR_k =&QR_0+QR_1+QR_2+\cdots+QR_n\\ =&(HR_0+HR_4e)+(HR_1+HR_5e)+\cdots+(HR_n+HR_{n+4}e)\\ =&(HR_1+HR_2+\cdots+HR_n)+(HR_4+HR_5+\cdots+HR_{n+4})e\\ =&(HR_1+HR_2+\cdots+HR_n)+(HR_1+HR_2+\cdots+HR_n)e\\ =&(HR_0+\cdots+HR_n)+(HR_0+\cdots+HR_{n+4}-HR_0-HR_1-HR_2-HR_3)e\\ =&\left(\sum_{k=0}^{n} HR_k\right)+\left(\sum_{k=0}^{n+4} HR_k-\sum_{k=0}^{3} HR_k\right)e\\ =&\frac{10R_n-n}{9}+\frac{10R_{n+1}-(n+1)}{9}i+\frac{10R_{n+2}-(n+11)}{9}j+\frac{10R_{n+3}-(n+109)}{9}k\\ &+\frac{10R_{n+4}-(n+1111)}{9}+\frac{10R_{n+5}-(n+11111)}{9}ie+\frac{10R_{n+6}-(n+111111)}{9}je\\ &+\frac{10R_{n+7}-(n+1111111)}{9}ke\end{aligned}$$

(b) In a similar way we have

$$\begin{aligned}
&\sum_{k=0}^{n} QR_{2k} = QR_0 + QR_2 + QR_4 + QR_6 + \ldots + QR_{2n} \\
=&(HR_0 + HR_4 e) + (HR_2 + HR_6 e) + \ldots + (HR_{2n} + HR_{2n+4} e) \\
=&\frac{10^2 R_{2n} - 11n}{99} + \frac{R_{2n+3} - 11(n+1) - 1}{99} i + \frac{10^2 R_{2n+2} - 11(n+1)}{99} j \\
&+ \frac{R_{2n+5} - 11(n+11) - 1}{99} k + \frac{10^2 R_{2n+4} - 11(n+101)}{99} e + \frac{R_{2n+7} - 11(n+1011) - 1}{99} ie \\
&+ \frac{10^2 R_{2n+6} - 11(n+10101)}{99} je + \frac{R_{2n+9} - 11(n+101011) - 1}{99} ke
\end{aligned}$$

(c) Note that

$$\begin{aligned}
&\sum_{k=0}^{n} QR_{2k+1} = QR_1 + QR_3 + \cdots + QR_{2n+1} \\
=&\sum_{k=0}^{n} HR_{2k+1} + \left(\sum_{k=0}^{n+2} HR_{2k+1} - HR_1 - HR_3 \right) e \\
=&\frac{R_{2n+3} - 11(n+1) - 1}{99} + \frac{10^2 R_{2n+2} - 11(n+1)}{99} i + \frac{R_{2n+5} - 11(n+11) - 1}{99} j \\
&+ \frac{10^2 R_{2n+4} - 11(n+101)}{99} k + \frac{R_{2n+7} - 11(n+1011) - 1}{99} e + \frac{10^2 R_{2n+6} - 11(n+10101)}{99} ie \\
&+ \frac{R_{2n+9} - 11(n+101011) - 1}{99} je + \frac{10^2 R_{2n+8} - 11(n+1010101)}{99} ke \,,
\end{aligned}$$

as required. □

A direct consequence of Proposition 4.7 is the next result.

