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INFINITELY MANY SOLUTIONS FOR A DOUBLE PHASE GRADIENT SYSTEM WITH NONLINEAR FLUX BOUNDARY CONDITIONS

A. Ahmed and G. Lorenzini

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Corresponding Author: G. Lorenzini

Abstract. This paper establishes the existence of infinitely many weak solutions for a class of gradient systems governed by double phase operators with variable exponents and nonlinear flux boundary conditions. The system under investigation is:

$$\begin{cases} -\operatorname{div}\big(a(\theta,\nabla\eta)\big)+a(\theta,\eta)-\operatorname{div}\big(b(\theta,\nabla\eta)\big)+b(\theta,\eta)=\frac{\partial\Phi}{\partial\eta}(\theta,\eta,\zeta) & \text{in } \mathcal{Q},\\ -\operatorname{div}\big(c(\theta,\nabla\zeta)\big)+c(\theta,\zeta)-\operatorname{div}\big(d(\theta,\nabla\zeta)\big)+d(\theta,\zeta)=\frac{\partial\Phi}{\partial\zeta}(\theta,\eta,\zeta) & \text{in } \mathcal{Q},\\ a(\theta,\nabla\eta)\cdot\gamma=\psi_1(\theta,\eta), \quad b(\theta,\nabla\eta)\cdot\gamma=\psi_2(\theta,\eta) & \text{on } \partial\mathcal{Q},\\ c(\theta,\nabla\zeta)\cdot\gamma=\psi_3(\theta,\zeta), \quad d(\theta,\nabla\zeta)\cdot\gamma=\psi_4(\theta,\zeta) & \text{on } \partial\mathcal{Q}, \end{cases}$$

where $\mathcal{Q}\subset\mathbb{R}^N$ is a bounded domain with a Lipschitz boundary $\partial\mathcal{Q}$ and γ is the unit outward normal. The approach is variational, combining critical point theory in the framework of variable exponent Sobolev spaces with a powerful variational principle due to Ricceri.

1 Introduction

Partial differential systems governed by nonlinear and nonhomogeneous differential operators with variable exponents have attracted considerable attention in recent years due to their ability to describe a wide range of physical, biological, and engineering phenomena. In particular, *double phase problems*whose energy density switches between two different growth behaviorshave emerged as a powerful modeling framework for materials exhibiting heterogeneities, such as composites, electrorheological fluids, and image restoration models (see, e.g., Mingione and Rădulescu [26], Colombo, De Filippis, and Mingione [9]). These problems are characterized by energy functionals of the type

$$\mathcal{H}(\theta,\eta)=|\eta|^{\ell(\theta)}+\mu(\theta)|\eta|^{\kappa(\theta)}, \quad 1<\ell(\theta)<\kappa(\theta),$$

where the measurable coefficient $\mu(\theta)$ controls the local transition between two different power-type behaviors. The interplay between the exponents $\ell(\cdot)$, $\kappa(\cdot)$, and the weight $\mu(\cdot)$ gives rise to nonstandard growth conditions and nonuniform ellipticity, posing significant analytical challenges in the study of existence, multiplicity, and regularity of weak solutions.

In this context, the present paper is devoted to the investigation of a class of gradient systems governed by double phase operators with variable exponents and nonlinear flux boundary conditions, of the form

$$\begin{cases} -\operatorname{div}\big(a(\theta,\nabla\eta)\big)+a(\theta,\eta)-\operatorname{div}\big(b(\theta,\nabla\eta)\big)+b(\theta,\eta)=\dfrac{\partial\Phi}{\partial\eta}(\theta,\eta,\zeta) & \text{in } \mathcal{Q},\\ -\operatorname{div}\big(c(\theta,\nabla\zeta)\big)+c(\theta,\zeta)-\operatorname{div}\big(d(\theta,\nabla\zeta)\big)+d(\theta,\zeta)=\dfrac{\partial\Phi}{\partial\zeta}(\theta,\eta,\zeta) & \text{in } \mathcal{Q},\\ a(\theta,\nabla\eta)\cdot\gamma=\psi_1(\theta,\eta), \quad b(\theta,\nabla\eta)\cdot\gamma=\psi_2(\theta,\eta) & \text{on } \partial\mathcal{Q},\\ c(\theta,\nabla\zeta)\cdot\gamma=\psi_3(\theta,\zeta), \quad d(\theta,\nabla\zeta)\cdot\gamma=\psi_4(\theta,\zeta) & \text{on } \partial\mathcal{Q}, \end{cases} \tag{1.1}$$

where $\mathcal{Q}\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\mathcal{Q}$ and γ denotes the unit outward normal. The nonlinear flux boundary terms ψ_i introduce an additional layer of complexity, modeling nonhomogeneous or reactive boundary effects relevant in several applied contexts, including heat conduction, nonlinear elasticity, and diffusion processes across permeable interfaces.

The gradient system studied in this paper arises naturally in the modeling of various nonlinear phenomena where different diffusion mechanisms coexist and interact, and the double phase operator captures transitions between distinct material phases or energetic responses depending on spatial position and the magnitude of deformation. Such models are particularly relevant in nonlinear elasticity, where the double phase term describes materials with spatially varying stiffness, allowing one to represent the coexistence of soft and rigid components in composites or biological tissues, in electrorheological fluids, where the variable exponents $\ell(\theta)$ and $\kappa(\theta)$ encode spatially dependent responses of the viscosity to an external electric field (see Rădulescu and Repovš [27]), and in heat transfer or diffusion through porous media, where nonlinear flux conditions on $\partial\mathcal{Q}$ model the exchange of energy or matter through reactive or semipermeable boundaries, leading to nonlinear Robin-type conditions. Mathematically, these systems are challenging due to the lack of homogeneity from the double phase operator, the failure of classical compact embeddings induced by variable exponents, and the additional complexity introduced by the coupling of equations and nonlinear boundary conditions, which affects compactness and variational structures. The motivation for this work is therefore twofold: theoretically, it extends the study of nonstandard growth problems and nonlinear systems by providing new multiplicity results for problems with variable exponents and nonlinear flux boundaries, practically, it offers a rigorous framework for analyzing heterogeneous and multiscale models in which the interplay between interior dynamics and boundary interactions plays a critical role in the overall behavior of the system.

The mathematical treatment of such problems naturally takes place in variable exponent Sobolev spaces $W^{1,\ell(\cdot)}(\mathcal{Q})$, which generalize the classical Sobolev spaces and allow the growth rate of the energy to vary with position. These spaces have been extensively developed over the last two decades (see Diening et al. [10], Rădulescu and Repovš [27]) and provide a flexible functional framework for handling anisotropic and nonstandard phenomena. The associated differential operators, often referred to as Leray-Lions-type operators with variable growth, require refined variational and topological tools to establish the existence of weak solutions.

The main contribution of this paper is to establish the existence of infinitely many weak solutions to the above gradient system. The proof is based on a variational approach, employing critical point theory within the variable exponent setting, and relies crucially on a variational principle due to Ricceri [28]. This method provides a robust and elegant way to detect multiple critical points of the associated energy functional under relatively mild conditions, without the need for symmetric assumptions or monotonicity requirements.

