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2-ABSORBING SEMI δ-PRIMARY IDEALS IN LATTICES

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Abstract. In this paper, we introduce the concept of 2-absorbing semi-δ-primary ideals which is an extension of 2-absorbing ideals. A proper ideal I of lattice L is called 2-absorbing semi-δ-primary ideal for a , b, c $\in$ L if a $\wedge$ b $\wedge$ c $\in$ I, then either a $\wedge$ b $\in \delta(I)$ or b $\wedge$ c $\in \delta(I)$ or a $\wedge$ c $\in \delta(I)$. We have obtained many properties and characterizations of semi-δ-primary ideals. In addition, we have studied these concepts in the product of lattices.

1 Introduction

The concept of prime ideals is a fundamental and extensively studied topic in both ring theory and lattice theory. While originating in ring theory, the notion of prime ideals has great importance and found significant applications in lattice theory. A. Badawi and A.Y. Darani have introduced some theorems and characterization of 2-absorbing δ-primary ideals in commutative rings. Wasadikar M.P. and Gaikwad K.T.[7] in 2019 introduced a primary ideal, 2-absorbing primary ideal and a weakly 2-absorbing primary ideal in a lattice. A proper ideal I of a lattice L is called a 2-absorbing primary ideal if whenever a , b, c $\in$ L if a $\wedge$ b $\wedge$ c $\in$ I, then either a $\wedge$ b $\in$ I or b $\wedge$ c $\in \sqrt{I}$ or a $\wedge$ c $\in \sqrt{I}$, where radical of I as the intersection of all prime ideals containing I and it denote as $\sqrt{I}$. Various work on generalization of prime and primary ideals are studied thoroughly in [1], [2]. In the work of Zhao [2], we find expansion of ideals and δ-primary ideals of commutative rings. Furthermore, δ-primary ideals, weakly δ-primary ideals and 2-absorbing δ-primary ideals in a lattice introduced by Nimbhorkar and Nehete [4]-[5]. Later on in 2021 Ece Yetkin Celikel presented paper on 2-absorbing δ-semiprimary ideals of commutative rings with important results [3].

In this paper, we define 2-absorbing semi δ-primary ideals and study some of related properties. Additionally, we define weakly 2-absorbing semi δ-primary ideals in lattices. Also, investigate related properties of a 2-absorbing semi δ-primary ideals with respect to a homomorphism. Furthermore, we studied to define weakly 2-absorbing semi δ-primary ideals in lattices and introduce the notion of a semi-δ-triple-zero. Also, we define strongly weakly 2-absorbing semi δ-primary ideals in lattice. In this Paper L denotes a lattice with a least element 0. It is known Id(L), the set of all ideals of a lattice L, forms a lattice under set inclusion.

2 PRELIMINARIES

The following defined by Nimbhorkar and Nehete [4].

Definition 2.1. An expansion of ideals or an ideal expansion is a function δ : Id(L) $\rightarrow$ Id(L), satisfying the conditions (i) $I \subseteq \delta(I)$ and (ii) $J \subseteq K$ implies $\delta(J) \subseteq \delta(K)$ for all I,J, K $\in$ Id(L).

Example 1. (i) The identity function δ_0 : Id(L) $\rightarrow$ Id(L), where $\delta_0(I) = I$ for every I $\in$ Id(L) is an expansion of ideals.

(ii) For each ideal I define $\delta_1(I) = \sqrt{I} = \bigcap\{P \in Id(L) \mid P$ is a prime ideal where $I \subseteq P\}$ is the radical of I. Then $\delta_1(I)$ is an expansion of ideals.
(iii) For each proper ideal P, the mapping M: Id(L) $\to$ Id(L) defined by
M(P)=$\bigcap\{I \in Id(L) \mid P \subseteq I$, I is a maximal ideal other than L $\}$ and M(L)=L. Then M is an expansion of ideals.

3 Properties of 2-absorbing semi δ-primary ideals

Definition 3.1. Let δ : Id(L) $\to$ Id(L) be an expansion function of lattice. I be a proper ideal of lattice L is called 2-absorbing semi-δ-primary ideal for a , b, c $\in$ L if a $\wedge$ b $\wedge$ c $\in$ I, then either a $\wedge$ b $\in$ δ (I) or b $\wedge$ c $\in$ δ (I) or a $\wedge$ c $\in$ δ (I).

Definition 3.2. Let δ : Id(L) $\to$ Id(L) be an expansion function of lattice. I be a proper ideal of lattice L is called weakly 2-absorbing semi-δ-primary ideal for a , b, c $\in$ L if 0 $\neq$ a $\wedge$ b $\wedge$ c $\in$ I, then either a $\wedge$ b $\in$ δ (I) or b $\wedge$ c $\in$ δ (I) or a $\wedge$ c $\in$ δ (I).

Theorem 3.3. *Let I be a proper ideal of lattice L. Then following statements holds:*

1. *If I is a 2-absorbing semi-δ-primary ideal then I is a weakly 2-absorbing semi-δ-primary ideal.*
2. *I is a 2-absorbing semi-δ_0-primary ideal if and only if I is a 2-absorbing ideal.*
3. *I is a weakly 2-absorbing semi-δ_0-primary ideal if and only if I is a weakly 2-absorbing ideal.*
4. *If I is a (weakly) semi-δ-primary ideal then I is a (weakly) 2-absorbing semi-δ-primary ideal.*
5. *If I is a (weakly) 2-absorbing δ-primary ideal then I is a (weakly) 2-absorbing semi-δ-primary ideal.*
6. *Every 2-absorbing ideal is an 2-absorbing semi-δ-primary ideal but the converse is not necessary true.*
7. *Let δ and γ be two ideal expansion functions with $\delta(I) \subseteq \gamma(I)$. If I is a (weakly) 2-absorbing semi-δ-primary ideal of lattice L then I is a (weakly) 2-absorbing semi-γ-primary ideal of lattice L.*

Example 3.4. Consider the lattice shown in Figure 1.