Proposition 4.8. *For all integers $n \geq 0$, we have the following identities:*
(a)

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k QR_k =& -\frac{10^{2n+2}}{99} - \frac{10^{2n+3} - 10}{99} i - \frac{10^{2n+4} - 100}{99} j - \frac{10^{2n+5} - 1000}{99} k \\
&- \frac{10^{2n+6} - 10000}{99} e - \frac{10^{2n+7} - 100000}{99} ei - \frac{10^{2n+8} - 1000000}{99} ej \\
&- \frac{10^{2n+9} - 10000000}{99} je - \frac{10^{2n+10} - 100000000}{99} ke;
\end{aligned}$$

if n is odd, and
(b)

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k QR_k =& \frac{10^{2n+4} - 10}{99} + \frac{10^{2n+4} - 1}{99} i - \frac{10^{2n+5} - 89}{99} j + \frac{10^{2n+7} + 978}{99} k \\
&+ \frac{10^{2n+7} - 10011}{99} e + \frac{10^{2n+8} + 1100989}{99} ei + \frac{10^{2n+9} + 999989}{99} je \\
&+ \frac{10^{2n+10} + 10999999}{99} ke;
\end{aligned}$$

if n is even, where $\{R_n\}_{n\geq 0}$ is the repunit sequence, and $\{QR_n\}_{n\geq 0}$ is the biquaternion repunit sequence.

4.2 The classical identities

In this section, we present the classical identities associated with the biquaternion repunit sequence. Since multiplication is not commutative in the set of biquaternions, for each identity, there are two versions; however, in these notes, we present only the first version of each of these identities.

Before presenting the identities of the biquaternion repunit sequence, we exhibit an auxiliary result known as the first Tagiuri-Vajda identity for the quaternion repunit sequence.

Proposition 4.9. *(Taguiri-Vajda's identity) Let m, n, s be any natural numbers. The following property holds:*

$$\begin{aligned}&HR_{m+n}HR_{m+s} - HR_mHR_{m+n+s}\\ =&10^m R_n\left[-1109R_s - \left(\frac{889\cdot 10^s+911}{9}\right)\mathbf{i} + \left(\frac{1091\cdot 10^s+889}{9}\right)\mathbf{j} + \left(\frac{911\cdot 10^s-1091}{9}\right)\mathbf{k}\right],\end{aligned}$$

where $\{HR_n\}_{n\geq 0}$ is the quaternion repunit sequence, and R_n is the n-th repunit number.

Proof. According to Lemma 4.3, considering $X = (1 + 10\mathbf{i} + 100\mathbf{j} + 1000\mathbf{k})$ and $Y = (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$, we obtain that

$$\begin{aligned}HR_{m+n}HR_{m+s} &= \left(\frac{X10^{m+n}-Y}{9}\right)\left(\frac{X10^{m+s}-Y}{9}\right)\\ &= \frac{X^2 10^{2m+n+s} - XY10^{m+n} - XY10^{m+s} + Y^2}{81}\end{aligned}$$

and, on the other hand,

$$\begin{aligned}HR_mHR_{m+n+s} &= \left(\frac{X10^{m}-Y}{9}\right)\left(\frac{X10^{m+n+s}-Y}{9}\right)\\ &= \frac{X^2 10^{2m+n+s} - YX10^{m} - XY10^{m+n+s} + Y^2}{81}\end{aligned}$$

Applying Equation (2.2), we have that

$$\begin{aligned}HR_{m+n}HR_{m+s} - HR_mHR_{m+n+s} =& XY\left(10^m - 10^{m+n}\right) + YX\left(10^{m+n+s} - 10^{m+s}\right)\\ =& \frac{XY\cdot 10^m(1-10^n) + YX\cdot 10^{m+s}(10^n-1)}{81}\\ =& 10^m R_n\frac{10^sYX - XY}{9}\end{aligned}$$

According to Lemma 15 in [4], it follows that

$$\begin{aligned}&10^m R_n\frac{10^sYX - XY}{9}\\ =&10^m R_n\frac{10^s(-1109 - 889i + 1091j + 911k) - (-1109 + 911i - 889j + 1091k)}{9}\\ =&10^m R_n\left[-1109R_s - \left(\frac{889\cdot 10^s+911}{9}\right)i + \left(\frac{1091\cdot 10^s+889}{9}\right)j + \left(\frac{911\cdot 10^s-1091}{9}\right)k\right],\end{aligned}$$

this completes the proof. □

Now we present the Tagiuri-Vajda identity for the biquaternion repunit sequence.