The study of multiplicity results for systems driven by double phase operators with variable exponents is still in its early stages. Existing literature has primarily focused on scalar problems or systems with homogeneous boundary conditions (see, e.g., Bahrouni and Rădulescu [6], Ho and Kim [22], Liu et al. [23]). Our work extends these investigations to a coupled gradient system with nonlinear boundary fluxes, filling an important gap in the current theory. In addition to its theoretical interest, the analysis developed here may have potential applications to models in nonlinear thermodynamics, non-Newtonian fluid mechanics, heterogeneous materials, and electrorheological fluid flows, where the energy density and boundary responses depend on spatial variability and material heterogeneity.

We observe that the analysis of double-phase systems is a relatively recent area of research, and the existing literature on this topic remains rather limited. For related developments, we refer the reader to [7, 17, 18, 24].

Moreover, it is worth emphasizing that a rich body of work exists concerning elliptic systems formulated under various structural assumptions and physical frameworks. For classical results and comprehensive surveys, the reader may consult [1, 3, 4, 5, 12].

The paper is organized as follows. In Section 2, we recall the functional framework and fundamental properties of Musielak-Orlicz-Sobolev spaces. Section 3 is devoted to the variational formulation of the problem and the verification of the assumptions required by Ricceri's variational principle. In Section 4, we establish the main existence and multiplicity results for weak solutions.

We now recall the following abstract result due to B. Ricceri, which plays a fundamental role in proving the existence of weak solutions for our main problem.

Theorem 1.1. *(See [28], Theorem 2.5.) Let X be a reflexive real Banach space, and let $\Upsilon_1, \Upsilon_2 : X \to \mathbb{R}$ be two Gâteaux differentiable, sequentially weakly lower semicontinuous functionals. Assume, in addition, that Υ_1 is strongly continuous and satisfies*

$$\lim_{\|\eta\| \to +\infty} \Upsilon_1(\eta) = +\infty.$$

For each $\vartheta > \inf_X \Upsilon_1$, define

$$\Lambda(\vartheta) = \inf_{\eta \in \Upsilon_1^{-1}((-\infty,\vartheta))} \frac{\Upsilon_2(\eta) - \inf_{\zeta \in \overline{(\Upsilon_1^{-1}((-\infty,\vartheta)))}_w} \Upsilon_2(\zeta)}{\vartheta - \Upsilon_1(\eta)}, \tag{1.2}$$

where $\overline{(\Upsilon_1^{-1}((-\infty,\vartheta)))}_w$ denotes the weak closure of $\Upsilon_1^{-1}((-\infty,\vartheta))$ in X. Next, set

$$\varphi = \liminf_{\vartheta \to +\infty} \Lambda(\vartheta), \tag{1.3}$$

and

$$\chi = \liminf_{\vartheta \to (\inf_X \Upsilon_1)^+} \Lambda(\vartheta). \tag{1.4}$$

Then the following statements hold:

- ***(a)*** *For every $\vartheta > \inf_X \Upsilon_1$ and every $\mu > \Lambda(\vartheta)$, the functional $\Upsilon_2 + \mu\Upsilon_1$ possesses at least one critical point in the set $\Upsilon_1^{-1}((-\infty,\vartheta))$.*
- ***(b)*** *If $\varphi < +\infty$, then for each $\mu > \varphi$, one of the following alternatives holds: either $\Upsilon_2 + \mu\Upsilon_1$ attains a global minimum in X, or there exists a sequence $\{\eta_n\} \subset X$ of critical points of $\Upsilon_2 + \mu\Upsilon_1$ such that*
$$\lim_{n \to \infty} \Upsilon_1(\eta_n) = +\infty.$$
- ***(c)*** *If $\chi < +\infty$, then for every $\mu > \chi$, one of the following occurs: either a global minimum of Υ_1 is also a local minimum of $\Upsilon_2 + \mu\Upsilon_1$, or there exists a sequence of pairwise distinct critical points of $\Upsilon_2 + \mu\Upsilon_1$ converging weakly to a global minimum of Υ_1.*

2 Preliminaries

Let $\mathcal{Q} \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. We define

$$\mathcal{C}_+(\overline{\mathcal{Q}}) = \left\{ \ell \in \mathcal{M} \ : \ \ell(\cdot) : \overline{\mathcal{Q}} \to \mathbb{R}, \ 1 < \ell^- := \operatorname*{ess\,inf}_{\theta \in \overline{\mathcal{Q}}} \ell(\theta) \leq \ell^+ := \operatorname*{ess\,sup}_{\theta \in \overline{\mathcal{Q}}} \ell(\theta) < \infty \right\},$$

where $\mathcal{M}$ denotes the class of measurable real-valued functions on $\overline{\mathcal{Q}}$.

The variable exponent Lebesgue space $L^{\ell(\cdot)}(\mathcal{Q})$ is defined as the collection of all measurable functions $\eta : \mathcal{Q} \to \mathbb{R}$ such that

$$\rho_{\ell(\cdot)}(\eta) := \int_{\mathcal{Q}} |\eta(\theta)|^{\ell(\theta)} \, d\theta < \infty.$$

This space is endowed with the Luxemburg norm

$$\|\eta\|_{L^{\ell(\cdot)}(\mathcal{Q})} = \inf\Big\{\upsilon > 0 : \rho_{\ell(\cdot)}\Big(\frac{\eta}{\upsilon}\Big) \le 1\Big\}.$$

The space $\big(L^{\ell(\cdot)}(\mathcal{Q}), \|\cdot\|_{L^{\ell(\cdot)}(\mathcal{Q})}\big)$ is a separable, reflexive, and uniformly convex Banach space. Moreover, its dual is isomorphic to $L^{\ell'(\cdot)}(\mathcal{Q})$, where $\ell'(\cdot)$ is given by the pointwise relation $\frac{1}{\ell(\cdot)} + \frac{1}{\ell'(\cdot)} = 1$.

A fundamental tool in this framework is the following Hölder-type inequality: for all $\eta \in L^{\ell(\cdot)}(\mathcal{Q})$ and $\zeta \in L^{\ell'(\cdot)}(\mathcal{Q})$,

$$\Big|\int_{\mathcal{Q}} \eta(\theta)\zeta(\theta)\,d\theta\Big| \le \Big(\frac{1}{\ell^-} + \frac{1}{(\ell^-)'}\Big)\|\eta\|_{L^{\ell(\cdot)}(\mathcal{Q})}\,\|\zeta\|_{L^{\ell'(\cdot)}(\mathcal{Q})} \le 2\,\|\eta\|_{L^{\ell(\cdot)}(\mathcal{Q})}\,\|\zeta\|_{L^{\ell'(\cdot)}(\mathcal{Q})}. \tag{2.1}$$

Proposition 2.1 ([10]). *Let $\eta \in L^{\ell(\cdot)}(\mathcal{Q})$. Then the following hold:*

(a) *If $\|\eta\|_{L^{\ell(\cdot)}(\mathcal{Q})} > 1$, then*

$$\|\eta\|^{\ell^-}_{L^{\ell(\cdot)}(\mathcal{Q})} < \rho_{\ell(\cdot)}(\eta) < \|\eta\|^{\ell^+}_{L^{\ell(\cdot)}(\mathcal{Q})}.$$