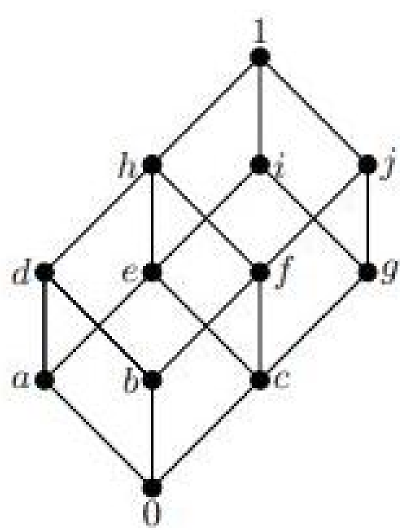

Figure 1. L1

In figure 1, consider the ideal P=(a] is 2-absorbing, weakly 2-absorbing, 2-absorbing primary, weakly 2-absorbing primary, 2-absorbing δ-primary ideal, 2-absorbing semi-δ-primary ideal,weakly 2-absorbing semi-δ-primary ideal. But the ideal T=(c] of the given lattice L1 is neither 2-absorbing δ-primary ideal nor 2-absorbing semi-δ-primary ideal as $i \wedge j \wedge h$ = c $\in T$ but $i \wedge j \notin \delta(T)$ or $j \wedge h \notin \delta(T)$ or $i \wedge h \notin \delta(T)$

Example 3.5. Consider the lattice shown in Figure 2.

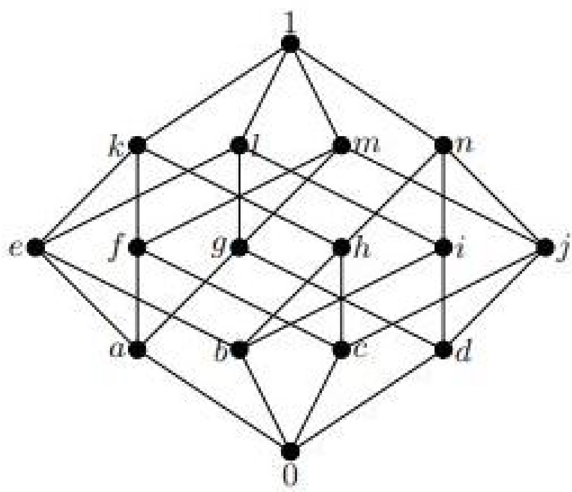

Figure 2. L2

Consider the ideal P=(l] is 2-absorbing δ-primary ideal, 2-absorbing semi-δ-primary ideal, weakly 2-absorbing semi-δ-primary ideal. But the ideal S=(a] of the given lattice L2 is neither 2-absorbing δ-primary ideal nor 2-absorbing semi-δ-primary ideal as $k \wedge l \wedge m$ = a but $k \wedge l = e \notin \delta(S)$ or $l \wedge m = g \notin \delta(S)$ or $k \wedge m = f \notin \delta(S)$.

Example 3.6. Consider the lattice shown in Figure 3.

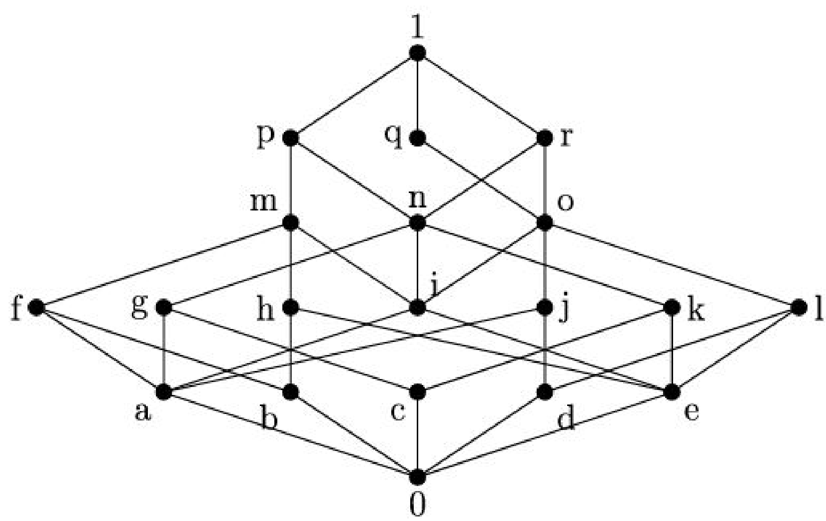

Figure 3. L3

Consider the ideal I=(0] in figure 3. Thus, $\sqrt{I} = (n]$. Here I is not 2-absorbing ideal. However, $i \wedge j \wedge l = 0 \in I$ but $i \wedge j = a \notin I$ or $j \wedge l = d \notin I$ or $i \wedge l = e \notin I$ but it is 2-absorbing semi-δ-primary ideal.