Theorem 4.10. *(Taguiri-Vajda's identity) Let m, n, s be any natural numbers. The following property is valid:*

$$\begin{aligned}
&QR_{m+n}QR_{m+s} - QR_m QR_{m+n+s}\\
=& -9999\cdot 10^m R_n\left[-1109R_s - \left(\frac{889\cdot 10^s+911}{9}\right)i + \left(\frac{1091\cdot 10^s+889}{9}\right)j - \left(\frac{911\cdot 10^s-1091}{9}\right)k\right]\\
&+10^m R_n\left[-11091109R_s - \left(\frac{8890889\cdot 10^s+9110911}{9}\right)i + \left(\frac{10911091\cdot 10^s+8890889}{9}\right)j\right.\\
&\left.+\left(\frac{9110911\cdot 10^s-10911091}{9}\right)k\right]e\,,
\end{aligned}$$

where $\{QR_n\}_{n\geq 0}$ is the biquaternion repunit sequence, and R_n is the n-th repunit number.

Proof. Note that

$$\begin{aligned}
&QR_{m+n}\cdot QR_{m+s} - QR_m\cdot QR_{m+n+s}\\
&=(HR_{m+n}+HR_{m+n+4}e)(HR_{m+s}+HR_{m+s+4}e) - (HR_m+HR_{m+4}e)(HR_{m+n+s}+HR_{m+n+s+4}e)\\
&=(HR_{m+n}\cdot HR_{m+s} - HR_m\cdot HR_{m+n+s}) - (HR_{(m+4)+n}\cdot HR_{(m+4)+s} - HR_{m+4}\cdot HR_{(m+4)+n+s})\\
&+(HR_{m+n}\cdot HR_{m+(s+4)}e - HR_m\cdot HR_{m+n+(s+4)}e) + (HR_{(m+4)+n}\cdot HR_{(m+4)}e - HR_{(m+4)+n+(s-4)}e)
\end{aligned}$$

Applying Proposition 4.9, we obtain

$$\begin{aligned}
&(HR_{m+n}\cdot HR_{m+s} - HR_m\cdot HR_{m+n+s}) - (HR_{(m+4)+n}\cdot HR_{(m+4)+s} - HR_{m+4}\cdot HR_{(m+4)+n+s})\\
=& -9999\cdot 10^m R_n\left[-1109R_s - \left(\frac{889\cdot 10^s+911}{9}\right)i + \left(\frac{1091\cdot 10^s+889}{9}\right)j + \left(\frac{911\cdot 10^s-1091}{9}\right)k\right].
\end{aligned} \tag{4.6}$$

Similary,

$$\begin{aligned}
&(HR_{m+n}\cdot HR_{m+(s+4)}e - HR_m\cdot HR_{m+n+(s+4)}e) + (HR_{(m+4)+n}\cdot HR_{(m+4)}e - HR_{(m+4)+n+(s-4)}e)\\
=&10^m R_n\left[-11091109R_s - \left(\frac{8890889\cdot 10^s+9110911}{9}\right)i + \left(\frac{10911091\cdot 10^s+8890889}{9}\right)j\right.\\
&\left.+\left(\frac{9110911\cdot 10^s-10911091}{9}\right)k\right]e
\end{aligned} \tag{4.7}$$

By combining Equations (4.6) and (4.7), we arrive at the desired result. □

The other identities will follow as a consequence of Tagiuri-Vajda's identity, Theorem 4.10, as we will see below.

Proposition 4.11. *(d'Ocagne's identity) Let $m,\ n$ be any natural number. The following property holds*

$$\begin{aligned}
&QR_nQR_{m+1} - QR_mQR_{n+1}\\
=& -10^n R_{m-n}(11088891 + 10888911i - 13308669j + 8909109k - 11091109e\\
&-2000200ei + 13111311ej + 200020ek),
\end{aligned}$$

where $\{QR_n\}_{n\geq 0}$ is the biquaternion repunit sequence, and R_n is the n-th repunit number.

