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GENERALIZED χ-CONTRACTION MAPPINGS ON SOFT NEUTROSOPHIC METRIC SPACES

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Abstract *This study examines the Banach contraction theorem in soft neutrosophic metric spaces, introducing a χ-contraction function by restricting the soft neutrosophic metric to soft points of the absolute soft set. We explore fixed-point results for soft mappings with χ-contraction, ensuring the continuity of soft t-norms and t-conorms. Examples demonstrate the effectiveness of the χ-contraction, leading to a new soft neutrosophic Banach contraction theorem and related fixed-point results. These findings advance fixed-point theory in this context.*

1 Introduction

Fuzzy set theory, introduced by Zadeh [20], has been instrumental in representing uncertainty in decision-making processes across various fields. Atanassov [2] extended this concept by introducing intuitionistic fuzzy sets, which added a degree of non-membership to the existing framework. Building on these foundational concepts, Molodtsov [15] proposed soft set theory, offering a more flexible approach to handling uncertainty without requiring additional parameters, a concept further developed by Maji et al. [14].

Several studies have explored the development and applications of Soft Fuzzy Metric Spaces (SFMSs), including contributions by Das et al. [4], Erduran et al. [5], and Yazar et al. [19], who worked on expanding the understanding of soft metrics and applying them to fixed-point theorems. In similar research, Sonam et al. [18] and Sabri et al. [16] investigated properties such as compactness and completeness in SFMSs.

Fuzzy and intuitionistic fuzzy metric spaces have also been explored by George and Veeramani [6], as well as Kramosil and Michalek [13], contributing significantly to the development of fuzzy topology and metric spaces. Jeyaraman et al. [12] investigate common fixed point theorems within the setting of intuitionistic generalized fuzzy cone metric spaces, focusing on (Φ, Ψ)-weak contractions and providing valuable insights into their properties and applications.

Additionally, Gupta and Gondhi [7] provided important results on tripled coincidence fixed points in SFMSs, further enhancing the application of soft set theory in complex mathematical problems.

In this study, a novel approach to the Banach contraction mapping in Soft Neutrosophic Metric Spaces (SNMSs) is presented, introducing the χ-contraction function and establishing fixed-point results supported by examples, with implications for various metric spaces and applications in handling uncertainty.

2 Preliminaries

Definition 2.1. [18] A soft set (Λ, Υ) defined over the universal set Γ consists of a function Λ mapping elements from a parameter set Υ to subsets of Γ.

Definition 2.2. [18] An absolute soft set (Λ, Υ) over Γ is characterized by the condition that $\Lambda(\zeta) = \Gamma$ holds for every ζ belonging to the parameter set Υ. The absolute soft set Γ defined over the parameter set Υ is denoted by $\tilde{\Gamma}_{\Upsilon}$.

Definition 2.3. [4] A soft set (Λ, Υ) defined over the universal set Γ is termed as a null or void soft set if $\Lambda(\zeta) = \{\}$ for every ζ in the parameter set Υ. This condition is denoted as $\tilde{\varphi}$.

Let $\mathbb{R}$ denote the set of all real numbers. We use $\breve{\mathfrak{B}}(\mathbb{R})$ to represent the collection of all non-empty bounded subsets of $\mathbb{R}$.

Definition 2.4. [18] A soft real set is defined as a set consisting of (Λ, Υ) where Λ chooses every member of Υ to a non-empty bounded subset of $\mathbb{R}$. A soft real number is defined by a soft real set (Λ, Υ) where $\Lambda(\zeta)$ denotes a single real number inside a finite subset of $\mathbb{R}$ for each ζ in Υ. The representation of the value for this is denoted as $\breve{\eta}$. The function $\breve{\eta}(\zeta) = \{k\}$ for a given k in the set $\mathbb{R}$ is represented as $\bar{k}$.

Definition 2.5. [5] When there is only one parameter $\zeta \in \Upsilon$ such that $\Lambda(\zeta) = \{x\}$, where $x \in \Gamma$, and $\Lambda(\mu) = \varphi$ for all $\mu \in \Upsilon$ except ζ, then a soft set defined over the universal collection Γ is called a soft point. It is represented by $\breve{\eta}\zeta$. A soft point $\breve{\eta}\zeta$ is considered to be a component of a soft set (Λ, Υ) if $\breve{\eta}\zeta = \{x\}$ is a subset of $\Lambda(\zeta)$. This relation is also represented as $\breve{\eta}\zeta \in (\Lambda, \Upsilon)$. The set containing all soft points of (Λ, Υ) is denoted by $SP(\Lambda, \Upsilon)$.

Definition 2.6. [5] The function ϕ, representing soft elements, transfers parameter sets Υ to the universal set Γ. That is $\phi : \Upsilon \to \Gamma$ signifies a soft element. The soft set formed by grouping the collection $\breve{\mathcal{U}}$ of soft elements is denoted as $SS(\breve{\mathcal{U}})$.

The set $\mathbb{R}(\Upsilon)$ represents all non-negative soft real numbers, whereas $\mathbb{R}(\Upsilon)*$ represents all soft real numbers with a parameter set Υ. The sets $[a, b](\Upsilon)$ and $[0, \infty)(\Upsilon)$ represent the collections of all soft real numbers corresponding to the intervals $[a, b]$ and $[0, \infty)$, respectively.

Definition 2.7. [18] The operations defined for two soft real numbers $\breve{\lambda}$ and $\breve{\varrho}$ are as follows:

$$(\breve{\lambda} \oplus \breve{\varrho})(\zeta) = \{\breve{\lambda}(\zeta) + \breve{\varrho}(\zeta), \zeta \in \Upsilon\},$$
$$(\breve{\lambda} \ominus \breve{\varrho})(\zeta) = \{\breve{\lambda}(\zeta) - \breve{\varrho}(\zeta), \zeta \in \Upsilon\},$$
$$(\breve{\lambda} \circ \breve{\varrho})(\zeta) = \{\breve{\lambda}(\zeta) \cdot \breve{\varrho}(\zeta), \zeta \in \Upsilon\}.$$

Definition 2.8. [18] On a universal set, we consider $\tilde{\Gamma}_{\Upsilon}$ to be an absolute-soft set. A metric over $\tilde{\Gamma}_{\Upsilon}$ is defined as a mapping $\Xi : SP(\tilde{\Gamma}_{\Upsilon}) \times SP(\tilde{\Gamma}_{\Upsilon}) \to \mathbb{R}(\Upsilon)*$ if the following conditions hold:

(SM1) $\Xi(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}) \tilde{\geq} \bar{0}$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}$,

(SM2) $\Xi(\breve{\varsigma}_{p_i}, \breve{\nu}_{\xi_j}) = \bar{0} \leftrightarrow \breve{\varsigma}_{\xi_i} = \breve{\nu}_{\xi_j}$

(SM3) $\Xi(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}) = \Xi(\breve{\nu}_{\xi_j}, \breve{\varsigma}_{\xi_i})$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \breve{\in} \tilde{\Gamma}_{\Upsilon}$,

(SM4) $\Xi(\breve{\varsigma}_{\xi_i}, \breve{\varpi}_{p_k}) \tilde{\leq} \Xi(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}) + \Xi(\breve{\nu}_{\xi_j}, \breve{\varpi}_{p_k})$ for each $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \breve{\varpi}_{p_k} \tilde{\in} \tilde{\Gamma}_{\Upsilon}$.

A soft metric space is formed by combining the soft metric Ξ with the absolute soft set $\tilde{\Gamma}$. It is represented as $(\tilde{\Gamma}_{\Upsilon}, \Xi)$ or $(\tilde{\Gamma}_{\Upsilon}, \Xi, \Upsilon)$.

Definition 2.9. [18] We examine a pair of soft metric spaces. $(\tilde{\Gamma}_{\Upsilon_{\mathcal{A}}}, \Xi, \Upsilon_{\mathcal{A}})$ and $(\tilde{\breve{\mathcal{U}}}_{\Upsilon_{\breve{\mathfrak{B}}}}, \Xi, \Upsilon_{\mathfrak{B}})$. We contemplate a function.$(\mathcal{U}, \Phi) : (\tilde{\Gamma}_{\Upsilon_{\mathcal{A}}}, \Xi, \Upsilon_{\mathcal{A}}) \to (\tilde{\breve{\mathcal{U}}}_{\Upsilon_{\breve{\mathfrak{B}}}}, \Xi, \Upsilon_{\breve{\mathfrak{B}}})$. A soft mapping is represented by the pair $(\mathcal{U}, \Phi)$, where $\mathcal{U} : \tilde{\Gamma}_{\Upsilon_{\mathcal{A}}} \to \tilde{M}_{\Upsilon_{\breve{\mathfrak{B}}}}$ and $\Phi : \Upsilon_{\mathcal{A}} \to \Upsilon_{\breve{\mathfrak{B}}}$.

Definition 2.10. We examine function $\ddot{\otimes} : [0, 1](\Upsilon) \times [0, 1](\Upsilon) \to [0, 1](\Upsilon)$. This function is proposed to be a continuous soft t-norm (CSTN) provided it satisfies the following conditions:

(i) The function $\ddot{\otimes}$ obeys both the commutative and associative properties.

(ii) $\ddot{\otimes}$ needs to be continuous.

(iii) $\breve{u} \ddot{\otimes} \bar{1} = \breve{u}$ for every $\breve{u} \in [0, 1](\Upsilon)$,

(iv) $\breve{u}\ddot{\otimes}\breve{v} \leq \breve{\sigma}\ddot{\otimes}\breve{\rho}$ whenever $\breve{u} \leq \breve{v}$ and $\breve{\sigma} \leq \breve{\rho}$ for $\breve{u}, \breve{v}, \breve{\sigma}, \breve{\rho} \in [0,1](\Upsilon)$.