Theorem 3.7. *Let δ be an expansion function of Id(L) and I is a proper ideal of lattice L.*

1. *If δ(I) is a (weakly) 2-absorbing ideal of lattice L then I is a (weakly) 2-absorbing semi-δ-primary ideal of lattice L.*
2. *Let $\delta(\delta(I)) = \delta(I)$. Then $\delta(I)$ is a (weakly) 2-absorbing semi-δ-primary ideal of L iff $\delta(I)$ is a (weakly) 2-absorbing ideal of L. Moreover, if δ (I) is 2-absorbing semi-δ-primary then $\mid Min\ \delta(I) \mid\leq 2$.*

Proof. 1. Suppose that (0 $\neq$ a $\wedge$ b $\wedge$ c $\in$ I) a $\wedge$ b $\wedge$ c $\in$ I and a $\wedge$ b $\notin$ δ (I). Since I $\subseteq$ $\delta(I)$ and $\delta(I)$ is 2-absorbing then we have a $\wedge$ c $\in$ $\delta(I)$ or b $\wedge$ c $\in$ $\delta(I)$. Thus I is a (weakly) 2-absorbing semi-δ-primary ideal.
2. Suppose that $\delta(I)$ is a (weakly) 2-absorbing semi-δ-primary ideal of lattice L. Since $\delta(\delta(I)) = \delta(I)$, $\delta(I)$ is (weakly) 2-absorbing by the definition. The converse is clear from (1). Suppose that $\delta(I)$ is a 2-absorbing semi-δ-primary ideal of L then $\delta(I)$ is 2-absorbing and so $\mid Min\ \delta(I) \mid\leq 2$. □

Theorem 3.8. *Let δ be an intersection preserving expansion function of Id(L). If $I_1, I_2, \ldots, I_n$ are 2-absorbing semi-δ-primary ideals of L with $\delta(I_i) = K$ for all $i \in 1, 2, \ldots n$ then $I = \bigcap_{i=1}^{n} I_i$ is a 2-absorbing semi-δ-primary ideal of L.*

Proof. Suppose a $\wedge$ b $\wedge$ c $\in$ I then a $\wedge$ b $\notin$ δ (I) and b $\wedge$ c $\notin$ δ (I) for some $a, b, c \in L$ since $\delta(I) = \delta(\bigcap_{i=1}^{n} I_i) = \bigcap_{i=1}^{n} \delta(I_i) = K$, we have a $\wedge$ b $\notin$ K and b $\wedge$ c $\notin$ K, As a $\wedge$ b $\wedge$ c $\in$ I_i and I_i is 2-absorbing semi-δ-primary a $\wedge$ c $\in$ $\delta(I_i) = K = \delta(I)$. □

Theorem 3.9. *Let δ be an expansion function of ideals of lattice L and I is (weakly) 2-absorbing semi-δ-primary ideal of L.*
If $J \subseteq I$ and $\delta(J) = \delta(I)$ then J is a (weakly) 2-absorbing semi-δ-primary ideal of L.

Proof. Assume that (0 $\neq$ a $\wedge$ b $\wedge$ c $\in$ J) a $\wedge$ b $\wedge$ c $\in$ J for some a, b, c $\in$ L. Since $J \subseteq I$, we have (0 $\neq$ a $\wedge$ b $\wedge$ c $\in$ I) a $\wedge$ b $\wedge$ c $\in$ I. As I is (weakly) 2-absorbing semi-δ-primary then we have a $\wedge$ b $\in$ δ (I) or b $\wedge$ c $\in$ δ (I) or a $\wedge$ c $\in$ δ (I). Since $\delta(I) = \delta(J)$, Here J is a (weakly) 2-absorbing semi-δ-primary ideal of lattice L. □

Theorem 3.10. *Let δ be an expansion of Id(L). Every proper principal ideal is a 2-absorbing semi-δ-primary ideal of L iff every proper ideal is a 2-absorbing semi-δ-primary ideal of lattice L.*

Proof. Suppose that every proper principal ideal is a 2-absorbing semi-δ-primary ideal of lattice L. Let I be a proper ideal of L and $a, b, c \in L$ with a $\wedge$ b $\wedge$ c $\in$ I. Then a $\wedge$ b $\wedge$ c $\in$ (a $\wedge$ b $\wedge$ c), Since (a $\wedge$ b $\wedge$ c) is 2-absorbing semi-δ-primary ideal of L with assumption we have either a $\wedge$ b $\in$ δ (a $\wedge$ b $\wedge$ c) $\subseteq$ δ (I) or b $\wedge$ c $\in$ δ (a $\wedge$ b $\wedge$ c) $\subseteq$ δ (I) or a $\wedge$ c $\in$ δ (a $\wedge$ b $\wedge$ c) $\subseteq$ δ (I). Thus, I is a 2-absorbing semi-δ-primary ideal of lattice L. □

Definition 3.11. Let δ be an expansion function of Id(L) and I is a weakly 2-absorbing semi-δ-primary ideal of lattice L. We have triple (a, b, c) a semi-δ-triple zero of I if a $\wedge$ b $\wedge$ c = 0 for some elements a, b, c $\in$ L and neither a $\wedge$ b $\in$ $\delta(I)$ nor b $\wedge$ c $\in$ $\delta(I)$ nor a $\wedge$ c $\in$ $\delta(I)$.

Remark 3.12. Let δ be an expansion function of Id(L) and I a weakly 2-absorbing semi-δ-primary ideal of L. Then I is not 2-absorbing semi-δ-primary ideal of L if and only if there exist at least one semi-δ-triple zero of I.