(b) *If $\|\eta\|_{L^{\ell(\cdot)}(\mathcal{Q})} < 1$, then*

$$\|\eta\|^{\ell^+}_{L^{\ell(\cdot)}(\mathcal{Q})} < \rho_{\ell(\cdot)}(\eta) < \|\eta\|^{\ell^-}_{L^{\ell(\cdot)}(\mathcal{Q})}.$$

The corresponding variable exponent Sobolev space is defined as

$$W^{1,\ell(\cdot)}(\mathcal{Q}) = \big\{\eta \in L^{\ell(\cdot)}(\mathcal{Q}) : |\nabla\eta| \in L^{\ell(\cdot)}(\mathcal{Q})\big\},$$

equipped with the norm

$$\|\eta\|_{W^{1,\ell(\cdot)}(\mathcal{Q})} = \|\eta\|_{L^{\ell(\cdot)}(\mathcal{Q})} + \|\nabla\eta\|_{L^{\ell(\cdot)}(\mathcal{Q})}.$$

It is well known that the space $\big(W^{1,\ell(\cdot)}(\mathcal{Q}), \|\cdot\|_{W^{1,\ell(\cdot)}(\mathcal{Q})}\big)$ is a separable and reflexive Banach space. For further details, we refer the reader to [10].

Remark 2.2. If $\ell \in C_+(\overline{\mathcal{Q}})$ and $N < \ell^-$, then the embedding

$$W^{1,\ell(\cdot)}(\mathcal{Q}) \hookrightarrow\hookrightarrow C^0(\overline{\mathcal{Q}}),$$

is both continuous and compact. Furthermore, $W^{1,\ell(\cdot)}(\mathcal{Q})$ is continuously embedded into $W^{1,\ell^-}(\mathcal{Q})$.

Let $\ell_i(\cdot), \kappa_i(\cdot) \in C_+(\overline{\mathcal{Q}})$, $i = 1, 2$, be variable exponents satisfying the growth conditions

$$N < \ell_i^- \le \ell_i(\theta) \le \ell_i^+ < \kappa_i^- \le \kappa_i(\theta) \le \kappa_i^+ < +\infty, \quad \text{for all } \theta \in \overline{\mathcal{Q}}. \tag{2.2}$$

Under condition (2.2), we define the embedding constants

$$c_1 = \sup_{\eta \in W^{1,\ell_1(\cdot)}(\mathcal{Q})\setminus\{0\}} \frac{\|\eta\|_\infty}{\|\eta\|_{W^{1,\ell_1(\cdot)}(\mathcal{Q})}}, \qquad c_2 = \sup_{\eta \in W^{1,\kappa_1(\cdot)}(\mathcal{Q})\setminus\{0\}} \frac{\|\eta\|_\infty}{\|\eta\|_{W^{1,\kappa_1(\cdot)}(\mathcal{Q})}}, \tag{2.3}$$

and

$$c_3 = \sup_{\zeta \in W^{1,\ell_2(\cdot)}(\mathcal{Q})\setminus\{0\}} \frac{\|\zeta\|_\infty}{\|\zeta\|_{W^{1,\ell_2(\cdot)}(\mathcal{Q})}}, \qquad c_4 = \sup_{\zeta \in W^{1,\kappa_2(\cdot)}(\mathcal{Q})\setminus\{0\}} \frac{\|\zeta\|_\infty}{\|\zeta\|_{W^{1,\kappa_2(\cdot)}(\mathcal{Q})}}. \tag{2.4}$$

We now introduce the Musielak-Orlicz-Sobolev spaces, which will play a fundamental role in the analytical framework of our main results. To this end, we first recall the notions of an Orlicz function and a Musielak function.

Definition 2.3. An Orlicz function, denoted by $\mathfrak{M} \in N(\mathcal{Q})$, is a mapping $\mathfrak{M} : \mathbb{R} \to [0, +\infty[$ that is even, continuous, and convex, satisfying $\mathfrak{M}(0) = 0$, $\mathfrak{M}(t) > 0$ for all $t > 0$, and

$$\lim_{t\to 0^+} \frac{\mathfrak{M}(t)}{t} = 0, \qquad \lim_{t\to +\infty} \frac{\mathfrak{M}(t)}{t} = +\infty.$$

A function $\mathfrak{M} : \mathcal{Q} \times \mathbb{R} \to [0, +\infty[$ is called a Musielak function, and is denoted by $\mathfrak{M} \in \Phi(\mathcal{Q})$, if for each $t \ge 0$, $\mathfrak{M}(\cdot, t)$ is a measurable function on $\mathcal{Q}$, and for almost every $\theta \in \mathcal{Q}$, the function $\mathfrak{M}(\theta, \cdot)$ is an Orlicz function.

Given $\mathfrak{M} \in \Phi(\mathcal{Q})$, the corresponding Musielak-Orlicz space $L_{\mathfrak{M}}(\mathcal{Q})$ is defined as

$$L_{\mathfrak{M}}(\mathcal{Q}) := \Big\{\eta \in \mathcal{M} : \exists\, \sigma > 0 \text{ such that } \int_{\mathcal{Q}} \mathfrak{M}\Big(\theta, \tfrac{|\eta(\theta)|}{\sigma}\Big)\, d\theta < \infty\Big\},$$

and it is equipped with the Luxemburg norm

$$\|\eta\|_{L_{\mathfrak{M}}(\mathcal{Q})} := \inf\Big\{\sigma > 0 : \int_{\mathcal{Q}} \mathfrak{M}\Big(\theta, \tfrac{|\eta(\theta)|}{\sigma}\Big)\, d\theta \le 1\Big\}.$$

The associated Musielak-Orlicz-Sobolev space is defined by

$$W^1L_{\mathfrak{M}}(\mathcal{Q}) := \big\{\eta \in L_{\mathfrak{M}}(\mathcal{Q}) : |\nabla \eta| \in L_{\mathfrak{M}}(\mathcal{Q})\big\},$$

endowed with the norm

$$\|\eta\|_{W^1L_{\mathfrak{M}}(\mathcal{Q})} = \|\eta\|_{L_{\mathfrak{M}}(\mathcal{Q})} + \|\nabla\eta\|_{L_{\mathfrak{M}}(\mathcal{Q})}, \qquad \|\nabla\eta\|_{L_{\mathfrak{M}}(\mathcal{Q})} := \||\nabla\eta|\|_{L_{\mathfrak{M}}(\mathcal{Q})}.$$

Definition 2.4. (i) A function $\mathfrak{M} \in \Phi(\mathcal{Q})$ is said to satisfy the Δ_2-condition, denoted by $\mathfrak{M} \in \Delta_2$, if there exist a constant $k > 0$ and a function $b \in L^1(\mathcal{Q})$, with $b \ge 0$, such that

$$\mathfrak{M}(\theta, 2t) \le k\, \mathfrak{M}(\theta, t) + b(\theta), \qquad \forall\, \theta \in \mathcal{Q},\ \forall\, t \in \mathbb{R}.$$

(ii) A function $\mathfrak{M} \in \Phi(\mathcal{Q})$ is called locally integrable if $\mathfrak{M}(\cdot, t_0) \in L^1(\mathcal{Q})$ for every $t_0 > 0$.