Definition 2.11. We consider function $\ddot{\oplus} : [0,1](\Upsilon) \times [0,1](\Upsilon) \to [0,1](\Upsilon)$; then, $\ddot{\otimes}$ is considered a continuous soft t-conorm (CSTCN) if it satisfies the conditions listed below:

(i) $\ddot{\oplus}$ obeys both commutativity and associativity.

(ii) $\ddot{\oplus}$ needs to be continuous.

(iii) $\breve{u}\ddot{\oplus}\bar{0} = \breve{u}$ for all $\breve{u} \in [0,1](\Upsilon)$,

(iv) $\breve{u}\ddot{\oplus}\breve{v} \leq \breve{\sigma}\ddot{\oplus}\breve{\rho}$ whenever it ocuurs $\breve{u} \leq \breve{v}$ and $\breve{\sigma} \geq \breve{\rho}$ for $\breve{u}, \breve{v}, \breve{\sigma}, \breve{\rho} \in [0,1](\Upsilon)$.

Example 2.12. $\tilde{\beta}\ddot{\otimes}\tilde{\delta} = \min\{\tilde{\beta}, \tilde{\delta}\}, \tilde{\beta}\ddot{\oplus}\tilde{\delta} = \max\{\tilde{\beta}, \tilde{\delta}\}$

Definition 2.13. We assume $\mathfrak{S}_{\mathfrak{G}}$ as a mapping $\mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}} : \mathcal{SP}(\tilde{\Gamma}_{\Upsilon}) \times \mathcal{SP}(\tilde{\Gamma}_{\Upsilon}) \times (0,\infty)(\Upsilon) \to [0,1](\Upsilon)$.Then, $\mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}$ is claimed to constitute a soft fuzzy metric on $\tilde{\Gamma}_{\Upsilon}$ if

(SNM1) $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) + \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) + \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) \leq \bar{3}$

(SNM2) $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) \tilde{\geq} \bar{0}$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM3) $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{1} \leftrightarrow \breve{\varsigma}_{\xi_i} = \breve{\nu}_{\xi_j}$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM4) $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{G}}(\breve{\nu}_{\xi_i}, \breve{\varsigma}_{\xi_j}, \acute{l})$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM5) $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\varpi}_{p_k}, \acute{l} \oplus \breve{q}) \tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) \ddot{\otimes} \mathfrak{S}_{\mathfrak{G}}(\breve{\nu}_{\xi_j}, \breve{\varpi}_{p_k}, \breve{q})$ forall $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \breve{\varpi}_{p_k} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l}, \breve{q} \tilde{>} \bar{0}$,

(SNM6) $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, .) : (0,\infty)(\Upsilon) \to [0,1](\Upsilon)$ is a continuous map.

(SNM7) $\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) \tilde{\leq} \bar{0}$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM8) $\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{0} \leftrightarrow \breve{\varsigma}_{\xi_i} = \breve{\nu}_{\xi_j}$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM9) $\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{H}}(\breve{\nu}_{\xi_i}, \breve{\varsigma}_{\xi_j}, \acute{l})$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM10) $\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\varpi}_{p_k}, \acute{l} \oplus \breve{q}) \tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) \ddot{\oplus} \mathfrak{S}_{\mathfrak{H}}(\breve{\nu}_{\xi_j}, \breve{\varpi}_{p_k}, \breve{q})$ forall $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \breve{\varpi}_{p_k} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l}, \breve{q} \tilde{>} \bar{0}$,

(SNM11) $\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, .) : (0,\infty)(\Upsilon) \to [0,1](\Upsilon)$ is a continuous map.

(SNM12) $\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) \tilde{\leq} \bar{0}$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM13) $\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{0} \leftrightarrow \breve{\varsigma}_{\xi_i} = \breve{\nu}_{\xi_j}$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM14) $\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{J}}(\breve{\nu}_{\xi_i}, \breve{\varsigma}_{\xi_j}, \acute{l})$ for all $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l} \tilde{>} \bar{0}$,

(SNM15) $\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\varpi}_{p_k}, \acute{l} \oplus \breve{q}) \tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) \ddot{\oplus} \mathfrak{S}_{\mathfrak{J}}(\breve{\nu}_{\xi_j}, \breve{\varpi}_{p_k}, \breve{q})$ forall $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \breve{\varpi}_{p_k} \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l}, \breve{q} \tilde{>} \bar{0}$,

(SNM16) $\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, .) : (0,\infty)(\Upsilon) \to [0,1](\Upsilon)$ is a function that exhibits continuity.

The Soft Neutrosophic Metric Space (SNMS) is defined as the combination of the soft fuzzy metrics $\mathfrak{S}_{\mathfrak{G}}$, $\mathfrak{S}_{\mathfrak{H}}$, and $\mathfrak{S}_{\mathfrak{J}}$, together with the absolute soft set $\tilde{\Gamma}_{\Upsilon}$. It is symbolized as $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$.

Example 2.14. We consider an SNMS $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$. We let $\tilde{\beta}\ddot{\otimes}\tilde{\delta} = \min\{\tilde{\beta}, \tilde{\delta}\}$ and $\tilde{\beta}\ddot{\otimes}\tilde{\delta} = \max\{\tilde{\beta}, \tilde{\delta}\}$ be defined in $(\tilde{\Gamma}_{\Upsilon}, \Xi)$. We define $\mathfrak{S}_{\mathfrak{G}}$, $\mathfrak{S}_{\mathfrak{H}}$ and $\mathfrak{S}_{\mathfrak{J}}$ are the mappings from $\mathcal{SP}(\tilde{\Gamma}_{\Upsilon}) \times \mathcal{SP}(\tilde{\Gamma}_{\Upsilon}) \times (0,\infty)(\Upsilon)$ to $[0,1](\Upsilon)$ as $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \frac{\acute{l}}{\acute{l} \oplus \Xi(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j})}$, $\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \frac{\Xi(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j})}{\acute{l} \oplus \Xi(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j})}$, $\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \frac{\Xi(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j})}{\acute{l}}$, where $\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}$ and $\acute{l} \tilde{>} \bar{0}$.
Then, $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is an SNMS. Furthermore, the soft fuzzy metrics $\mathfrak{S}_G$, $\mathfrak{S}_H$, and $\mathfrak{S}_J$, which are derived from the soft metric Ξ, are referred to as the standard soft neutrosophic metric.

Definition 2.15. In the context of $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ being an SNMS, let's introduce a collection of soft sets denoted as Ω. The collection Ω is considered a soft open cover of $\tilde{\Gamma}_{\Upsilon}$ if every set in Ω is a soft open set that fully contains $\tilde{\Gamma}_{\Upsilon}$. An SNMS is defined as compact if, for any soft open cover of $\tilde{\Gamma}_{\Upsilon}$ within $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \ddot{\otimes})$, there exists a finite set of soft open sets that collectively cover $\tilde{\Gamma}_{\Upsilon}$.

Definition 2.16. Any soft sequence $\{\breve{\varsigma}^{m}_{\xi_i}\}$ in SNMS $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is considered to converge towards a soft point $\breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}$ if $\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{1}$, $\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{0}$, $\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{0}, \forall \acute{l} \tilde{>} \bar{0}$. Similarly, for any given value $\tilde{\tau}$ belonging to the interval $(\bar{0}, \bar{1})$ and $\acute{l}$ greater than $\bar{0}$, there exists a positive integer N_0 such that $\breve{\varsigma}^{m}_{\xi_i}$ belongs to the soft set $\mathcal{SS}(\breve{\mathfrak{B}}_{\mathfrak{S}_{\mathfrak{G}}}(\breve{\nu}_{\xi_j}, \tilde{\tau}, \acute{l}))$ for all m greater than or equal to N_0, where $\breve{\mathfrak{B}}_{\mathfrak{S}_{\mathfrak{G}}}(\breve{\nu}_{\xi_j}, \tilde{\tau}, \acute{l})$ represents a soft open ball centered at $\breve{\nu}_{\xi_j}$ with radius $\tilde{\tau}$ with respect to $\acute{l}$. This means $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) \tilde{>} \bar{1} \ominus \tilde{\tau}, \forall \acute{l} \tilde{>} \bar{0}, m \geq N_0$.

Definition 2.17. Any soft sequence $\{\breve{\varsigma}^{m}_{\xi_i}\}$ in SNMS $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is considered to be a Cauchy sequence in SNMS if $\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{t}_{\xi_i}, \acute{l}) = \bar{1}$, $\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{t}_{\xi_i}, \acute{l}) = \bar{0}$, $\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{t}_{\xi_i}, \acute{l}) = \bar{0}, \forall \acute{l} \tilde{>} \bar{0}$. Alternatively, for any specified $\tilde{\tau}$ belonging to the interval $(\bar{0}, \bar{1})$ and $\acute{l}$ greater than $\bar{0}$, there exists N_0 in the set of positive integers such that $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{t}_{\xi_i}, \acute{l}) \tilde{>} \bar{1} \ominus \tilde{\tau}, \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{t}_{\xi_i}, \acute{l}) \tilde{>} \bar{0} \ominus \tilde{\tau}, \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{t}_{\xi_i}, \acute{l}) \tilde{>} \bar{0} \ominus \tilde{\tau}, \forall m, t \geq N_0$.

Definition 2.18. An SNMS $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is considered complete if all Cauchy sequences in the SNMS converge.

Definition 2.19. In a Soft Neutrosophic Metric Space (SNMS) $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$, compactness implies that every sequence of soft fuzzy elements within $\tilde{\Gamma}_{\Upsilon}$ has at least one subsequence which converges.