Theorem 3.13. *Let δ be an expansion function of Id(L) and I weakly 2-absorbing semi-δ-primary ideal of lattice L. If I is not a 2-absorbing semi-δ-primary ideal of L then $I^3 = 0$.*

Proof. Suppose that I is a weakly 2-absorbing semi-δ-primary ideal of L which is not 2-absorbing semi-δ-primary. Hence there exist a semi-δ-triple zero (x, y, z) of I by remark 3.12 then $x \wedge y \wedge z = 0$ and neither $x \wedge y \in \delta(I)$ nor $y \wedge z \in \delta(I)$ nor $x \wedge z \in \delta(I)$ for some $y, z \in L$. Now we show that $x\wedge y\wedge I = 0$. Assume that $x\wedge y\wedge i \neq 0$ for some $i \in I$. Since $0 \neq x\wedge y\wedge(z\vee i) \in I$ and I is a weakly 2-absorbing semi-δ-primary ideal, we conclude $x\wedge y \in \delta(I)$ or $x\wedge(z\vee i) \in \delta(I)$ or $y\wedge(z\vee i) \in \delta(I)$ is a contradiction. Similarly it is easy to show that $x\wedge z\wedge I = y\wedge z\wedge I = 0$. Now, we have to show that $x\wedge I^2 = 0$. Assume that $x\wedge i_1\wedge i_2 \neq 0$ for some $i_1, i_2 \in I$. Since $x\wedge y\wedge I = y\wedge z\wedge I = x\wedge z\wedge I = 0$, we have $0 \neq x\wedge(y\vee i_1)\wedge(z\vee i_2) = x\wedge i_1\wedge i_2 \in I$. As I is a weakly 2-absorbing semi-δ-primary which contradicts our assumption that neither $x\wedge y \in \delta(I)$ nor $y\wedge z \in \delta(I)$ nor $x\wedge z \in \delta(I)$. Thus, $x\wedge I^2 = 0$ can easily show that $y\wedge I^2 = z\wedge I^2 = 0$. Now, we show that $I^3 = 0$. Assume that $i_1\wedge i_2\wedge i_3 \neq 0$ for some $i_1, i_2, i_3 \in I$. Since $x\wedge y\wedge I = y\wedge z\wedge I = x\wedge z\wedge I = x\wedge I^2 = y\wedge I^2 = z\wedge I^2 = 0$. Observe that $0 \neq (x\vee i_1)\wedge(y\vee i_2)\wedge(z\vee i_3) = i_1\wedge i_2\wedge i_3 \in I$. Since I is a weakly 2-absorbing semi-δ-primary which is a contradiction. Thus, $I^3 = 0$. □

Nehete and Nimbhorkar defined $\gamma\delta$-lattice homomorphism [5].

Definition 3.14. Let f: L→ K be a lattice homomorphism δ be an ideal expansion of K and γ be an ideal expansion of L. We say that f is a homomorphism of the $\gamma\delta$-lattice if $\gamma(f^{-1}(I)) = f^{-1}(\delta(I))$ for all $I \in Id(K)$.

If f is a surjective $\gamma\delta$-lattice homomorphism and $I \in Id(L)$ is such that $ker(f) \subseteq I$ then $f(\gamma(I)) = \delta(f(I))$.
Particularly if f is a $\gamma\delta$-lattice isomorphism, then $f(\gamma(I)) = \delta(f(I))$

Theorem 3.15. *Let f: $L \to K$ be a $\gamma\delta$-lattice homomorphism where γ and δ are ideal expansion function of Id(L) and Id(K) respectively. Then the following holds:*

1. *If P is a 2-absorbing semi-δ-primary ideal of K then $f^{-1}(P)$ is a 2-absorbing semi-γ-primary ideal of lattice L.*
2. *If P is a weakly 2-absorbing semi-δ-primary ideal of K and Ker(f) is a weakly 2-absorbing semi-γ-primary ideal of L then $f^{-1}(P)$ is a weakly 2-absorbing semi-γ-primary ideal of L.*

Proof. 1. Let $x\wedge y\wedge z \in f^{-1}(P)$ for some x, y, z ∈ L. Then $f(x\wedge y\wedge z) = f(x)\wedge f(y)\wedge f(z) \in P$ which implies $f(x)\wedge f(y) = f(x\wedge y) \in \delta(P)$ or $f(y)\wedge f(z) = f(y\wedge z) \in \delta(P)$ or $f(x)\wedge f(z) = f(x\wedge z) \in \delta(P)$. Thus, we have $x\wedge y \in f^{-1}(\delta(P))$ or $y\wedge z \in f^{-1}(\delta(P))$ or $x\wedge z \in f^{-1}(\delta(P))$. Since $f^{-1}(\delta(P)) = \gamma(f^{-1}(P)$ then $f^{-1}(P)$ is a 2-absorbing semi-γ-primary ideal of L.
2. when $0 \neq x\wedge y\wedge z \in f^{-1}(P)$ for some $x, y, z \in$ L. Then $f(x\wedge y\wedge z) = f(x)\wedge f(y)\wedge f(z) \in P$. If $f(x\wedge y\wedge z) \neq 0$ it can be proved as part 1 that $f^{-1}(P)$ is a weakly 2-absorbing semi-γ-primary ideal of L. Assume that $f(x\wedge y\wedge z) = 0$. Hence $x\wedge y\wedge z \in Ker(f)$. Since Ker(f) is a weakly 2-absorbing semi-γ-primary. We get $x\wedge y \in \gamma(Ker(f))$ or $y\wedge z \in \gamma(Ker(f))$ or $x\wedge z \in \gamma(Ker(f))$. since $\gamma(Ker(f)) = \gamma(f^{-1}(0) \subseteq \gamma(f^{-1}(P)$. □

We introduce the notion of 2-absorbing semi-δ-primary ideals in the context of direct products of lattices.

Let $L_1, L_2, \dots L_n$ where $n \geq 2$ be lattices with $1 \neq 0$. Assume that $\delta_1, \delta_2, \dots \delta_n$ are expansion functions of ideals of $L_1, L_2, \dots L_n$ respectively.