Proposition 4.12. *(Catalan's identity) Let $m,\ n$ be any non-negative integer. For $m\geq n$ the property holds*

$$\begin{aligned}
&QR_{m+n}QR_{m-n} - (QR_m)^2\\
=&10^{m-n}R_n\left[11088891R_n - (1012121\cdot 10^n + 987679)i + (987679\cdot 10^n - 1212101)j\right.\\
&-(1212101\cdot 10^n - 1012121)k + 11091109R_ne - \left(\frac{9110911\cdot 10^n+8890889}{9}\right)ie\\
&\left.+\left(\frac{8890889\cdot 10^n+10911091}{9}\right)je - \left(\frac{10911091\cdot 10^n-9110911}{9}\right)ke\right],
\end{aligned}$$

where $\{QR_n\}_{n\geq 0}$ *is the biquaternion repunit sequence, and* R_n *is the* n*-th repunit number.*

Making $n = 1$, without effort, follows directly from Proposition 4.12 we have that:

Proposition 4.13. *(Cassini's identity) For all integers* $m \geq 1$*, the following property holds*

$$\begin{aligned} QR_{m+1}QR_{m-1} - (QR_m)^2 =& 10^{m+1}\,(11088891 - 11108889i + 11088891j - 11108889k \\ & +11091109e - 2000200ie + 11091109je - 2000200ke)\ , \end{aligned}$$

where QR_n *is the biquaternion repunit sequence, and* R_n *is the* n*-th repunit number.*

Next, we present the generating function for the biquaternion repunit sequence.

Proposition 4.14. *The generating function for the biquaternion repunit sequence* $\{QR_n\}_{n\geq 0}$*, denoted by* $G_{QR_n}(x)$*, is*

$$\begin{aligned} G_{QR_n}(x) =& \frac{x}{1-11x+10x^2} + \frac{1}{1-11x+10x^2}i + \frac{11-10x}{1-11x+10x^2}j + \frac{1111-110x}{1-11x+10x^2}k \\ & \frac{1111-110x}{1-11x+10x^2}e + \frac{1111-1110x}{1-11x+10x^2}ei + \frac{111111-111110x}{1-11x+10x^2}ej + \frac{111111-111110x}{1-11x+10x^2}ek\ . \end{aligned}$$

Proof. Let $G_{QR_n}(x) = \sum_{n=0}^{\infty} QR_n x^n$ be the generating function for the biquaternion repunit sequence. Combining the expressions $-11xG_{QR_n}(x)$ and $10x^2 G_{QR_n}(x)$, we have

$$\begin{aligned} G_{QR_n} =& \frac{QR_0 + (QR_1 - 11QR_0)x}{1-11x+10x^2} \\ =& \frac{x}{1-11x+10x^2} + \frac{1}{1-11x+10x^2}i + \frac{11-10x}{1-11x+10x^2}j + \frac{1111-110x}{1-11x+10x^2}k \\ & \frac{1111-110x}{1-11x+10x^2}e + \frac{1111-1110x}{1-11x+10x^2}ei + \frac{111111-111110x}{1-11x+10x^2}ej + \frac{111111-111110x}{1-11x+10x^2}ek\ . \end{aligned}$$

which verifies the result. □

5 Conclusion

In this work, we explore two new extensions of Horadam-type sequences of: bicomplex repunit numbers and biquaternion repunit numbers. These extensions are formulated using advanced algebraic concepts involving bicomplex and biquaternion numbers, which are, in turn, generalizations of complex and quaternion numbers. Additionally, we present a detailed analysis of quaternion repunit numbers, highlighting, in particular, the Taiguri-Vajda identity and some partial sums of the terms of these sequences.

For each of these new sequences, we provide a series of fundamental properties, including their recurrence relations and generating functions. We also explore important associated identities, such as the Taiguri-Vajda, Catalan, Cassini, and d'Ocagne identities. This work makes a significant contribution to the literature by enlarging the scope of Horadam sequences and their applications in various areas of mathematics.

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