3 Main Results

Definition 3.1. Considering $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ as an SNMS, a soft mapping $(\mathcal{U}, \Phi)$ from this SNMS to itself is considered a soft neutrosophic contraction if there exists a value $\bar{\alpha}$ in the interval $[\bar{0}, \bar{1})$ that satisfies a certain condition.

$$\mathfrak{S}_{\mathfrak{G}}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \acute{l}) \tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}\bar{\alpha})$$
$$\mathfrak{S}_{\mathfrak{H}}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \acute{l}) \tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}\bar{\alpha}) \text{ and}$$
$$\mathfrak{S}_{\mathfrak{J}}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \acute{l}) \tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}\bar{\alpha}) \forall \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon} \text{ and } \acute{l} \tilde{>} \bar{0}.$$

Definition 3.2. A function $\theta : \mathbb{R}(\Upsilon) \to [0, \infty)(\Upsilon)$ is considered a χ-function if it meets the following requirements:

(i) $\theta(\acute{l}) = \bar{0} \leftrightarrow \acute{l} = \bar{0}$,

(ii) The function θ is monotonically increasing, and as $\acute{l}$ approaches infinity, $\theta(\acute{l})$ tends to positive infinity,

(iii) The function θ is left-continuous whenever $\acute{l}$ exceeds $\bar{0}$,

(iv) At $\acute{l} = \bar{0}$, θ exhibits continuity.

Example 3.3. Let $\mathbb{R}(\Upsilon)$ denote the set of all soft real numbers equipped with a soft topology, and let $[0, \infty)(\Upsilon)$ represent the non-negative subset of $\mathbb{R}(\Upsilon)$. We define the function $\theta : \mathbb{R}(\Upsilon) \to [0, \infty)(\Upsilon)$ as follows:

$$\theta(\acute{l}) = \begin{cases} \bar{0} & \text{if} \quad \acute{l} = \bar{0} \\ \sqrt{3}k & if \quad \acute{l} \neq \bar{0}. \end{cases}$$

Thus, θ satisfies all the criteria to be considered a χ-function.

Definition 3.4. Considering $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ as an SNMS, a soft mapping $(\mathcal{U}, \Phi) : (\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus}) \to (\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is called a χ-contraction mapping on an SNMS if there exists a soft real number $\bar{\alpha} \in [\bar{0}, \bar{1})$ that satisfies the following condition:

$$\mathfrak{S}_{\mathfrak{G}}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \theta(\acute{l})) \tilde{\geq} \mathfrak{S}_{\mathfrak{G}} \ominus \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \theta(\acute{l}\bar{\alpha}) \ominus,$$
$$\mathfrak{S}_{\mathfrak{H}}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \theta(\acute{l})) \tilde{\leq} \mathfrak{S}_{\mathfrak{H}} \ominus \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \theta(\acute{l}\bar{\alpha}) \ominus, \text{ and}$$
$$\mathfrak{S}_{\mathfrak{J}}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \theta(\acute{l})) \tilde{\leq} \mathfrak{S}_{\mathfrak{J}} \ominus \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \theta(\acute{l}\bar{\alpha}) \ominus, \forall \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon} \text{ and } \acute{l} \tilde{>} \bar{0},$$

where θ is defined as a χ-function.

Theorem 3.5. *We consider* $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ *as a complete SNMS wherein*

$$\begin{aligned}
&\lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{1},\\
&\lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{0},\\
&\lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{0} \textit{for all} \, \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}. \qquad (3.1)
\end{aligned}$$

Hence, the soft neutrosophic contraction mapping $(\mathcal{U}, \Phi)$ *on* $\tilde{\Gamma}_{\Upsilon}$ *possesses exactly one unique soft fixed point.*

Proof. We start with a soft point $\breve{\varsigma}^0_{\xi_i}$ in $\tilde{\Gamma}_{\Upsilon}$ and create a soft sequence $\{\breve{\varsigma}^m_{\xi_i}\}$ where each $\breve{\varsigma}^m_{\xi_i}$ is obtained by applying $(\mathcal{U}, \Phi)$ repeatedly to $\breve{\varsigma}^0_{\xi_i}$.

By induction, we derive:

$$\begin{aligned}
\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) &\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{\alpha}^m),\\
\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{\alpha}^m),\\
\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{\alpha}^m). \qquad (3.2)
\end{aligned}$$

Now, by conditions (3.2) and (SNM5,SNM10,SNM15), for any $c \in \mathbb{Z}^+$, we have

$$\begin{aligned}
\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) &\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}/\bar{c}) \underbrace{\ddot{\otimes} \ldots \ddot{\otimes}}_{c-times} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m+c-1}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}/\bar{c})\\
&\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{c}\bar{\alpha}m) \underbrace{\ddot{\otimes} \ldots \ddot{\otimes}}_{c-times} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{c}\bar{\alpha}^{m+c-1}),\\
\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}/\bar{c}) \underbrace{\ddot{\oplus} \ldots \ddot{\oplus}}_{c-times} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m+c-1}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}/\bar{c})\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{c}\bar{\alpha}m) \underbrace{\ddot{\oplus} \ldots \ddot{\oplus}}_{c-times} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{c}\bar{\alpha}^{m+c-1}),\\
\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}/\bar{c}) \underbrace{\ddot{\oplus} \ldots \ddot{\oplus}}_{c-times} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m+c-1}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}/\bar{c})\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{c}\bar{\alpha}m) \underbrace{\ddot{\oplus} \ldots \ddot{\oplus}}_{c-times} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \acute{l}/\bar{c}\bar{\alpha}^{m+c-1}).
\end{aligned}$$

Now, using (3.1), we obtain

$$\begin{aligned}
&\lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) \tilde{\geq} \underbrace{\bar{1} \ddot{\otimes} \bar{1} \ddot{\otimes} \ldots \ddot{\otimes} \bar{1}}_{c-times} = \bar{1},\\
&\lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) \tilde{\leq} \underbrace{\bar{0} \ddot{\oplus} \bar{0} \ddot{\oplus} \ldots \ddot{\oplus} \bar{0}}_{c-times} = \bar{0},\\
&\lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) \tilde{\leq} \underbrace{\bar{0} \ddot{\oplus} \bar{0} \ddot{\oplus} \ldots \ddot{\oplus} \bar{0}}_{c-times} = \bar{0}.
\end{aligned}$$

Therefore, the soft fuzzy sequence $\{\breve{\varsigma}^m_{\xi_i}\}$ is considered to be convergent in $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ because it satisfies the Cauchy property. Additionally, as $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is complete, this implies that the sequence converges. We let $\{\breve{\varsigma}^m_{\xi_i}\} \to \breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}$, i.e.,

$$\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^m_{\xi_i}, \breve{\nu}^t_{p_j}, \acute{l}) = \bar{1}, \lim_{m\to\infty} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^m_{\xi_i}, \breve{\nu}^t_{p_j}, \acute{l}) = \bar{0}, \lim_{m\to\infty} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^m_{\xi_i}, \breve{\nu}^t_{p_j}, \acute{l}) = \bar{0}. \qquad (3.3)$$

Then,

$$\begin{aligned}
\mathfrak{S}_{\mathfrak{G}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l}) &\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},(\mathcal{U},\Phi)\breve{\varsigma}^{m}_{\xi_i},\acute{l}/\bar{2}) \ddot{\otimes} \mathfrak{S}_{\mathfrak{G}}((\mathcal{U},\Phi)\breve{\varsigma}^{m}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l}/\bar{2})\\
&\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\nu}_{\xi_j},\breve{\varsigma}^{m}_{\xi_i},\acute{l}/\bar{2}\bar{\alpha}) \ddot{\otimes} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m+1}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l}/\bar{2})\\
\mathfrak{S}_{\mathfrak{H}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},(\mathcal{U},\Phi)\breve{\varsigma}^{m}_{\xi_i},\acute{l}/\bar{2}) \ddot{\oplus} \mathfrak{S}_{\mathfrak{H}}((\mathcal{U},\Phi)\breve{\varsigma}^{m}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l}/\bar{2})\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\nu}_{\xi_j},\breve{\varsigma}^{m}_{\xi_i},\acute{l}/\bar{2}\bar{\alpha}) \ddot{\oplus} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m+1}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l}/\bar{2})\\
\mathfrak{S}_{\mathfrak{J}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},(\mathcal{U},\Phi)\breve{\varsigma}^{m}_{\xi_i},\acute{l}/\bar{2}) \ddot{\oplus} \mathfrak{S}_{\mathfrak{J}}((\mathcal{U},\Phi)\breve{\varsigma}^{m}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l}/\bar{2})\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\nu}_{\xi_j},\breve{\varsigma}^{m}_{\xi_i},\acute{l}/\bar{2}\bar{\alpha}) \ddot{\oplus} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m+1}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l}/\bar{2})
\end{aligned}$$

From (3.3), we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{G}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l})\tilde{\geq}\bar{1}\ddot{\otimes}\bar{1}=\bar{1}, \text{ or } \lim_{m\to\infty} \mathfrak{S}_{\mathfrak{G}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l})=\bar{1},\\
&\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{H}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l})\tilde{\leq}\bar{0}\ddot{\oplus}\bar{0}=\bar{0}, \text{ or } \lim_{m\to\infty} \mathfrak{S}_{\mathfrak{H}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l})=\bar{0},\\
&\lim_{m\to\infty} \mathfrak{S}_{\mathfrak{J}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l})\tilde{\leq}\bar{0}\ddot{\oplus}\bar{0}=\bar{0}, \text{ or } \lim_{m\to\infty} \mathfrak{S}_{\mathfrak{J}}((\mathcal{U},\Phi)\breve{\nu}_{\xi_j},\breve{\nu}_{\xi_j},\acute{l})=\bar{0}.
\end{aligned}$$