Let $L = L_1 \times L_2 \times \dots \times L_n$. Define a function $\delta_\times : Id(L) \to Id(L)$ such that $\delta_\times(I_1 \times I_2 \times \dots \times I_n) = \delta_1(I_1) \times \delta_2(I_2) \times \dots \times \delta_n(I_n)$ for every $I_i \in Id(L_i)$, where $1 \leq i \leq n$. Clearly $\delta_\times$ is an expansion function of ideals of L. Note that every ideals of L is of the form $I_1 \times I_2 \times \dots \times I_n$, where each I_i is an ideal of L_i for $1 \leq i \leq n$ [6].

The next four theorems are devoted to characterizing the 2-absorbing and weakly 2-absorbing semi-δ-primary ideals in the finite direct product of lattices.

Theorem 3.16. *Let L_1 and L_2 be lattices with $1 \neq 0$ and $L = L_1 \times L_2$ and let δ_1, δ_2 be expansion functions of $Id(L_1)$ and $Id(L_2)$ respectively. Suppose $\delta_\times(I)$ is a proper ideal of L for any proper ideal of L then the following statements are equivalent:*

1. *$I = I_1 \times I_2$ is a 2-absorbing semi-$\delta_\times$-primary ideal of lattice L.*
2. *Either I_1 is a 2-absorbing semi-δ_1-primary ideal of L_1 and $\delta_2(I_2) = L_2$ or I_2 is a 2-absorbing semi-δ_2-primary ideal of L_2 and $\delta_1(I_1) = L_1$ or I_1, I_2 are semi-δ_1-primary ideal of L_1 and semi-δ_2-primary ideal of L_2 respectively.*

Proof. (1) $\implies$ (2) Suppose $I = I_1 \times I_2$ is a 2-absorbing semi-$\delta_\times$-primary ideal of lattice L. Since I is a proper ideal of lattice L such that $\delta_\times(I) = \delta_1(I_1) \times \delta_2(I_2)$. So we have three cases:
Case(1): Let $\delta_1(I_1) \neq L_1$ and $\delta_2(I_2) = L_2$ we have to show that I_1 is a 2-absorbing semi-δ_1-primary ideal of L_1. Suppose that $x \wedge y \wedge z \in I_1$ and $x \wedge y \notin \delta_1(I_1)$ then $(x, 0) \wedge (y, 0) \wedge (z, 0) \in I$ and $(x, 0) \wedge (y, 0) \notin \delta_\times(I)$ implies that $(y, 0) \wedge (z, 0) \in \delta_\times(I)$ or $(x, 0) \wedge (z, 0) \in \delta_\times(I)$. Thus $y \wedge z \in \delta_1(I_1)$ or $x \wedge z \in \delta_1(I_1)$. Hence I_1 is 2-absorbing semi-δ_1-primary ideal of L_1 with $\delta_2(I_2) = L_2$.
Case(2): Let $\delta_1(I_1) = L_1$ and $\delta_2(I_2) \neq L_2$. we have to show that I_2 is a 2-absorbing semi-δ_2-primary ideal of L_2. Suppose that $x \wedge y \wedge z \in I_2$ and $x \wedge y \notin \delta_2(I_2)$.Then $(0, x) \wedge (0, y) \wedge (0, z) \in I$ and $(0, x) \wedge (0, y) \notin \delta_\times(I)$ implies that $(0, y) \wedge (0, z) \notin \delta_\times(I)$ or $(0, x) \wedge (0, z) \notin \delta_\times(I)$. Thus $y \wedge z \in \delta_2(I_2)$ or $x \wedge z \in \delta_2(I_2)$. Hence proved.
Case(3): Let $\delta_1(I_1) \neq L_1$ and $\delta_2(I_2) \neq L_2$. Suppose that $x \wedge y \in I_1$ and $x \notin \delta_1(I_1)$ for some $x, y \in L_1$. Observe that $(x, 1) \wedge (y, 1) \wedge (1, 0) \in I$. $(x, 1) \wedge (y, 1) \notin \delta_\times(I)$ and $(x, 1) \wedge (1, 0) \notin \delta_\times(I)$. Since I is a 2-absorbing semi-$\delta_\times$-primary, we conclude that $(y, 1) \wedge (1, 0) \in \delta_\times(I)$. Thus $y \in \delta_1(I_1)$ and so I_1 is semi-δ_1-primary ideal of L_1. Similarly it can be shown that I_2 is semi-δ_2-primary ideal of L_2.
(2) $\implies$ (1) If I_1 is a 2-absorbing semi-δ_1-primary ideal of L_1 and $\delta_2(I_2) = L_2$ or I_2 is 2-absorbing semi-δ_2-primary ideal of L_2 and $\delta_1(I_1) = L_1$ then clearly I is 2-absorbing semi-$\delta_\times$-primary ideal of L. Now suppose that I_1 and I_2 are semi-δ_1-primary ideals of L_1 and semi-δ_2-primary ideals of L_2 respectively.
Suppose $(x_1, x_2) \wedge (y_1, y_2) \wedge (z_1, z_2) \in I = I_1 \times I_2, (x_1, x_2) \wedge (y_1, y_2) \notin \delta_\times(I)$ and $(x_1, x_2) \wedge (z_1, z_2) \notin \delta_\times(I)$. Hence we have four cases:
Case(1): Let $x_1 \wedge y_1 \notin \delta_1(I_1)$ and $x_1 \wedge z_1 \notin \delta_1(I_1)$. Since $x_1 \wedge y_1 \wedge z_1 \in I_1$. It contradicts with the assumption that I_1 is semi-δ_1-primary ideal.
Case(2): Let $x_2 \wedge y_2 \notin \delta_2(I_2)$ and $x_2 \wedge z_2 \notin \delta_2(I_2)$ since $x_2 \wedge y_2 \wedge z_2 \in I_2$. It contradicts with the assumption I_2 is a semi-δ_2-primary ideal.
Case(3): Let $x_1 \wedge y_1 \notin \delta_1(I_1)$ and $x_2 \wedge z_2 \notin \delta_2(I_2)$. Since $x_1 \wedge y_1 \wedge z_1 \in I_1$ and I_1 is a semi-δ_1-primary. We have $z_1 \in \delta_1(I_1)$. Since $x_2 \wedge y_2 \wedge z_2 \in I_2$ and I_2 is semi-δ_2-primary. We get $y_2 \in \delta_2(I_2)$.Thus $(y_1, y_2) \wedge (z_1, z_2) \in \delta_\times(I)$.
Case(4): Let $x_1 \wedge z_1 \notin \delta_1(I_1)$ and $x_2 \wedge y_2 \notin \delta_2(I_2)$. As $x_1 \wedge y_1 \wedge z_1 \in I_1$ and I_1 is semi-δ_1-primary. We have $y_1 \in \delta_1(I_1)$. As $x_2 \wedge y_2 \wedge z_2 \in I_2$ and I_2 is semi-δ_2-primary, implies $z_2 \in \delta_2(I_2)$. Thus $(y_1, y_2) \wedge (z_1, z_2) \in \delta_\times(I)$. Therefore I is a 2-absorbing semi-$\delta_\times$-primary ideal of lattice L. □