For $t \ge 0$, the right-hand derivative of $\mathfrak{M}(\theta, \cdot)$ is defined by

$$\mathfrak{M}'_d(\theta, t) = \lim_{h \to 0^+} \frac{\mathfrak{M}(\theta, t+h) - \mathfrak{M}(\theta, t)}{h},$$

and it is extended to $t < 0$ by setting $\mathfrak{M}'_d(\theta, t) = -\mathfrak{M}'_d(\theta, -t)$. Consequently,

$$\mathfrak{M}(\theta, t) = \int_0^{|t|} \mathfrak{M}'_d(\theta, s)\, ds, \qquad \forall\, t \in \mathbb{R},\ \theta \in \mathcal{Q}.$$

The complementary function $\mathfrak{M}^* : \mathcal{Q} \times \mathbb{R} \to [0, +\infty[$ is defined by

$$\mathfrak{M}^*(\theta, s) = \sup_{t \in \mathbb{R}} \big(st - \mathfrak{M}(\theta, t)\big), \qquad \forall\, s \in \mathbb{R},\ \theta \in \mathcal{Q}.$$

According to Young's theory, $\mathfrak{M}^*$ is also a Musielak function, and $\mathfrak{M}$ is the complementary function of $\mathfrak{M}^*$.

For the fundamental properties of these spaces, we refer to [8].

Lemma 2.5 ([8]). *The following norms are equivalent on $W^1L_{\mathfrak{M}}(\mathcal{Q})$:*

$$\begin{aligned}
\|\eta\|_{W^1L_{\mathfrak{M}}(\mathcal{Q})} &= \|\eta\|_{L_{\mathfrak{M}}(\mathcal{Q})} + \|\nabla\eta\|_{L_{\mathfrak{M}}(\mathcal{Q})},\\
\|\eta\|_{2,\mathfrak{M}} &= \max\big\{\|\eta\|_{L_{\mathfrak{M}}(\mathcal{Q})}, \|\nabla\eta\|_{L_{\mathfrak{M}}(\mathcal{Q})}\big\},\\
\|\eta\| &= \inf\Big\{\upsilon > 0 : \int_{\mathcal{Q}} \Big[\mathfrak{M}\Big(\theta, \tfrac{|\eta(\theta)|}{\upsilon}\Big) + \mathfrak{M}\Big(\theta, \tfrac{|\nabla\eta(\theta)|}{\upsilon}\Big)\Big] d\theta \le 1\Big\}.
\end{aligned}$$

Lemma 2.6 ([25]). *Assume that $\mathfrak{M}$ and $\mathfrak{M}^*$ are complementary Musielak functions satisfying the Δ_2-condition. Then there exist constants $1 < m_* \le m^* < \infty$ such that*

$$m_* \le \frac{t\, \mathfrak{M}'_d(\theta, t)}{\mathfrak{M}(\theta, t)} \le m^*, \qquad \forall\, \theta \in \mathcal{Q},\ t > 0.$$

Moreover:

(i) If $\|\eta\| \le 1$, then

$$\|\eta\|^{m^*} \le \int_{\mathcal{Q}} \big(\mathfrak{M}(\theta, |\eta(\theta)|) + \mathfrak{M}(\theta, |\nabla\eta(\theta)|)\big)\, d\theta \le \|\eta\|^{m_*}.$$

(ii) If $\|\eta\| > 1$, then

$$\|\eta\|^{m_*} \leq \int_{\mathcal{Q}} \big(\mathfrak{M}(\theta, |\eta(\theta)|) + \mathfrak{M}(\theta, |\nabla\eta(\theta)|)\big)\, d\theta \leq \|\eta\|^{m^*}.$$

(iii) If $\eta_n \to \eta$ in $W^1L_{\mathfrak{M}}(\mathcal{Q})$, then

$$\int_{\mathcal{Q}} \big(\mathfrak{M}(\theta, |\eta_n(\theta)|) + \mathfrak{M}(\theta, |\nabla\eta_n(\theta)|)\big)\, d\theta \to \int_{\mathcal{Q}} \big(\mathfrak{M}(\theta, |\eta(\theta)|) + \mathfrak{M}(\theta, |\nabla\eta(\theta)|)\big)\, d\theta.$$

For additional details and comprehensive studies on Musielak-Orlicz-Sobolev spaces, we refer the reader to [11, 13, 14, 15, 16, 19, 20].

Throughout this work, let $\ell_i(\cdot), \kappa_i(\cdot) \in \mathcal{C}_+(\overline{\mathcal{Q}})$ $(i = 1, 2)$ be variable exponents satisfying condition (2.2). We define the associated Musielak functions by

$$\mathfrak{M}_i(\theta, t) = t^{\ell_i(\theta)} + t^{\kappa_i(\theta)}, \qquad (\theta, t) \in \mathcal{Q} \times \mathbb{R}_+^*, \quad i = 1, 2. \tag{2.5}$$

It is readily seen that each $\mathfrak{M}_i$ and its complementary function fulfill the Δ_2-condition. We then consider the Musielak-Orlicz-Sobolev spaces

$$W^{1,\ell_i(\cdot),\kappa_i(\cdot)}(\mathcal{Q}) := W^1L_{\mathfrak{M}_i}(\mathcal{Q}), \qquad i = 1, 2,$$

endowed with the norm

$$\|\eta\|_{W^1L_{\mathfrak{M}_i}(\mathcal{Q})} = \|\eta\|_{1,\ell_i(\cdot)} + \|\eta\|_{1,\kappa_i(\cdot)}.$$

Proposition 2.7 ([11]). *Assume that condition (2.2) holds. Then the embedding*

$$W^1L_{\mathfrak{M}_i}(\mathcal{Q}) \hookrightarrow W^{1,m_*}(\mathcal{Q})$$

is continuous, and it is compact into $W^{1,m_*}(\mathcal{Q})$. In particular, we have the compact embedding

$$W^1L_{\mathfrak{M}_i}(\mathcal{Q}) \hookrightarrow\hookrightarrow C^0(\overline{\mathcal{Q}}).$$

Moreover, we define

$$d_i = \sup_{\eta \in W^1L_{\mathfrak{M}_i}(\mathcal{Q})\setminus\{0\}} \frac{\|\eta\|_{L^\infty(\mathcal{Q})}}{\|\eta\|_{W^1L_{\mathfrak{M}_i}(\mathcal{Q})}}. \tag{2.6}$$

Finally, we define the product space

$$X := \prod_{i=1}^{2} W^{1,\ell_i(\cdot),\kappa_i(\cdot)}(\mathcal{Q}) = W^1L_{\mathfrak{M}_1}(\mathcal{Q}) \times W^1L_{\mathfrak{M}_2}(\mathcal{Q}), \tag{2.7}$$

endowed with the norm

$$\|(\eta, \zeta)\|_X = \sqrt{\|\eta\|^2_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})} + \|\zeta\|^2_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}}, \tag{2.8}$$

or with any equivalent norm.

For each $\theta \in \partial\mathcal{Q}$, we define the corresponding boundary exponent as

$$\ell^\partial(\theta) = \begin{cases} \dfrac{(N-1)\,\ell(\theta)}{N - \ell(\theta)}, & \text{if } \ell(\theta) < N, \\ \infty, & \text{if } \ell(\theta) \geq N. \end{cases}$$

Next, we introduce the boundary Musielak function $\mathfrak{M}_i^\partial : \partial\mathcal{Q} \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\mathfrak{M}_i^\partial(\theta, t) = t^{\ell_i^\partial(\theta)} + t^{\kappa_i^\partial(\theta)},$$

where $\ell_i^\partial(\cdot)$ and $\kappa_i^\partial(\cdot)$ are obtained from $\ell_i(\cdot)$ and $\kappa_i(\cdot)$, respectively, as described above for $i = 1, 2$.