Hence, $(\mathcal{U},\Phi)\breve{\nu}_{\xi_j}=\breve{\nu}_{\xi_j}$. Therefore, $\breve{\nu}_{\xi_j}$ serves as a soft fixed point of $(\mathcal{U},\Phi)$.
It is easy to verify that there is only one soft fixed point for the soft neutrosophic contraction mapping $(\mathcal{U},\Phi)$. □

Theorem 3.6. *Regard* $(\tilde{\Gamma}_{\Upsilon},\mathfrak{S}_{\mathfrak{G}},\mathfrak{S}_{\mathfrak{H}},\mathfrak{S}_{\mathfrak{J}},\ddot{\otimes},\ddot{\oplus})$ *as a complete SNMS wherein*

$$\lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l})=\bar{1},\ \lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l})=\bar{0},\ \lim_{\acute{l}\to+\infty_s} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i},\breve{\nu}_{\xi_j},\acute{l})=\bar{0}\,\forall\acute{l}\tilde{>}\bar{0}. \tag{3.4}$$

Also, we consider a χ*-contraction* $(\mathcal{U},\Phi):(\tilde{\Gamma}_{\Upsilon},\mathfrak{S}_{\mathfrak{G}},\mathfrak{S}_{\mathfrak{H}},\mathfrak{S}_{\mathfrak{J}},\ddot{\otimes},\ddot{\oplus})\to(\tilde{\Gamma}_{\Upsilon},\mathfrak{S}_{\mathfrak{G}},\mathfrak{S}_{\mathfrak{H}},\mathfrak{S}_{\mathfrak{J}},\ddot{\otimes},\ddot{\oplus})$. *Then,* $(\mathcal{U},\Phi)$ *possesses only one soft fixed point.*

Proof. Start with a soft point $\breve{\varsigma}^{0}_{\xi_i}$ belonging to $\tilde{\Gamma}_{\Upsilon}$ and create a soft sequence $\{\breve{\varsigma}^{m}_{\xi_i}\}$ where each $\breve{\varsigma}^{m}_{\xi_i}$ is obtained by applying $(\mathcal{U},\Phi)$ repeatedly to $\breve{\varsigma}^{0}_{\xi_i}$.
Based on conditions i) and iv) in Definition (2.11), for any $\acute{l}$ greater than $\bar{0}$, there exists a $\breve{q}$ greater than $\bar{0}$ such that $\theta(\breve{q})$ is less than $\acute{l}$. Now, through the process of induction, we derive:

$$\begin{aligned}
\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i},\breve{\varsigma}^{m+1}_{\xi_i},\acute{l}) &\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{0}_{\xi_i},\breve{\varsigma}^{1}_{\xi_i},\theta(\breve{q}/\bar{\alpha}^{m})),\\
\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i},\breve{\varsigma}^{m+1}_{\xi_i},\acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{0}_{\xi_i},\breve{\varsigma}^{1}_{\xi_i},\theta(\breve{q}/\bar{\alpha}^{m})),\\
\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i},\breve{\varsigma}^{m+1}_{\xi_i},\acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{0}_{\xi_i},\breve{\varsigma}^{1}_{\xi_i},\theta(\breve{q}/\bar{\alpha}^{m})).
\end{aligned} \tag{3.5}$$

By utilising conditions (3.5), we have, for any $c \in \mathbb{Z}^+$,

$$\begin{aligned}
\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) &\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \theta(\breve{q})),\\
&\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \theta(\breve{q}/\bar{c}))\, \underbrace{\ddot{\otimes} \ldots \ddot{\otimes}}_{c-times}\, \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m+c-1}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \theta(\breve{q}/\bar{c})),\\
&\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{0}_{\xi_i}, \breve{\varsigma}^{1}_{\xi_i}, \theta(\breve{q}/\bar{c}\bar{\alpha}^{m}))\, \underbrace{\ddot{\otimes} \ldots \ddot{\otimes}}_{c-times}\, \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{0}_{\xi_i}, \breve{\varsigma}^{1}_{\xi_i}, \theta(\breve{q}/\bar{c}\bar{\alpha}^{m+c-1})),\\
\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \theta(\breve{q})),\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \theta(\breve{q}/\bar{c}))\, \underbrace{\ddot{\oplus} \ldots \ddot{\oplus}}_{c-times}\, \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m+c-1}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \theta(\breve{q}/\bar{c})),\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{0}_{\xi_i}, \breve{\varsigma}^{1}_{\xi_i}, \theta(\breve{q}/\bar{c}\bar{\alpha}^{m}))\, \underbrace{\ddot{\oplus} \ldots \ddot{\oplus}}_{c-times}\, \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{0}_{\xi_i}, \breve{\varsigma}^{1}_{\xi_i}, \theta(\breve{q}/\bar{c}\bar{\alpha}^{m+c-1})),\\
\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \theta(\breve{q})),\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \theta(\breve{q}/\bar{c}))\, \underbrace{\ddot{\oplus} \ldots \ddot{\oplus}}_{c-times}\, \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m+c-1}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \theta(\breve{q}/\bar{c})),\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{0}_{\xi_i}, \breve{\varsigma}^{1}_{\xi_i}, \theta(\breve{q}/\bar{c}\bar{\alpha}^{m}))\, \underbrace{\ddot{\oplus} \ldots \ddot{\oplus}}_{c-times}\, \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{0}_{\xi_i}, \breve{\varsigma}^{1}_{\xi_i}, \theta(\breve{q}/\bar{c}\bar{\alpha}^{m+c-1})).
\end{aligned}$$

Now, letting $\acute{l} \to +\infty_s$ and using (3.4), we obtain

$$\lim_{\acute{l} \to +\infty_s} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) \tilde{\geq} \underbrace{\bar{1} \ddot{\otimes} \bar{1} \ddot{\otimes} \ldots \ddot{\otimes} \bar{1}}_{c\text{-times}} = \bar{1},$$

$$\lim_{\acute{l} \to +\infty_s} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) \tilde{\leq} \underbrace{\bar{0} \ddot{\oplus} \bar{0} \ddot{\oplus} \ldots \ddot{\oplus} \bar{0}}_{c\text{-times}} = \bar{0},$$

$$\lim_{\acute{l} \to +\infty_s} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\varsigma}^{m+c}_{\xi_i}, \acute{l}) \tilde{\leq} \underbrace{\bar{0} \ddot{\oplus} \bar{0} \ddot{\oplus} \ldots \ddot{\oplus} \bar{0}}_{c\text{-times}} = \bar{0}.$$

Thus, the soft fuzzy sequence $\{\breve{\varsigma}^{m}_{\xi_i}\}$ is Cauchy in $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ and hence it is convergent as $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is complete. We let $\{\breve{\varsigma}^{m}_{\xi_i}\} \to \breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}$, i.e.,

$$\lim_{m \to \infty} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}^{t}_{p_j}, \acute{l}) = \bar{1}, \lim_{m \to \infty} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}^{t}_{p_j}, \acute{l}) = \bar{0}, \lim_{m \to \infty} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}^{t}_{p_j}, \acute{l}) = \bar{0}. \tag{3.6}$$

Also,

$$\begin{aligned}
\mathfrak{S}_{\mathfrak{G}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) &\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, (\mathcal{U}, \Phi)\breve{\varsigma}^{m}_{\xi_i}, \acute{l}/\bar{2}) \ddot{\otimes} \mathfrak{S}_{\mathfrak{G}}((\mathcal{U}, \Phi)\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),\\
&\tilde{\geq} \mathfrak{S}_{\mathfrak{G}}(\breve{\nu}_{\xi_j}, \breve{\varsigma}^{m}_{\xi_i}, \theta(\breve{q}/\bar{2}\bar{\alpha})) \ddot{\otimes} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}^{m+1}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),\\
\mathfrak{S}_{\mathfrak{H}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, (\mathcal{U}, \Phi)\breve{\varsigma}^{m}_{\xi_i}, \acute{l}/\bar{2}) \ddot{\oplus} \mathfrak{S}_{\mathfrak{H}}((\mathcal{U}, \Phi)\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{H}}(\breve{\nu}_{\xi_j}, \breve{\varsigma}^{m}_{\xi_i}, \theta(\breve{q}/\bar{2}\bar{\alpha})) \ddot{\oplus} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}^{m+1}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),\\
\mathfrak{S}_{\mathfrak{J}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) &\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, (\mathcal{U}, \Phi)\breve{\varsigma}^{m}_{\xi_i}, \acute{l}/\bar{2}) \ddot{\oplus} \mathfrak{S}_{\mathfrak{J}}((\mathcal{U}, \Phi)\breve{\varsigma}^{m}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),\\
&\tilde{\leq} \mathfrak{S}_{\mathfrak{J}}(\breve{\nu}_{\xi_j}, \breve{\varsigma}^{m}_{\xi_i}, \theta(\breve{q}/\bar{2}\bar{\alpha})) \ddot{\oplus} \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}^{m+1}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}).
\end{aligned}$$

From (3.6) and the fact that $\ddot{\otimes}$ is a CSTN and $\ddot{\oplus}$ is CSTCN, we get $\mathfrak{S}_{\mathfrak{G}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) \to \bar{1}$, $\mathfrak{S}_{\mathfrak{H}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) \to \bar{0}$, $\mathfrak{S}_{\mathfrak{J}}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) \to \bar{0}$, as $m \to \infty$.
Therefore, $\breve{\nu}_{\xi_j}$ serves as a soft fixed point for the mapping $(\mathcal{U}, \Phi)$. The uniqueness of the soft fixed point for the χ-contraction mapping $(\mathcal{U}, \Phi)$ on $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is easily demonstrable. □