Theorem 3.17. *Let $L_1, L_2, \ldots L_n$ are lattices with $1 \neq 0$ and $L = L_1 \times L_2 \times \ldots \times L_n$ where $n \geq 2$. Let δ_i be an expansion function of Id(L_i) for each $i = 1, 2, \ldots n$. Then the following statements are equivalent:*

1. *I is a 2-absorbing semi-$\delta_\times$-primary ideal of lattice L.*
2. *$I = I_1 \times I_2 \times \ldots \times I_n$ and either for some $K \in \{1, 2, \ldots n\}$ such that I_K is a 2-absorbing semi-$\delta_\times$-primary ideal of lattice L and $\delta_j(I_j) = L_j$ for all $j \in \{1, 2, \ldots n\} \backslash \{K\}$ or I_K and I_S are semi-δ_K-primary and δ_S-primary ideals of L_K and L_S respectively for some K,S $\in \{1, 2, \ldots n\}$ and $\delta_j(I_j) = L_j$ for all $j \in \{1, 2 \ldots n\} \backslash \{K, S\}$*

Proof. By using mathematical Induction on n and Theorem 3.16 □

Theorem 3.18. *Let L_1 and L_2 are lattices. Let $L = L_1 \times L_2$ and δ_1, δ_2 be expansion functions of $Id(L_1)$ and $Id(L_2)$ respectively. Then the following statements are equivalent:*

1. *$I = I_1 \times L_2$ is a weakly 2-absorbing semi-$\delta_\times$-primary ideal of L.*
2. *$I = I_1 \times L_2$ is a 2-absorbing semi-$\delta_\times$-primary ideal of L.*
3. *I_1 is a 2-absorbing semi-δ_1-primary ideal of L_1*

Proof. (1) $\implies$ (2) Suppose that $I = I_1 \times L_1$ is a weakly 2-absorbing semi $\delta_\times$-primary ideal of L. since $I^3 \neq 0, I = I_1 \times L_2$ is 2-absorbing semi-$\delta_\times$-primary ideal of L by Theorem 3.13.
(2) $\implies$ (3) $\implies$ (1) is can be proved with the Theorem 3.16. □

Following definition is introduced by Nehete and Nimbhorkar [5].

Definition 3.19. Let P and Q be ideals of a lattice L, the residual division of P by Q is defined to be the ideal $P : Q = \{x \in L \mid x \wedge y \in P \,\forall\, y \in Q\}$.

Theorem 3.20. *Let δ be an expansion function of Id(L) and I a proper ideal of L. Then following are equivalent:*

1. *I is a 2-absorbing semi-δ-primary ideal of lattice L.*
2. *For every elements x, $y \in L$ with $x \wedge y \notin \delta(I), (I : x \wedge y) \subseteq (\delta(I) : x) \vee (\delta(I) : y)$.*
3. *For every elements $x, y \in L$ with $x \wedge y \notin \delta(I), (I : x \wedge y) \subseteq (\delta(I) : x)$ or $(I : x \wedge y) \subseteq (\delta(I) : y)$.*
4. *For every elements $x, y \in L$ with $x \wedge y \wedge P \subseteq I$ and $x \wedge y \notin \delta(I)$ implies either $x \wedge P \subseteq \delta(I)$ or $y \wedge P \subseteq \delta(I)$.*
5. *For any ideals P,S and T of lattice L with $P \wedge S \wedge T \subseteq I$ implies $P \wedge S \subseteq \delta(I)$ or $P \wedge T \subseteq \delta(I)$ or $S \wedge T \subseteq \delta(I)$.*