For a detailed account of the theory of Musielak-Orlicz-Sobolev spaces and their trace embeddings, we refer the reader to [8, 21].

Theorem 2.8 (Compact Trace Embedding). *Let* $\mathcal{Q} \subset \mathbb{R}^N$ *be a bounded domain with smooth boundary* $\partial\mathcal{Q}$. *Assume that* $\ell_i(\cdot), \kappa_i(\cdot) \in \mathcal{C}_+(\overline{\mathcal{Q}})$ *for* $i = 1, 2$, *and let* $r(\cdot) \in \mathcal{C}_+(\partial\mathcal{Q})$ *be an exponent satisfying*

$$1 \leq r(\theta) \leq \ell_i^{\partial}(\theta) \quad \textit{for all } \theta \in \partial\mathcal{Q} \textit{ and } i = 1, 2,$$

with strict inequality $r(\theta) < \ell_i^{\partial}(\theta)$ *whenever* $\ell_i^{\partial}(\theta) = \infty$.

Then the trace operator

$$T : W^1 L_{\mathfrak{M}_i}(\mathcal{Q}) \longrightarrow L^{r(\cdot)}(\partial\mathcal{Q})$$

is compact. In particular, the following compact embedding holds:

$$W^1 L_{\mathfrak{M}_i}(\mathcal{Q}) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\partial\mathcal{Q}).$$

Proof. The proof follows the classical arguments used for trace embeddings in variable exponent and Musielak-Orlicz-Sobolev spaces. First, note that for each $i = 1, 2$, the space $W^1 L_{\mathfrak{M}_i}(\mathcal{Q})$ is continuously embedded into $W^{1,\ell_i^-}(\mathcal{Q})$, where $\ell_i^- := \inf_{\theta \in \mathcal{Q}} \ell_i(\theta)$. It is well known that the trace operator from $W^{1,\ell_i^-}(\mathcal{Q})$ into $L^s(\partial\mathcal{Q})$ is compact whenever $s < (\ell_i^-)^{\partial}$. Since $r(\cdot)$ satisfies $r(\theta) \leq \ell_i^{\partial}(\theta)$ for all $\theta \in \partial\mathcal{Q}$, a localization and density argument allows this compactness result to extend to the variable exponent framework. Consequently, the trace operator T is compact from $W^1 L_{\mathfrak{M}_i}(\mathcal{Q})$ into $L^{r(\cdot)}(\partial\mathcal{Q})$. □

3 Essential Assumptions

The functions $a, b, c, d : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ are assumed to be Carathéodory mappings. They are continuous on $\overline{\mathcal{O}} \times \mathbb{R}^N$ and satisfy the following hypotheses for all $\theta \in \overline{\mathcal{O}}$ and $\xi, \eta \in \mathbb{R}^N$:

(H1) **Normalization:**

$$A(\theta, 0) = B(\theta, 0) = C(\theta, 0) = D(\theta, 0) = 0. \tag{3.1}$$

(H2) **Growth Conditions:** There exist positive constants r_1, r_2, r_3, r_4 such that

$$\begin{aligned} |a(\theta,s)| &\leq r_1\big(1 + |s|^{\ell_1(\theta)-1}\big), & |b(\theta,s)| &\leq r_2\big(1 + |s|^{\kappa_1(\theta)-1}\big), \\ |c(\theta,s)| &\leq r_3\big(1 + |s|^{\ell_2(\theta)-1}\big), & |d(\theta,s)| &\leq r_4\big(1 + |s|^{\kappa_2(\theta)-1}\big). \end{aligned} \tag{3.2}$$

(H3) **Coercivity and Boundedness:** For all $\theta \in \overline{\mathcal{O}}$ and $\xi \in \mathbb{R}^N$,

$$\begin{aligned} |s|^{\ell_1(\theta)} &\leq a(\theta,s) \cdot \xi \leq \ell_1(\theta)\, A(\theta,s), \\ |s|^{\kappa_1(\theta)} &\leq b(\theta,s) \cdot \xi \leq \kappa_1(\theta)\, B(\theta,s), \\ |s|^{\ell_2(\theta)} &\leq c(\theta,s) \cdot \xi \leq \ell_2(\theta)\, C(\theta,s), \\ |s|^{\kappa_2(\theta)} &\leq d(\theta,s) \cdot \xi \leq \kappa_2(\theta)\, D(\theta,s). \end{aligned} \tag{3.3}$$

(H4) **Monotonicity:** The mappings $\xi \mapsto a(\theta, s)$, etc., are monotone, i.e.,

$$\begin{aligned} &\big(a(\theta,s) - a(\theta,t)\big) \cdot (s-t) \geq 0, \\ &\big(b(\theta,s) - b(\theta,t)\big) \cdot (s-t) \geq 0, \\ &\big(c(\theta,s) - c(\theta,t)\big) \cdot (s-t) \geq 0, \\ &\big(d(\theta,s) - d(\theta,t)\big) \cdot (s-t) \geq 0, \end{aligned} \tag{3.4}$$

with equality if and only if $s = t$.

(H5) **Uniform Convexity:** There exist constants $e_1, e_2, e_3, e_4 > 0$ such that for all $\eta, \zeta \in \mathbb{R}^N$,

$$\begin{aligned} A\Big(\theta, \tfrac{\eta+\zeta}{2}\Big) &\le \tfrac{1}{2}A(\theta,\eta) + \tfrac{1}{2}A(\theta,\zeta) - e_1|\eta-\zeta|^{\ell_1(\theta)}, \\ B\Big(\theta, \tfrac{\eta+\zeta}{2}\Big) &\le \tfrac{1}{2}B(\theta,\eta) + \tfrac{1}{2}B(\theta,\zeta) - e_2|\eta-\zeta|^{\kappa_1(\theta)}, \\ C\Big(\theta, \tfrac{\eta+\zeta}{2}\Big) &\le \tfrac{1}{2}C(\theta,\eta) + \tfrac{1}{2}C(\theta,\zeta) - e_3|\eta-\zeta|^{\ell_2(\theta)}, \\ D\Big(\theta, \tfrac{\eta+\zeta}{2}\Big) &\le \tfrac{1}{2}D(\theta,\eta) + \tfrac{1}{2}D(\theta,\zeta) - e_4|\eta-\zeta|^{\kappa_2(\theta)}, \end{aligned} \tag{3.5}$$

where the functions $A, B, C, D : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ are defined by

$$A(\theta,s) = \int_0^s a(\theta,t)\,dt, \quad B(\theta,s) = \int_0^s b(\theta,t)\,dt,$$

$$C(\theta,s) = \int_0^s c(\theta,t)\,dt, \quad D(\theta,s) = \int_0^s d(\theta,t)\,dt.$$