Theorem 3.7. *Consider* $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus})$ *as a complete SNMS, and let* $(\mathcal{U}, \Phi) : (\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus}$ $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus})$ *be a* χ*-contraction mapping. Additionally, assume that for a soft point* $\breve{\varsigma}^0_{\xi_i} \in \tilde{\Gamma}_\Upsilon$*, the iterated soft sequence* $\{\breve{\varsigma}^m_{\xi_i}\}$*, defined by* $\breve{\varsigma}^m_{\xi_i} = (\mathcal{U}, \Phi)\breve{\varsigma}^{m-1}_{\xi_i}$ *for* $m = 1, 2, 3, \ldots$*, is convergent. As a result, there exists a unique soft fixed point of* $(\mathcal{U}, \Phi)$ *in* $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus})$ *to which the sequence* $\{\breve{\varsigma}^m_{\xi_i}\}$ *converges.*

Proof. We examine a χ-contraction mapping $(\mathcal{U}, \Phi)$ operating on $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus})$. Subsequently, a soft real number $\bar{\alpha}$ exists within the range $[\bar{0}, \bar{1})$ that fulfills the specified condition.

$$\mathfrak{S}_\mathfrak{G}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \theta(\acute{l})) \tilde{\geq} \mathfrak{S}_\mathfrak{G}\left(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \theta\left(\frac{\acute{l}}{\bar{\alpha}}\right)\right),$$

$$\mathfrak{S}_\mathfrak{H}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \theta(\acute{l})) \tilde{\leq} \mathfrak{S}_\mathfrak{H}\left(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \theta\left(\frac{\acute{l}}{\bar{\alpha}}\right)\right),$$

$$\mathfrak{S}_\mathfrak{J}((\mathcal{U}, \Phi)\breve{\varsigma}_{\xi_i}, (\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \theta(\acute{l})) \tilde{\leq} \mathfrak{S}_\mathfrak{J}\left(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \theta\left(\frac{\acute{l}}{\bar{\alpha}}\right)\right), \forall \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_\Upsilon, \acute{l} \tilde{>} \bar{0},$$

where θ is a χ-function.
According to conditions (i) and (iv) outlined in Definition (2.11), for any $\acute{l} > \bar{0}$, there exists a $\breve{q} > \bar{0}$ such that $\acute{l} > \theta(\breve{q})$. As a result, we conclude

$$\begin{aligned}
\mathfrak{S}_\mathfrak{G}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) &\tilde{\geq} \mathfrak{S}_\mathfrak{G}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \theta(\breve{q}/\bar{\alpha}m)),\\
\mathfrak{S}_\mathfrak{H}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) &\tilde{\leq} \mathfrak{S}_\mathfrak{H}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \theta(\breve{q}/\bar{\alpha}m)),\\
\mathfrak{S}_\mathfrak{J}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) &\tilde{\leq} \mathfrak{S}_\mathfrak{J}(\breve{\varsigma}^0_{\xi_i}, \breve{\varsigma}^1_{\xi_i}, \theta(\breve{q}/\bar{\alpha}m)).
\end{aligned} \tag{3.7}$$

We let $m \to \infty$ in condition (3.7). Then, $\mathfrak{S}_\mathfrak{G}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) \to \bar{1}, \mathfrak{S}_\mathfrak{H}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) \to \bar{0}$, $\mathfrak{S}_\mathfrak{J}(\breve{\varsigma}^m_{\xi_i}, \breve{\varsigma}^{m+1}_{\xi_i}, \acute{l}) \to \bar{0}$.
Given that $\{\breve{\varsigma}^m_{\xi_i}\}$ is convergent, there exists a soft point $\breve{\nu}_{\xi_j} \in \tilde{\Gamma}_\Upsilon$ such that $\{\breve{\varsigma}^m_{\xi_i}\} \to \breve{\nu}_{\xi_j}$, meaning that,

$$\mathfrak{S}_\mathfrak{G}(\breve{\varsigma}^m_{\xi_i}, \breve{\nu}^t_{p_j}, \acute{l}) \to \bar{1}, \mathfrak{S}_\mathfrak{H}(\breve{\varsigma}^m_{\xi_i}, \breve{\nu}^t_{p_j}, \acute{l}) \to \bar{0}, \mathfrak{S}_\mathfrak{J}(\breve{\varsigma}^m_{\xi_i}, \breve{\nu}^t_{p_j}, \acute{l}) \to \bar{0} \text{ as } m \to \infty. \tag{3.8}$$

Thus,

$$\begin{aligned}
\mathfrak{S}_\mathfrak{G}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) &\tilde{\geq} \mathfrak{S}_\mathfrak{G}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, (\mathcal{U}, \Phi)\breve{\varsigma}^m_{\xi_i}, \acute{l}/\bar{2}) \ddot{\otimes} \mathfrak{S}_\mathfrak{G}((\mathcal{U}, \Phi)\breve{\varsigma}^m_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),\\
&\tilde{\geq} \mathfrak{S}_\mathfrak{G}(\breve{\nu}_{\xi_j}, \breve{\varsigma}^m_{\xi_i}, \theta(\breve{q}/\bar{2}\bar{\alpha})) \ddot{\otimes} \mathfrak{S}_\mathfrak{G}(\breve{\varsigma}^{m+1}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{S}_\mathfrak{H}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) &\tilde{\leq} \mathfrak{S}_\mathfrak{H}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, (\mathcal{U}, \Phi)\breve{\varsigma}^m_{\xi_i}, \acute{l}/\bar{2}) \ddot{\oplus} \mathfrak{S}_\mathfrak{H}((\mathcal{U}, \Phi)\breve{\varsigma}^m_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),\\
&\tilde{\leq} \mathfrak{S}_\mathfrak{H}(\breve{\nu}_{\xi_j}, \breve{\varsigma}^m_{\xi_i}, \theta(\breve{q}/\bar{2}\bar{\alpha})) \ddot{\oplus} \mathfrak{S}_\mathfrak{H}(\breve{\varsigma}^{m+1}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{S}_\mathfrak{J}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) &\tilde{\leq} \mathfrak{S}_\mathfrak{J}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, (\mathcal{U}, \Phi)\breve{\varsigma}^m_{\xi_i}, \acute{l}/\bar{2}) \ddot{\oplus} \mathfrak{S}_\mathfrak{J}((\mathcal{U}, \Phi)\breve{\varsigma}^m_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}),\\
&\tilde{\leq} \mathfrak{S}_\mathfrak{J}(\breve{\nu}_{\xi_j}, \breve{\varsigma}^m_{\xi_i}, \theta(\breve{q}/\bar{2}\bar{\alpha})) \ddot{\oplus} \mathfrak{S}_\mathfrak{J}(\breve{\varsigma}^{m+1}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}/\bar{2}).
\end{aligned}$$

From (3.8) and the fact that $\ddot{\otimes}$ is a CSTN, $\ddot{\oplus}$ is CSTCN
$\mathfrak{S}_\mathfrak{G}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) \to \bar{1}, \mathfrak{S}_\mathfrak{H}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) \to \bar{0}, \mathfrak{S}_\mathfrak{J}((\mathcal{U}, \Phi)\breve{\nu}_{\xi_j}, \breve{\nu}_{\xi_j}, \acute{l}) \to \bar{0}$ as $m \to \infty$.
Therefore, $\breve{\nu}_{\xi_j}$ is a fixed point of $(\mathcal{U}, \Phi)$. Lastly, the uniqueness of the soft fixed point for the χ-contraction map $(\mathcal{U}, \Phi)$ on $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus})$ can be easily confirmed. □

Theorem 3.8. *Considering* $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus})$ *as a complete SNMS with* $\tilde{\beta}\ddot{\otimes}\tilde{\delta} = \min\{\tilde{\beta}, \tilde{\delta}\}, \tilde{\beta}\ddot{\oplus}\tilde{\delta} =$ $\max\{\tilde{\beta}, \tilde{\delta}\}$ *and a* χ*-contraction* $(\mathcal{U}, \Phi) : (\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus}) \to (\tilde{\Gamma}_\Upsilon, \mathfrak{S}_\mathfrak{G}, \mathfrak{S}_\mathfrak{H}, \mathfrak{S}_\mathfrak{J}, \ddot{\otimes}, \ddot{\oplus})$*, then* $(\mathcal{U}, \Phi)$ *possesses a unique soft fixed point.*

Proof. We begin with a soft point $\breve{\varsigma}_{\xi_i}^{0}$ in $\tilde{\Gamma}_{\Upsilon}$ and generate a soft sequence $\{\breve{\varsigma}_{\xi_i}^{m}\}$ defined by $\breve{\varsigma}_{\xi_i}^{m} = (\mathcal{U}, \Phi)^m \breve{\varsigma}_{\xi_i}^{0}$. According to Theorem (3.7), the proof is concluded, establishing that $\{\breve{\varsigma}_{\xi_i}^{m}\}$ is a Cauchy soft sequence. Now, suppose that $\{\breve{\varsigma}_{\xi_i}^{m}\}$ is not a Cauchy soft sequence. In this case, there exist soft real numbers $\acute{l} > \bar{0}$ and $\tilde{\tau} > \bar{0}$, such that for any $N_0 \in \mathbb{Z}^+$, there exist $m(N_0), t(N_0) \geq N_0$ that satisfy:

$$\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \acute{l})\tilde{>}\bar{1} \ominus \tilde{\tau}, \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \acute{l})\tilde{<}\bar{0} \ominus \tilde{\tau}, \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \acute{l})\tilde{<}\bar{0} \ominus \tilde{\tau}, \tag{3.9}$$

choosing $m(N_0) < t(N_0)$. In this case, $t(N_0)$ represents the smallest positive integer concerning $m(N_0)$ that fulfills condition (3.9).
Consequently, there are $\acute{l} > \bar{0}$ and $\tilde{\tau} > \bar{0}$ for which two increasing sequences $\{t(N_0)\}$ and $\{m(N_0)\}$, with $t(N_0) > m(N_0)$, can be generated, satisfying the following:

$$\begin{aligned}
&\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \acute{l})\tilde{\leq}\bar{1} \ominus \tilde{\tau},\\
&\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \acute{l})\tilde{\geq}\bar{0} \ominus \tilde{\tau},\\
&\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \acute{l})\tilde{\geq}\bar{0} \ominus \tilde{\tau},
\end{aligned} \tag{3.10}$$

and

$$\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \acute{l})\tilde{>}\bar{1} \ominus \tilde{\tau}, \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \acute{l})\tilde{<}\bar{0} \ominus \tilde{\tau}, \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \acute{l})\tilde{<}\bar{0} \ominus \tilde{\tau}. \tag{3.11}$$