Proof. (1) $\implies$ (2) Let $z \in (I : x \wedge y)$. Since $x \wedge y \wedge z \in I, x \wedge y \notin \delta(I)$ and I is a 2-absorbing semi-δ-primary. We have $x \wedge z \in \delta(I)$ or $y \wedge z \in \delta(I)$. Hence $z \in (\delta(I) : x) \vee (\delta(I) : y)$. Therefore $(I : x \wedge y) \subseteq (\delta(I) : x) \vee (\delta(I) : y)$
(2) $\implies$ (3) Obviously.
(3) $\implies$ (4) Suppose (3) holds and $x \wedge y \wedge P \subseteq I$ and $x \wedge y \notin \delta(I)$. Hence, we have $P \subseteq (I : x \wedge y) \subseteq (\delta(I) : x))$ or $P \subseteq (I : x \wedge y) \subseteq (\delta(I) : y)$ by assumption. Thus $x \wedge P \subseteq \delta(I)$ or $y \wedge P \subseteq \delta(I)$.
(4) $\implies$ (5) Suppose that $P \wedge S \wedge T \subseteq I$ and $S \wedge T \nsubseteq \delta(I)$. Then $x \wedge y \notin \delta(I)$ for some $x \in S$ and $y \in T$. Hence, $x \wedge P \subseteq \delta(I)$ or $y \wedge P \subseteq \delta(I)$.
Case(i) Assume that $x \wedge P \subseteq \delta(I)$ and $y \wedge P \nsubseteq \delta(I)$. We show that $P \wedge S \subseteq \delta(I)$. If $S \wedge P \nsubseteq \delta(I)$, for some $s \in S$, then $(x \vee s) \wedge y \wedge P \subseteq I$. As $y \wedge P \nsubseteq \delta(I)$, we have $(x \vee s) \wedge P \subseteq \delta(I)$ and so we get $S \wedge P \subseteq \delta(I)$, a contradiction.
Case(ii) Assume that $x \wedge P \nsubseteq \delta(I)$ and $y \wedge P \subseteq \delta(I)$ as previous argument, we can conclude $P \wedge T \subseteq \delta(I)$. Now, suppose that $x \wedge P \subseteq \delta(I)$ and $y \wedge P \subseteq \delta(I)$.
Case(iii) Assume that neither $P \wedge S \subseteq \delta(I)$ nor $P \wedge T \subseteq \delta(I)$ then there exist $s \in S$ and $t \in T$ such that $s \wedge P \nsubseteq \delta(I)$ and $t \wedge P \nsubseteq \delta(I)$. As $s \wedge t \wedge P \subseteq I$, we conclude $s \wedge t \in \delta(I)$. Since $(x \vee s) \wedge t \wedge P \subseteq I$, $t \wedge P \nsubseteq \delta(I)$ and $(x \vee s) \wedge P = (x \wedge P) \vee (s \wedge P) \nsubseteq \delta(I)$ we have $(x \vee s) \wedge t \in \delta(I)$. As $s \wedge t \in \delta(I)$ so we get $x \wedge t \in \delta(I)$. As $(y \vee t) \wedge s \wedge P \subseteq I, s \wedge P \nsubseteq \delta(I)$ and $(y \vee t) \wedge P = (y \wedge P) \vee (t \wedge P) \nsubseteq \delta(I)$. We have $(y \vee t) \wedge P \in \delta(I)$. Since $s \wedge t \in \delta(I)$ we have, $y \wedge s \in \delta(I)$. As $(x \vee s) \wedge (y \vee t) \wedge P \subseteq I$, $(x \vee s) \wedge P \nsubseteq \delta(I)$ and $(y \vee t) \wedge P \nsubseteq \delta(I)$ we get $(x \vee s) \wedge (y \vee t) \in \delta(I)$. As all $x \wedge t$, $y \wedge s$, $s \wedge t \in \delta(I)$, so we conclude $x \wedge y \in \delta(I)$, a contradiction. Thus $S \subseteq \delta(I)$ or $P \wedge T \subseteq \delta(I)$.
(5) $\implies$ (1) suppose that $x \wedge y \wedge z \in I$ for some $x, y, z \in L$, put $P = (x), S = (y)$ and $T = (z)$ in (5) the result is clear. □

We proceed to introduce the concepts of strongly(weakly) 2-absorbing semi-δ-primary ideals as defined below.

Definition 3.21. Let δ be an expansion function of Id(L). A proper ideal I of lattice L is strongly (weakly) 2-absorbing semi-δ-primary ideal if $P, S, T \in Id(L)$ with $(0 \neq P \wedge S \wedge T \subseteq I)$ $P \wedge S \wedge T \subseteq I$ implies $P \wedge S \subseteq \delta(I)$ or $S \wedge T \subseteq \delta(I)$ or $P \wedge T \subseteq \delta(I)$.

Definition 3.22. Let I be a weakly 2-absorbing semi-δ-primary ideal of lattice L and suppose $P \wedge S \wedge T \subseteq I$ for some ideals P, S and T of lattice L. I is a free semi-δ-triple-zero with respect to $P \wedge S \wedge T$ if (x, y, z) is not a semi-δ-triple-zero of I for every $x \in P, y \in S, z \in T$ or in other words whenever $x \in P, y \in S, z \in T$, we have $x \wedge y \in \delta(I)$ or $y \wedge z \in \delta(I)$ or $x \wedge z \in \delta(I)$.