(H6) **Boundary Growth Conditions:** There exist constants $h_1, h_2, h_3, h_4 > 0$ and functions $\nu_1, \nu_2, \nu_3, \nu_4 \in C_+(\overline{\partial\mathcal{Q}})$ with

$$\nu_1(\theta) \le \ell_1^{\partial}(\theta), \quad \nu_2(\theta) \le \kappa_1^{\partial}(\theta), \quad \nu_3(\theta) \le \ell_2^{\partial}(\theta), \quad \nu_4(\theta) \le \kappa_2^{\partial}(\theta)$$

for a.e. $\theta \in \partial\mathcal{Q}$, and such that for a.e. $\theta \in \partial\mathcal{Q}$ and all $s \in \mathbb{R}$

$$\begin{aligned} |\psi_1(\theta,s)| &\le h_1\big(f_1(\theta) + |s|^{\nu_1(\theta)-1}\big), \\ |\psi_2(\theta,s)| &\le h_2\big(f_2(\theta) + |s|^{\nu_2(\theta)-1}\big), \\ |\psi_3(\theta,s)| &\le h_3\big(f_3(\theta) + |s|^{\nu_3(\theta)-1}\big), \\ |\psi_4(\theta,s)| &\le h_4\big(f_4(\theta) + |s|^{\nu_4(\theta)-1}\big). \end{aligned} \tag{3.6}$$

Moreover assume $f_i \ge 0$ are measurable on $\partial\mathcal{Q}$ and

$$f_1 \in L^{\nu_1'(\cdot)}(\partial\mathcal{Q}), \quad f_2 \in L^{\nu_2'(\cdot)}(\partial\mathcal{Q}), \quad f_3 \in L^{\nu_3'(\cdot)}(\partial\mathcal{Q}), \quad f_4 \in L^{\nu_4'(\cdot)}(\partial\mathcal{Q}),$$

where $\nu_i'(\theta) = \dfrac{\nu_i(\theta)}{\nu_i(\theta)-1}$ are the variable conjugates (a.e.).

(H7) **Nonlinear Coupling Term:** Let $\Phi \in \mathcal{C}^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ be such that $\Phi(\theta, 0, 0) = 0$ for all $\theta \in \mathbb{R}$, and suppose that

$$\begin{aligned} \left|\frac{\partial\Phi}{\partial\eta}(\theta,\eta,\zeta)\right| &\le \phi_1(\theta) + |\eta|^{p(\theta)-1}, \\ \left|\frac{\partial\Phi}{\partial\zeta}(\theta,\eta,\zeta)\right| &\le \phi_2(\theta) + |\zeta|^{q(\theta)-1}, \end{aligned} \tag{3.7}$$

where $p(\cdot), q(\cdot) \in C_+(\overline{\mathcal{Q}})$ satisfy

$$p^- \ge \kappa_1^+, \qquad q^- \ge \kappa_2^+, \tag{3.8}$$

and ϕ_1, ϕ_2 are positive functions belonging to $L^1(\mathcal{Q})$.

We introduce the functionals

$$\Upsilon_1,\, \Upsilon_2 : X \longrightarrow \mathbb{R}$$

defined by

$$\begin{aligned} \Upsilon_1(\eta,\zeta) \;=\; & \int_{\mathcal{Q}} A(\theta,\nabla\eta)\,d\theta + \int_{\mathcal{Q}} A(\theta,\eta)\,d\theta + \int_{\mathcal{Q}} B(\theta,\nabla\eta)\,d\theta + \int_{\mathcal{Q}} B(\theta,\eta)\,d\theta \\ & + \int_{\mathcal{Q}} C(\theta,\nabla\zeta)\,d\theta + \int_{\mathcal{Q}} C(\theta,\zeta)\,d\theta + \int_{\mathcal{Q}} D(\theta,\nabla\zeta)\,d\theta + \int_{\mathcal{Q}} D(\theta,\zeta)\,d\theta \\ & - \int_{\partial\mathcal{Q}} \Psi_1(\theta,\eta)\,d\Gamma - \int_{\partial\mathcal{Q}} \Psi_2(\theta,\eta)\,d\Gamma - \int_{\partial\mathcal{Q}} \Psi_3(\theta,\zeta)\,d\Gamma - \int_{\partial\mathcal{Q}} \Psi_4(\theta,\zeta)\,d\Gamma \end{aligned} \tag{3.9}$$

and

$$\Upsilon_2(\eta,\zeta) = -\int_Q \Phi(\theta,\eta,\zeta)\,d\theta, \tag{3.10}$$

where

$$\Psi_i(\theta,s) = \int_0^s \psi_i(\theta,t)\,dt, \quad i = 1,2,3,4.$$

Lemma 3.1 ([2, 4]). *The functionals* Υ_1 *and* Υ_2 *are well defined on* X. *Moreover, both* Υ_1 *and* Υ_2 *belong to the class* $\mathcal{C}^1(X,\mathbb{R})$, *and their Fréchet derivatives are given by*

$$\begin{aligned}
&\Upsilon_1'(\eta,\zeta)(\omega_1,\omega_2)\\
&= \int_Q a(\theta,\nabla\eta)\cdot\nabla\omega_1\,d\theta + \int_Q a(\theta,\eta)\,\omega_1\,d\theta + \int_Q b(\theta,\nabla\eta)\cdot\nabla\omega_1\,d\theta + \int_Q b(\theta,\eta)\,\omega_1\,d\theta\\
&+ \int_Q c(\theta,\nabla\zeta)\cdot\nabla\omega_2\,d\theta + \int_Q c(\theta,\zeta)\,\omega_2\,d\theta + \int_Q d(\theta,\nabla\zeta)\cdot\nabla\omega_2\,d\theta + \int_Q d(\theta,\zeta)\,\omega_2\,d\theta\\
&- \int_{\partial Q} \psi_1(\theta,\eta)\,\omega_1\,d\Gamma - \int_{\partial Q} \psi_2(\theta,\eta)\,\omega_1\,d\Gamma - \int_{\partial Q} \psi_3(\theta,\zeta)\,\omega_2\,d\Gamma - \int_{\partial Q} \psi_4(\theta,\zeta)\,\omega_2\,d\Gamma,
\end{aligned}$$

and

$$\Upsilon_2'(\eta,\zeta)(\omega_1,\omega_2) = -\int_Q \left(\frac{\partial\Phi}{\partial\eta}(\theta,\eta,\zeta)\,\omega_1 + \frac{\partial\Phi}{\partial\zeta}(\theta,\eta,\zeta)\,\omega_2\right) d\theta,$$

for all $(\eta,\zeta),(\omega_1,\omega_2)\in X$.

Lemma 3.2 ([2, 4]). *Assume that conditions* (3.1)-(3.8) *and* (2.2) *hold. Then the functionals* Υ_1 *and* Υ_2 *are weakly lower semicontinuous on* X.