To construct these sequences, we need to locate a soft point $\breve{\varsigma}_{\xi_i}^{t(N_0)}$ such that:

$$\begin{aligned}
&\breve{\varsigma}_{\xi_i}^{t(N_0)} \tilde{\notin} \in \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\nu}_{\xi_j}, \acute{l})\tilde{\leq}\bar{1} \ominus \tilde{\tau}\}, \ \breve{\varsigma}_{\xi_i}^{t(N_0)-1} \tilde{\in} \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\nu}_{\xi_j}, \acute{l})\tilde{\leq}\bar{1} \ominus \tilde{\tau}\},\\
&\breve{\varsigma}_{\xi_i}^{t(N_0)} \tilde{\notin} \in \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\nu}_{\xi_j}, \acute{l})\tilde{\geq}\bar{0} \ominus \tilde{\tau}\}, \ \breve{\varsigma}_{\xi_i}^{t(N_0)-1} \tilde{\in} \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\nu}_{\xi_j}, \acute{l})\tilde{\geq}\bar{0} \ominus \tilde{\tau}\},\\
&\breve{\varsigma}_{\xi_i}^{t(N_0)} \tilde{\notin} \in \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\nu}_{\xi_j}, \acute{l})\tilde{\geq}\bar{0} \ominus \tilde{\tau}\}, \ \breve{\varsigma}_{\xi_i}^{t(N_0)-1} \tilde{\in} \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\nu}_{\xi_j}, \acute{l})\tilde{\geq}\bar{0} \ominus \tilde{\tau}\}.
\end{aligned}$$

The construction of such a sequence is feasible since we assumed that $\{\breve{\varsigma}_{\xi_i}^{m}\}$ is not a Cauchy soft sequence.
Since for $\breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_{\Upsilon}$, $\tilde{\tau} \tilde{>} \bar{0}$ and $\bar{0} \tilde{<} \acute{l}_1 \tilde{<} \acute{l}_2$,

$$\begin{aligned}
&\{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}_1)\tilde{\leq}\bar{1} \ominus \tilde{\tau}\} \tilde{\subset} \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}_2)\tilde{\leq}\bar{1} \ominus \tilde{\tau}\},\\
&\{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}_1)\tilde{\geq}\bar{1} \ominus \tilde{\tau}\} \tilde{\supset} \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}_2)\tilde{\geq}\bar{1} \ominus \tilde{\tau}\},\\
&\{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}_1)\tilde{\geq}\bar{1} \ominus \tilde{\tau}\} \tilde{\supset} \{\breve{\nu}_{\xi_j} : \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}_2)\tilde{\geq}\bar{1} \ominus \tilde{\tau}\},
\end{aligned}$$

Therefore, whenever it is possible to form such a sequence for $\acute{l} > \bar{0}$ and $\tilde{\tau} > \bar{0}$, the construction of $\{\breve{\varsigma}_{\xi_i}^{m(N_0)}\}$ and $\{\breve{\varsigma}_{\xi_i}^{t(N_0)}\}$ satisfies Conditions (3.10) and (3.11) for any $\breve{q} > \bar{0}$, where $\breve{q} < \acute{l}$. Since θ is a χ-function, for any $\acute{l} > \bar{0}$, there exists $\breve{q} > \bar{0}$ such that $\acute{l} > \theta(\breve{q})$. Therefore, we choose $\acute{l}$ in (3.10) and (3.11) as $\acute{l} = \theta(\acute{l}_1)$ for some $\acute{l}_1 > \bar{0}$ such that $\theta(\acute{l}_1/\breve{\alpha}) > \theta(\acute{l}_1)$. This selection is feasible due to the conditions i) and iv) stipulated in Definition (2.11)..
From Conditions (3.10) and (3.11), we derive:

$$\begin{aligned}
&\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1))\tilde{\leq}\bar{1} \ominus \tilde{\tau},\\
&\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1))\tilde{\geq}\bar{0} \ominus \tilde{\tau},\\
&\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1))\tilde{\geq}\bar{0} \ominus \tilde{\tau},
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \theta(\acute{l}_1))\tilde{>}\bar{1} \ominus \tilde{\tau},\\
&\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \theta(\acute{l}_1))\tilde{<}\bar{0} \ominus \tilde{\tau},\\
&\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \theta(\acute{l}_1))\tilde{<}\bar{0} \ominus \tilde{\tau}.
\end{aligned} \tag{3.13}$$

Thus,

$$\begin{aligned}
\bar{1} \ominus \tilde{\tau} \tilde{<} & \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \theta(\acute{l}_1)),\\
& \tilde{\leq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})), \text{ or}\\
\bar{1} \ominus \tilde{\tau} \tilde{<} & \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})),\\
\bar{0} \ominus \tilde{\tau} \tilde{>} & \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \theta(\acute{l}_1)),\\
& \tilde{\geq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})), \text{ or}\\
\bar{0} \ominus \tilde{\tau} \tilde{>} & \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})),\\
\bar{0} \ominus \tilde{\tau} \tilde{>} & \mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)}, \theta(\acute{l}_1)),\\
& \tilde{\geq} \mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})), \text{ or}\\
\bar{0} \ominus \tilde{\tau} \tilde{>} & \mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})).
\end{aligned}$$

As $\theta(\acute{l}_1/\bar{\alpha}) \tilde{>} \theta(\acute{l}_1)$, choosing $\tilde{\Gamma}$ as $\tilde{\Gamma} \tilde{<} \{\theta(\acute{l}_1/\bar{\alpha}) \ominus \theta(\acute{l}_1)\}$.
This means $\theta(\acute{l}_1/\bar{\alpha}) \ominus \tilde{\Gamma} \tilde{>} \theta(\acute{l}_1)$.
Utilizing Condition (3.7) in Theorem (3.7), we select N_0 to be sufficiently large such that:

$$\begin{aligned}
&\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \tilde{\Gamma}) \tilde{<} \bar{1} \ominus \tilde{\tau}_1,\\
&\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \tilde{\Gamma}) \tilde{>} \bar{0} \ominus \tilde{\tau}_1,\\
&\mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \tilde{\Gamma}) \tilde{>} \bar{0} \ominus \tilde{\tau}_1 \text{ for } \bar{0} \tilde{<} \tilde{\tau}_1 \tilde{<} \tilde{\tau}.
\end{aligned} \tag{3.14}$$

With this selection of N_0 and $\tilde{\Gamma}$, and considering Conditions (3.12) to (3.14), we deduce:

$$\begin{aligned}
\bar{1} \ominus \tilde{\tau} & \tilde{<} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})),\\
& \tilde{\leq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, (\theta(\acute{l}_1/\bar{\alpha}) \ominus \tilde{\Gamma})) \ddot{\otimes} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{m(N_0)}, \tilde{\Gamma}),\\
& \tilde{\leq} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1)) \ddot{\otimes} \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{m(N_0)}, \tilde{\Gamma}),\\
& \tilde{\leq} (\bar{1} \ominus \tilde{\tau}) \ddot{\otimes} (\bar{1} \ominus \tilde{\tau}_1),\\
\bar{0} \ominus \tilde{\tau} & \tilde{>} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})),\\
& \tilde{\geq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, (\theta(\acute{l}_1/\bar{\alpha}) \ominus \tilde{\Gamma})) \ddot{\oplus} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{m(N_0)}, \tilde{\Gamma}),\\
& \tilde{\geq} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1)) \ddot{\oplus} \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{m(N_0)}, \tilde{\Gamma}),\\
& \tilde{\geq} (\bar{1} \ominus \tilde{\tau}) \ddot{\oplus} (\bar{1} \ominus \tilde{\tau}_1),\\
\bar{0} \ominus \tilde{\tau} & \tilde{>} \mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1/\bar{\alpha})),\\
& \tilde{\geq} \mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, (\theta(\acute{l}_1/\bar{\alpha}) \ominus \tilde{\Gamma})) \ddot{\oplus} \mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{m(N_0)}, \tilde{\Gamma}),\\
& \tilde{\geq} \mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)}, \breve{\varsigma}_{\xi_i}^{t(N_0)-1}, \theta(\acute{l}_1)) \ddot{\oplus} \mathfrak{S}_{\mathfrak{I}}(\breve{\varsigma}_{\xi_i}^{m(N_0)-1}, \breve{\varsigma}_{\xi_i}^{m(N_0)}, \tilde{\Gamma}),\\
& \tilde{\geq} (\bar{1} \ominus \tilde{\tau}) \ddot{\oplus} (\bar{1} \ominus \tilde{\tau}_1),
\end{aligned}$$

and using the fact that $\tilde{\tau}_1 \tilde{<} \tilde{\tau}$, we have $(\bar{1} \ominus \tilde{\tau}) \tilde{<} (\bar{1} \ominus \tilde{\tau}_1)$.
This introduces a contradiction. As a result, $\{\breve{\varsigma}_{\xi_i}^{m}\}$ is Cauchy. The proof follows Theorem (3.7) after that. □