Lemma 3.23. *Let I be a weakly 2-absorbing semi-δ-primary ideal of lattice L such that $x \wedge y \wedge S \subseteq I$ for some $x, y \in L$ and an ideal S of R. If (x, y, s) is not a semi-δ-triple-zero of I for all $s \in S$ and $x \wedge y \notin \delta(I)$ then $x \wedge S \subseteq \delta(I)$ or $y \wedge S \subseteq \delta(I)$.*

Proof. Assume that $x \wedge y \wedge S \subseteq I$ but neither $x \wedge y \in \delta(I)$ nor $x \wedge S \in \delta(I)$ nor $y \wedge S \in \delta(I)$. Hence there exist $s_1, s_2 \in S$ such that $x \wedge s_1 \notin \delta(I)$ and $y \wedge s_2 \notin \delta(I)$. Since $x \wedge y \wedge s_1 \in I$, but neither $x \wedge y \in \delta(I)$ nor $x \wedge s_1 \in \delta(I)$. We have $y \wedge s_1 \in \delta(I)$ by given statement (x, y, s_1) is not a semi-δ-triple-zero of I. Similarly, since $x \wedge y \in \wedge s_2 \in \ I$ but neither $x \wedge y \in \delta(I)$ nor $y \wedge s_2 \in \delta(I)$. We get $x \wedge s_2 \in \delta(I)$. Now $x \wedge y \wedge (s_1 \vee s_2) \in I$ and since $x \wedge y \notin \delta(I)$. We have $x \wedge (s_1 \vee s_2) \in \delta(I)$ or $y \wedge (s_1 \vee s_2) \in \delta(I)$. Thus, we conclude that $x \wedge s_2 \in \delta(I)$ or $y \wedge s_2 \in \delta(I)$ a contradiction. Thus, $x \wedge S \subseteq \delta(I)$ or $y \wedge S \subseteq \delta(I)$. □

Lemma 3.24. *Let I be a weakly 2-absorbing semi-δ-primary ideal of L and let $x \wedge P \wedge S \subseteq I$ for some $x \in L$ and for an ideal T of L. If (x, p, s) is not a semi-δ-triple zero of I for all $p \in P, s \in S$ then $x \wedge P \subseteq \delta(I)$ or $x \wedge S \subseteq \delta(I)$ or $P \wedge S \subseteq \delta(I)$.*

Proof. Assume that neither $x \wedge P \subseteq \delta(I)$ nor $x \wedge S \subseteq \delta(I)$ nor $P \wedge S \subseteq \delta(I)$. Thus, there exist $p, p_1 \in P$ such that $x \wedge p \notin \delta(I)$ and $p_1 \wedge S \nsubseteq \delta(I)$. Since $x \wedge p \wedge S \subseteq I$, $x \wedge p \notin \delta(I)$ and $x \wedge S \nsubseteq \wedge(I)$. We have $P \wedge S \subseteq (I)$ by Lemma 3.23. Since $x \wedge p_1 \wedge S \subseteq I, x \wedge S \nsubseteq \delta(I)$ and $p_1 \wedge S \nsubseteq \delta(I)$. We have by $x \wedge p_1 \in \delta(I)$ Lemma 3.23. Now, since $x \wedge (p \vee p_1) \wedge S \subseteq I$ and $x \wedge S \nsubseteq \delta(I)$ from Lemma 3.23. we conclude that either $x \wedge (p \vee p_1) \in \delta(I)$ or $(p \vee p_1) \wedge S \subseteq I$. Hence $x \wedge p \in \delta(I)$ or $p_1 \wedge S \subseteq I$ a contradiction. Thus, $x \wedge P \subseteq \delta(I)$ or $x \wedge S \subseteq \delta(I)$ or $P \wedge S \subseteq \delta(I)$. □

Theorem 3.25. *Let I be a weakly 2-absorbing semi-δ-primary ideal of lattice L and suppose that $0 \neq P \wedge S \wedge T \subseteq I$ for some ideals P, S and T of L. If I is a free δ-triple zero with respect to $P \wedge S \wedge T$ then $P \wedge S \subseteq \delta(I)$ or $S \wedge T \subseteq \delta(I)$ or $P \wedge T \subseteq \delta(I)$.*

Proof. Suppose $P \wedge S \nsubseteq \delta(I)$ or $S \wedge T \nsubseteq \delta(I)$ or $P \wedge T \nsubseteq \delta(I)$. Hence, there exist $x, y \in P$ such that $x \wedge S \nsubseteq \delta(I)$ and $y \wedge T \nsubseteq \delta(I)$. Since $x \wedge S \wedge T \subseteq I, S \wedge T \nsubseteq \delta(I)$ and $x \wedge S \nsubseteq \delta(I)$. We have $x \wedge T \subseteq \delta(I)$ by Lemma 3.24. Since $y \wedge S \wedge T \subseteq I, S \wedge T \nsubseteq \delta(I)$ and $y \wedge T \nsubseteq \delta(I)$, we get $y \wedge S \subseteq \delta(I)$ by Lemma 3.24. Now $(x \vee y) \wedge S \wedge T \subseteq I$ and since $S \wedge T \nsubseteq \delta(I)$. We get $(x \vee y) \wedge S \subseteq \delta(I)$ or $(x \vee y) \wedge T \subseteq \delta(I)$. Hence, we get $x \wedge S \subseteq \delta(I)$ or $y \wedge T \subseteq \delta(I)$ a contradiction. $P \wedge S \subseteq \delta(I)$ or $S \wedge T \subseteq \delta(I)$ or $P \wedge T \subseteq \delta(I)$. □

Now, we define weakly n-absorbing semi-δ-primary ideal.

Definition 3.26. Let L be a lattice with mapping $\delta : Id(L) \to Id(L)$ an expansion of ideals of L and n is a positive integer. A proper ideal I of L is a (weakly) n-absorbing semi-δ-primary ideal if whenever $(0 \neq a_1 \wedge a_2 \wedge \ldots \wedge a_{n+1})$ $a_1 \wedge a_2 \wedge \ldots \wedge a_{n+1} \in I$ for some $a_1, a_2, \ldots, a_{n+1} \in L$ there exist $1 \leq i \leq n$ such that $a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_{n+1} \in \delta(I)$.

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