Lemma 3.3. *Under the assumptions* (3.3), (3.6), *and* (2.2), *the functional* Υ_1 *is coercive, that is,*

$$\Upsilon_1(\eta,\zeta) \longrightarrow +\infty \quad \text{as } \|(\eta,\zeta)\|_X \longrightarrow +\infty.$$

Proof. Throughout this proof, we denote by C_i a generic positive constant that may vary from line to line. Let $(\eta,\zeta)\in X$ be such that $\|\eta\|_{W^1L_{\mathfrak{M}_1}(Q)} > 1$ and $\|\zeta\|_{W^1L_{\mathfrak{M}_2}(Q)} > 1$. From the definition of Υ_1, we have

$$\begin{aligned}
\Upsilon_1(\eta,\zeta) \;=\; & \int_Q A(\theta,\nabla\eta)\,d\theta + \int_Q A(\theta,\eta)\,d\theta + \int_Q B(\theta,\nabla\eta)\,d\theta + \int_Q B(\theta,\eta)\,d\theta\\
&+ \int_Q C(\theta,\nabla\zeta)\,d\theta + \int_Q C(\theta,\zeta)\,d\theta + \int_Q D(\theta,\nabla\zeta)\,d\theta + \int_Q D(\theta,\zeta)\,d\theta\\
&- \int_{\partial Q} \Psi_1(\theta,\eta)\,d\Gamma - \int_{\partial Q} \Psi_2(\theta,\eta)\,d\Gamma - \int_{\partial Q} \Psi_3(\theta,\zeta)\,d\Gamma - \int_{\partial Q} \Psi_4(\theta,\zeta)\,d\Gamma.
\end{aligned}$$

By applying condition (3.3), we obtain

$$\begin{aligned}
\Upsilon_1(\eta,\zeta) \;\geq\; & \int_Q \frac{1}{\ell_1(\theta)}\left(|\nabla\eta|^{\ell_1(\theta)} + |\eta|^{\ell_1(\theta)}\right) d\theta + \int_Q \frac{1}{\kappa_1(\theta)}\left(|\nabla\eta|^{\kappa_1(\theta)} + |\eta|^{\kappa_1(\theta)}\right) d\theta\\
&+ \int_Q \frac{1}{\ell_2(\theta)}\left(|\nabla\zeta|^{\ell_2(\theta)} + |\zeta|^{\ell_2(\theta)}\right) d\theta + \int_Q \frac{1}{\kappa_2(\theta)}\left(|\nabla\zeta|^{\kappa_2(\theta)} + |\zeta|^{\kappa_2(\theta)}\right) d\theta\\
&- \sum_{i=1}^{4} \int_{\partial Q} \Psi_i(\theta,\eta,\zeta)\,d\Gamma.
\end{aligned} \tag{3.11}$$

Using the variable exponent inequalities and the definition of the Sobolev modular norms, it follows that

$$\begin{aligned}\Upsilon_1(\eta,\zeta) \quad \geq \quad & \frac{1}{\ell_1^+}\left(\|\eta\|^{\ell_1^-}_{W^{1,\ell_1(\cdot)}(\mathcal{Q})}-1\right)+\frac{1}{\kappa_1^+}\left(\|\eta\|^{\kappa_1^-}_{W^{1,\kappa_1(\cdot)}(\mathcal{Q})}-1\right)\\ & +\frac{1}{\ell_2^+}\left(\|\zeta\|^{\ell_2^-}_{W^{1,\ell_2(\cdot)}(\mathcal{Q})}-1\right)+\frac{1}{\kappa_2^+}\left(\|\zeta\|^{\kappa_2^-}_{W^{1,\kappa_2(\cdot)}(\mathcal{Q})}-1\right)\\ & -\sum_{i=1}^{4}\int_{\partial\mathcal{Q}}\Psi_i(\theta,\eta,\zeta)\,d\Gamma.\end{aligned}$$

After standard estimates and collecting lower-order constants, we obtain

$$\Upsilon_1(\eta,\zeta) \quad \geq \quad C_1\left(\|\eta\|^{\ell_1^-}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}+\|\zeta\|^{\ell_2^-}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}\right)-C_2-\sum_{i=1}^{4}\int_{\partial\mathcal{Q}}\Psi_i(\theta,\eta,\zeta)\,d\Gamma. \tag{3.12}$$

Now, by condition (3.6), for every $\varepsilon>0$, there exists $\delta>0$ such that for all $|s|>\delta$,

$$|\psi_1(\theta,s)|\leq\frac{\varepsilon}{\ell_1^-}|s|^{\ell_1^-}.$$

Hence,

$$\int_{\partial\mathcal{Q}}\Psi_1(\theta,\eta)\,d\Gamma\leq\int_{\{\theta\in\partial\mathcal{Q}:|\eta(\theta)|\leq\delta\}}\Psi_1(\theta,\eta)\,d\Gamma+\frac{\varepsilon}{\ell_1^-}\int_{\{\theta\in\partial\mathcal{Q}:|\eta(\theta)|>\delta\}}|\eta(\theta)|^{\ell_1^-}\,d\Gamma.$$

Since Ψ_1 is continuous and the set $\{\theta\in\partial\mathcal{Q}:|\eta(\theta)|\leq\delta\}$ is compact, it follows that

$$\int_{\partial\mathcal{Q}}\Psi_1(\theta,\eta)\,d\Gamma\leq C_3+\frac{\varepsilon}{\ell_1^-}\|\eta\|^{\ell_1^-}_{\ell_1^-,\partial\mathcal{Q}}.$$

Because $\ell_1^-<\ell_1^\partial(\cdot)$, we have the compact trace embedding $W^1L_{\mathfrak{M}_1}(\mathcal{Q})\hookrightarrow L^{\ell_1^-}(\partial\mathcal{Q})$, and thus

$$\int_{\partial\mathcal{Q}}\Psi_1(\theta,\eta)\,d\Gamma\leq C_3+C_4\frac{\varepsilon}{\ell_1^-}\|\eta\|^{\ell_1^-}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}. \tag{3.13}$$

Analogously, we have

$$\int_{\partial\mathcal{Q}}\Psi_2(\theta,\eta)\,d\Gamma\leq C_5+C_6\frac{\varepsilon}{\kappa_1^-}\|\eta\|^{\kappa_1^-}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}, \tag{3.14}$$

$$\int_{\partial\mathcal{Q}}\Psi_3(\theta,\zeta)\,d\Gamma\leq C_7+C_8\frac{\varepsilon}{\ell_2^-}\|\zeta\|^{\ell_2^-}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}, \tag{3.15}$$

$$\int_{\partial\mathcal{Q}}\Psi_4(\theta,\zeta)\,d\Gamma\leq C_9+C_{10}\frac{\varepsilon}{\kappa_2^-}\|\zeta\|^{\kappa_2^-}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}. \tag{3.16}$$

Combining (3.11)-(3.16), we obtain

$$\begin{aligned}\Upsilon_1(\eta,\zeta) \quad \geq \quad & C_1\left(\|\eta\|^{\ell_1^-}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}+\|\zeta\|^{\ell_2^-}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}\right)-C_4\frac{\varepsilon}{\ell_1^-}\|\eta\|^{\ell_1^-}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}-C_6\frac{\varepsilon}{\kappa_1^-}\|\eta\|^{\kappa_1^-}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}\\ & -C_8\frac{\varepsilon}{\ell_2^-}\|\zeta\|^{\ell_2^-}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}-C_{10}\frac{\varepsilon}{\kappa_2^-}\|\zeta\|^{\kappa_2^-}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}-C_{11}.\end{aligned}$$