4 Illustrations

Example 4.1. Considering, a set $\Gamma = \{r, s, t\}$ and a parameter set $\Upsilon = \{a, b\}$ with a CSTN and CSTCN defined as $\breve{u} \ddot{\otimes} \breve{v} = \min\{\breve{u}, \breve{v}\}$ and $\breve{u} \ddot{\oplus} \breve{v} = \max\{\breve{u}, \breve{v}\}$. Then, $\mathcal{SP}(\tilde{\Gamma}_{\Upsilon}) =$

$\{\breve{\varsigma}_\iota, \breve{\varsigma}_\mu, \breve{\nu}_a, \breve{\nu}_b, \breve{\varpi}_\iota, \breve{\varpi}_\mu\}$.
We define $\mathfrak{S}_{\mathfrak{G}} : \mathcal{SP}(\tilde{\Gamma}_\Upsilon) \times \mathcal{SP}(\tilde{\Gamma}_\Upsilon) \times (0,\infty)(\Upsilon) \to [0,1](\Upsilon)$ as follows: for any $p_i, p_j \in \Upsilon$,

$$\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{G}}(\breve{\nu}_{\xi_j}, \breve{\varsigma}_{\xi_i}, \acute{l}) = \begin{cases} \bar{0} & \text{if} \quad \acute{l} = \bar{0} \\ 0.8 & \text{if} \quad \bar{0} \tilde{<} \acute{l} \tilde{\leq} \bar{2} \\ \bar{1} & \text{if} \quad \acute{l} > \bar{2}, \end{cases},$$

$$\mathfrak{S}_{\mathfrak{G}}(\breve{\varpi}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{G}}(\breve{\nu}_{\xi_j}, \breve{\varpi}_{\xi_i}, \acute{l}) = \mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\varpi}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{G}}(\breve{\varpi}_{\xi_j}, \breve{\varsigma}_{\xi_i}, \acute{l}) = \begin{cases} \bar{0} & \text{if} \quad \acute{l} = \bar{0} \\ 0.5 & \text{if} \quad \bar{0} \tilde{<} \acute{l} \tilde{\leq} \bar{4} \\ \bar{1} & \text{if} \quad \acute{l} \tilde{>} \bar{4}, \end{cases}$$

$$\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{1} \leftrightarrow \breve{\varsigma}_{\xi_i} = \breve{\nu}_{\xi_j} \quad \text{for all } \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_\Upsilon \text{ and } \acute{l} \tilde{>} \bar{0}.$$

$$\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{H}}(\breve{\nu}_{\xi_j}, \breve{\varsigma}_{\xi_i}, \acute{l}) = \begin{cases} \bar{0}, & \text{if } \acute{l} = \bar{1}, \\ 0.1, & \text{if } \bar{0} \tilde{<} \acute{l} \tilde{\leq} \bar{2}, \\ \bar{1}, & \text{if } \acute{l} > \bar{2}, \end{cases}$$

$$\mathfrak{S}_{\mathfrak{H}}(\breve{\varpi}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{H}}(\breve{\nu}_{\xi_j}, \breve{\varpi}_{\xi_i}, \acute{l}) = \mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\varpi}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{H}}(\breve{\varpi}_{\xi_j}, \breve{\varsigma}_{\xi_i}, \acute{l}) = \begin{cases} \bar{0}, & \text{if } \acute{l} = \bar{1}, \\ 0.4, & \text{if } \bar{1} \tilde{<} \acute{l} \tilde{\leq} \bar{4}, \\ \bar{1}, & \text{if } \acute{l} \tilde{>} \bar{4}, \end{cases}$$

$$\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{0} \leftrightarrow \breve{\varsigma}_{\xi_i} = \breve{\nu}_{\xi_j} \quad \text{for all } \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_\Upsilon \text{ and } \acute{l} \tilde{>} \bar{0}.$$

$$\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{J}}(\breve{\nu}_{\xi_j}, \breve{\varsigma}_{\xi_i}, \acute{l}) = \begin{cases} \bar{0}, & \text{if } \acute{l} = \bar{1}, \\ 0.6, & \text{if } \bar{0} \tilde{<} \acute{l} \tilde{\leq} \bar{2}, \\ \bar{1}, & \text{if } \acute{l} > \bar{2}, \end{cases}$$

$$\mathfrak{S}_{\mathfrak{J}}(\breve{\varpi}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{J}}(\breve{\nu}_{\xi_j}, \breve{\varpi}_{\xi_i}, \acute{l}) = \mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\varpi}_{\xi_j}, \acute{l}) = \mathfrak{S}_{\mathfrak{J}}(\breve{\varpi}_{\xi_j}, \breve{\varsigma}_{\xi_i}, \acute{l}) = \begin{cases} \bar{0}, & \text{if } \acute{l} = \bar{1}, \\ 0.5, & \text{if } \bar{1} \tilde{<} \acute{l} \tilde{\leq} \bar{4}, \\ \bar{1}, & \text{if } \acute{l} \tilde{>} \bar{4}, \end{cases}$$

$$\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j}, \acute{l}) = \bar{0} \leftrightarrow \breve{\varsigma}_{\xi_i} = \breve{\nu}_{\xi_j} \quad \text{for all } \breve{\varsigma}_{\xi_i}, \breve{\nu}_{\xi_j} \tilde{\in} \tilde{\Gamma}_\Upsilon \text{ and } \acute{l} \tilde{>} \bar{0}.$$

Therefore, $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ forms a complete SNMS. Next, consider a soft self-mapping $(\mathcal{U}, \Phi)$ defined on $\tilde{\Gamma}_\Upsilon$ as follows: $(\mathcal{U}, \Phi)(\breve{\varsigma}_\iota) = \breve{\nu}_a, (\mathcal{U}, \Phi)(\breve{\varsigma}_\mu) = \breve{\nu}_b, (\mathcal{U}, \Phi)(\breve{\nu}_a) = \breve{\nu}_b$, $(\mathcal{U}, \Phi)(\breve{\nu}_b) = \breve{\nu}_b, (\mathcal{U}, \Phi)(\breve{\varpi}_\iota) = \breve{\varsigma}_\mu, (\mathcal{U}, \Phi)(\breve{\varpi}_\mu) = \breve{\varsigma}_\iota$.
Thus, $(\mathcal{U}, \Phi)$ forms a soft contraction map on the SNMS $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$, satisfying all the conditions specified in Theorem (3.5). Additionally, it has a unique fixed point, denoted as $\breve{\nu}_b$.

Example 4.2. Take a set $\Gamma = \mathcal{U} \cup \mathcal{V}$, where $\mathcal{U} = \left\{\frac{1}{2}, \frac{1}{3}\right\}, \mathcal{V} = [4,5]$, and a parameter set $\Upsilon = \{1, 2\}$. We describe $\Xi : SP(\tilde{\Gamma}_\Upsilon) \times SP(\tilde{\Gamma}_\Upsilon) \to \mathbb{R}(\Upsilon)^*$ as follows: $\Xi(\breve{\varsigma}_p, \breve{\nu}_q) = |\dot{\alpha} - \dot{\beta}| + |\dot{\gamma} - \dot{\delta}| \forall \breve{\varsigma}_p, \breve{\nu}_q \in SP(\tilde{\Gamma}_\Upsilon)$.
In $(\tilde{\Gamma}_\Upsilon, \Xi)$, we define $\breve{\varsigma} \ddot{\otimes} \breve{\nu} = \breve{\varsigma} \circ \breve{\nu}$ or $\breve{\varsigma} \ddot{\otimes} \breve{\nu} = \min\{\breve{\varsigma}, \breve{\nu}\}$ and $\breve{\varsigma} \ddot{\otimes} \breve{\nu} = \max\{\breve{\varsigma}, \breve{\nu}\}$. We define $\mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}} : SP(\tilde{\Gamma}_\Upsilon) \times SP(\tilde{\Gamma}_\Upsilon) \times (0,\infty)(\Upsilon) \to [0,1](\Upsilon)$ as follows: $\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_p, \breve{\nu}_q, \acute{l}) = \frac{\acute{l}}{\acute{l} \oplus \Xi(\breve{\varsigma}_p, \breve{\nu}_q)}$, $\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_p, \breve{\nu}_q, \acute{l}) = \frac{\Xi(\breve{\varsigma}_p, \breve{\nu}_q)}{\acute{l} \oplus \Xi(\breve{\varsigma}_p, \breve{\nu}_q)}$, $\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_p, \breve{\nu}_q, \acute{l}) = \frac{\Xi(\breve{\varsigma}_p, \breve{\nu}_q)}{\acute{l}}$ for each $\breve{\varsigma}_p, \breve{\nu}_q \tilde{\in} \tilde{\Gamma}_\Upsilon$ and $\acute{l} \tilde{>} \bar{0}$.
Then, $(\tilde{\Gamma}_\Upsilon, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ is a complete SNMS.

Now, we consider map $(\mathcal{U}, \Phi) : \tilde{\Gamma}_\Upsilon \to \tilde{\Gamma}_\Upsilon$ defined by $(\mathcal{U}, \Phi)(\breve{\varsigma}_p) = \begin{cases} \left(\frac{\tilde{1}}{2}\right)_1 & \text{if} \qquad \breve{\varsigma}_p \in SP(\tilde{V}_P) \\ \left(\frac{\tilde{1}}{3}\right)_2 & \text{otherwise.} \end{cases}$.

Then, $(\mathcal{U}, \Phi)$ is a soft contraction map on SNMS $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ and it follows all the conditions specified in Theorem (3.5). Moreover, it admits only one fixed point, i.e., $\left(\frac{\tilde{1}}{3}\right)_2$.

Example 4.3. Cconsider $\Gamma = \left\{\frac{5}{8}, \frac{3}{4}, \frac{8}{9}\right\}$ and parameter set $\Upsilon = \{1, 2\}$ with a soft t-norm and soft t-conorm defined as $\breve{u}\ddot{\otimes}\breve{v} = \min\{\breve{u}, \breve{v}\}$ and $\breve{u}\ddot{\oplus}\breve{v} = \min\{\breve{u}, \breve{v}\}$ for $\breve{u}, \breve{v} \in [0,1](\Upsilon)$. Then, $SP(\tilde{\Gamma}_{\Upsilon}) = \left\{\frac{\tilde{5}}{81}, \frac{\tilde{5}}{82}, \frac{\tilde{3}}{41}, \frac{\tilde{3}}{42}, \frac{\tilde{8}}{91}, \frac{\tilde{8}}{92}\right\}$. We define $\mathfrak{S}_{\mathfrak{G}} : SP(\tilde{\Gamma}_{\Upsilon}) \times SP(\tilde{\Gamma}_{\Upsilon}) \times (0, \infty)(\Upsilon) \to [0,1](\Upsilon)$ as follows: for any $p, q \in \Upsilon$,

$$\mathfrak{S}_{\mathfrak{G}}\left(\frac{\tilde{5}}{8_p}, \frac{\tilde{3}}{4_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{G}}\left(\frac{\tilde{3}}{4_q}, \frac{\tilde{5}}{8_p}, \acute{l}\right) = \begin{cases} \bar{0} & \text{if} \quad \bar{k} = \bar{0} \\ 0\bar{.}9 & \text{if} \quad \bar{0}\tilde{<}\acute{l}\tilde{\leq}\bar{3} \\ \bar{1} & \text{if} \quad \acute{l} > \bar{3}, \end{cases}$$

$$\mathfrak{S}_{\mathfrak{G}}\left(\frac{\tilde{8}}{9_p}, \frac{\tilde{3}}{4_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{G}}\left(\frac{\tilde{3}}{4_q}, \frac{\tilde{8}}{9_p}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{G}}\left(\frac{\tilde{5}}{8p}, \frac{\tilde{8}}{9_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{G}}\left(\frac{\tilde{8}}{9_q}, \frac{\tilde{5}}{8_p}, \acute{l}\right) = \begin{cases} \bar{0} & \text{if}\, \acute{l} = \bar{0} \\ 0\bar{.}6 & \text{if}\, \bar{0}\tilde{<}\acute{l}\tilde{\leq}\bar{8} \\ \bar{1}\,\text{if}\, \acute{l} > \bar{8}, \end{cases}$$

$\mathfrak{S}_{\mathfrak{G}}(\breve{\varsigma}_p, \breve{\nu}_q, \acute{l}) = \bar{1} \leftrightarrow \breve{\varsigma}_p = \breve{\nu}_p$, for all $\breve{\varsigma}_p, \breve{\nu}_q \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l}\tilde{>}\bar{0}$.
We define $\mathfrak{S}_{\mathfrak{H}} : SP(\tilde{\Gamma}_{\Upsilon}) \times SP(\tilde{\Gamma}_{\Upsilon}) \times (0, \infty)(\Upsilon) \to [0,1](\Upsilon)$ as follows: for any $p, q \in \Upsilon$,

$$\mathfrak{S}_{\mathfrak{H}}\left(\frac{\tilde{5}}{8_p}, \frac{\tilde{3}}{4_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{H}}\left(\frac{\tilde{3}}{4_q}, \frac{\tilde{5}}{8_p}, \acute{l}\right) = \begin{cases} \bar{1} & \text{if} \quad \bar{k} = \bar{0} \\ 0\bar{.}1 & \text{if} \quad \bar{0}\tilde{<}\acute{l}\tilde{\leq}\bar{3} \\ \bar{0} & \text{if} \quad \acute{l} > \bar{3}, \end{cases}$$

$$\mathfrak{S}_{\mathfrak{H}}\left(\frac{\tilde{8}}{9_p}, \frac{\tilde{3}}{4_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{H}}\left(\frac{\tilde{3}}{4_q}, \frac{\tilde{8}}{9_p}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{H}}\left(\frac{\tilde{5}}{8p}, \frac{\tilde{8}}{9_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{H}}\left(\frac{\tilde{8}}{9_q}, \frac{\tilde{5}}{8_p}, \acute{l}\right) = \begin{cases} \bar{1} & \text{if}\, \acute{l} = \bar{0} \\ 0\bar{.}2 & \text{if}\, \bar{0}\tilde{<}\acute{l}\tilde{\leq}\bar{8} \\ \bar{0}\,\text{if}\, \acute{l} > \bar{8}, \end{cases}$$

$\mathfrak{S}_{\mathfrak{H}}(\breve{\varsigma}_p, \breve{\nu}_q, \acute{l}) = \bar{0} \leftrightarrow \breve{\varsigma}_p = \breve{\nu}_p$, for all $\breve{\varsigma}_p, \breve{\nu}_q \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l}\tilde{>}\bar{0}$.
We define $\mathfrak{S}_{\mathfrak{J}} : SP(\tilde{\Gamma}_{\Upsilon}) \times SP(\tilde{\Gamma}_{\Upsilon}) \times (0, \infty)(\Upsilon) \to [0,1](\Upsilon)$ as follows: for any $p, q \in \Upsilon$,

$$\mathfrak{S}_{\mathfrak{J}}\left(\frac{\tilde{5}}{8_p}, \frac{\tilde{3}}{4_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{J}}\left(\frac{\tilde{3}}{4_q}, \frac{\tilde{5}}{8_p}, \acute{l}\right) = \begin{cases} \bar{1} & \text{if} \quad \bar{k} = \bar{0} \\ 0\bar{.}4 & \text{if} \quad \bar{0}\tilde{<}\acute{l}\tilde{\leq}\bar{3} \\ \bar{0} & \text{if} \quad \acute{l} > \bar{3}, \end{cases}$$

$$\mathfrak{S}_{\mathfrak{J}}\left(\frac{\tilde{8}}{9_p}, \frac{\tilde{3}}{4_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{J}}\left(\frac{\tilde{3}}{4_q}, \frac{\tilde{8}}{9_p}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{J}}\left(\frac{\tilde{5}}{8p}, \frac{\tilde{8}}{9_q}, \acute{l}\right) = \mathfrak{S}_{\mathfrak{J}}\left(\frac{\tilde{8}}{9_q}, \frac{\tilde{5}}{8_p}, \acute{l}\right) = \begin{cases} \bar{1} & \text{if}\, \acute{l} = \bar{0} \\ 0\bar{.}6 & \text{if}\, \bar{0}\tilde{<}\acute{l}\tilde{\leq}\bar{8} \\ \bar{0}\,\text{if}\, \acute{l} > \bar{8}, \end{cases}$$

$\mathfrak{S}_{\mathfrak{J}}(\breve{\varsigma}_p, \breve{\nu}_q, \acute{l}) = \bar{0} \leftrightarrow \breve{\varsigma}_p = \breve{\nu}_p$, for all $\breve{\varsigma}_p, \breve{\nu}_q \tilde{\in} \tilde{\Gamma}_{\Upsilon}, \acute{l}\tilde{>}\bar{0}$.

Then, $(\tilde{\Gamma}_{\Upsilon}, \mathfrak{S}_{\mathfrak{G}}, \mathfrak{S}_{\mathfrak{H}}, \mathfrak{S}_{\mathfrak{J}}, \ddot{\otimes}, \ddot{\oplus})$ *is a complete SNMS.*
Now, take a soft self-map $(\mathcal{U}, \Phi)$ *on* $\tilde{\Gamma}_{\Upsilon}$ *as* $(\mathcal{U}, \Phi)\left(\frac{\tilde{5}}{81}\right) = \frac{\tilde{3}}{41}; (\mathcal{U}, \Phi)\left(\frac{\tilde{5}}{82}\right) = \frac{\tilde{3}}{41};$
$(\mathcal{U}, \Phi)\left(\frac{\tilde{3}}{41}\right) = \frac{\tilde{3}}{41}; (\mathcal{U}, \Phi)\left(\frac{\tilde{3}}{42}\right) = \frac{\tilde{5}}{82};$
$(\mathcal{U}, \Phi)\left(\frac{\tilde{8}}{91}\right) = \frac{\tilde{8}}{92}; (\mathcal{U}, \Phi)\frac{\tilde{8}}{92} = \frac{\tilde{8}}{91}$. *We choose* $\theta(\acute{l}) = \bar{k}^{\frac{1}{3}}$. *This function* θ *fits the criteria for a* χ*-function. Consequently, the mapping* $(\mathcal{U}, \Phi)$ *meets the conditions outlined in Theorem (3.8), and uniquely converges to the fixed point* $\frac{\tilde{3}}{41}$.

5 Conclusion remarks

This work introduces a novel approach to Banach contraction in SNMSs by applying a specific constraint to link the soft points of the absolute soft set. It extends the contraction concept through χ*-contraction mapping, leading to unique fixed points with continuity of soft t-norms and t-conorms. The fixed-point results are supported by examples, demonstrating their practical applications in managing uncertainty. These results can also be extended to various metric spaces, such as soft fuzzy partial and b-metric spaces. The findings have important implications for both theoretical and computational mathematics.*

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