Choosing

$$\varepsilon=\tfrac{1}{2}\min\left(\frac{C_1\ell_1^-}{C_4},\frac{C_1\kappa_1^-}{C_6},\frac{C_1\ell_2^-}{C_8},\frac{C_1\kappa_2^-}{C_{10}}\right),$$

and using the fact that $\|\eta\|_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}>1$, $\|\zeta\|_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}>1$, and $\ell_1^-,\ell_2^->N$, we deduce that

$$\begin{aligned}\Upsilon_1(\eta,\zeta) \quad & \geq \quad C_{12}\left(\|\eta\|^{\ell_1^-}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}+\|\zeta\|^{\ell_2^-}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}\right)-C_{13}\\ & \geq \quad C_{12}\left(\|\eta\|^{N}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}+\|\zeta\|^{N}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}\right)-C_{13}\\ & = \quad C_{12}\left(\left(\|\eta\|^{2}_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}\right)^{\frac{N}{2}}+\left(\|\zeta\|^{2}_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}\right)^{\frac{N}{2}}\right)-C_{13}.\end{aligned}$$

Using the classical inequality

$$(a+b)^p \leq 2^{p-1}(a^p+b^p), \quad a,b\geq 0,\ 1\leq p<\infty, \tag{3.17}$$

with $a=\|\eta\|^2_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}$, $b=\|\zeta\|^2_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}$, and $p=\frac{N}{2}$, we obtain

$$\begin{aligned}\Upsilon_1(\eta,\zeta) &\geq C_{12}2^{1-\frac{N}{2}}\left(\|\eta\|^2_{W^1L_{\mathfrak{M}_1}(\mathcal{Q})}+\|\zeta\|^2_{W^1L_{\mathfrak{M}_2}(\mathcal{Q})}\right)^{\frac{N}{2}}-C_{13}\\ &= C_{12}2^{1-\frac{N}{2}}\|(\eta,\zeta)\|_X^N-C_{13}.\end{aligned}$$

Hence, as $\|(\eta,\zeta)\|_X\to+\infty$, we conclude that $\Upsilon_1(\eta,\zeta)\to+\infty$. □

The sets $\Theta_1(\tau)$ and $\Theta_2(\tau)$, for $\tau>0$, defined below, play a crucial role in our analysis:

$$\begin{aligned}\Theta_1(\tau) &= \Bigg\{(u,v)\in\mathbb{R}^2 : \frac{1}{\ell_1^+}\mathcal{F}_{\ell_1(\cdot)}(u)+\frac{1}{\kappa_1^+}\mathcal{F}_{\kappa_1(\cdot)}(u)+\frac{1}{\ell_2^+}\mathcal{F}_{\ell_2(\cdot)}(v)+\frac{1}{\kappa_2^+}\mathcal{F}_{\kappa_2(\cdot)}(v)\\ &\quad -h_1\mathcal{G}_{\nu_1'(\cdot)}(f_1)-2h_1\mathcal{G}_{\nu_1(\cdot)}(u)-h_2\mathcal{G}_{\nu_2'(\cdot)}(f_2)-2h_2\mathcal{G}_{\nu_2(\cdot)}(u)\\ &\quad -h_3\mathcal{G}_{\nu_3'(\cdot)}(f_3)-2h_3\mathcal{G}_{\nu_3(\cdot)}(v)-h_4\mathcal{G}_{\nu_4'(\cdot)}(f_4)-2h_4\mathcal{G}_{\nu_4(\cdot)}(v)\leq\tau\Bigg\},\end{aligned}$$

and

$$\begin{aligned}\Theta_2(\tau) &= \Bigg\{(u,v)\in\mathbb{R}^2 : \int_{\mathcal{Q}}A(\theta,u)\,d\theta+\int_{\mathcal{Q}}B(\theta,u)\,d\theta+\int_{\mathcal{Q}}C(\theta,v)\,d\theta+\int_{\mathcal{Q}}D(\theta,v)\,d\theta\\ &\quad -\int_\Gamma\Psi_1(\theta,u)\,d\Gamma-\int_\Gamma\Psi_2(\theta,u)\,d\Gamma-\int_\Gamma\Psi_3(\theta,v)\,d\Gamma-\int_\Gamma\Psi_4(\theta,v)\,d\Gamma\leq\tau\Bigg\}.\end{aligned}$$

Here,

$$\mathcal{G}_{s(\cdot)}(t)=\max\left(|t|^{s^+},|t|^{s^-}\right) \quad\text{and}\quad \mathcal{F}_{s(\cdot)}(t)=\min\left(|t|^{s^+},|t|^{s^-}\right),$$

for $s(\cdot)\in\{\ell_1(\cdot),\kappa_1(\cdot),\ell_2(\cdot),\kappa_2(\cdot),\nu_1(\cdot),\nu_2(\cdot),\nu_3(\cdot),\nu_4(\cdot)\}$ and $t\in\{u,v\}$.

Lemma 3.4. *For all $\tau>0$, we have*

$$\Theta_2(\tau)\subset\Theta_1(\tau).$$

Proof. Using assumptions (3.3) and (3.6), together with Young's inequality, we obtain

$$\begin{aligned}&\int_{\mathcal{Q}}A(\theta,u)\,d\theta+\int_{\mathcal{Q}}B(\theta,u)\,d\theta+\int_{\mathcal{Q}}C(\theta,v)\,d\theta+\int_{\mathcal{Q}}D(\theta,v)\,d\theta\\ &-\int_\Gamma\Psi_1(\theta,u)\,d\Gamma-\int_\Gamma\Psi_2(\theta,u)\,d\Gamma-\int_\Gamma\Psi_3(\theta,v)\,d\Gamma-\int_\Gamma\Psi_4(\theta,v)\,d\Gamma\\ \geq\ &\frac{1}{\ell_1^+}\int_{\mathcal{Q}}|u|^{\ell_1(\theta)}\,d\theta+\frac{1}{\kappa_1^+}\int_{\mathcal{Q}}|u|^{\kappa_1(\theta)}\,d\theta+\frac{1}{\ell_2^+}\int_{\mathcal{Q}}|v|^{\ell_2(\theta)}\,d\theta+\frac{1}{\kappa_2^+}\int_{\mathcal{Q}}|v|^{\kappa_2(\theta)}\,d\theta\\ &-\int_\Gamma h_1\left(f_1(\theta)|u|+\frac{|u|^{\nu_1(\theta)}}{\nu_1(\theta)}\right)d\Gamma-\int_\Gamma h_2\left(f_2(\theta)|u|+\frac{|u|^{\nu_2(\theta)}}{\nu_2(\theta)}\right)d\Gamma\\ &-\int_\Gamma h_3\left(f_3(\theta)|v|+\frac{|v|^{\nu_3(\theta)}}{\nu_3(\theta)}\right)d\Gamma-\int_\Gamma h_4\left(f_4(\theta)|v|+\frac{|v|^{\nu_4(\theta)}}{\nu_4(\theta)}\right)d\Gamma\\ \geq\ &\frac{1}{\ell_1^+}\mathcal{F}_{\ell_1(\cdot)}(u)+\frac{1}{\kappa_1^+}\mathcal{F}_{\kappa_1(\cdot)}(u)+\frac{1}{\ell_2^+}\mathcal{F}_{\ell_2(\cdot)}(v)+\frac{1}{\kappa_2^+}\mathcal{F}_{\kappa_2(\cdot)}(v)\\ &-h_1\mathcal{G}_{\nu_1'(\cdot)}(f_1)-2h_1\mathcal{G}_{\nu_1(\cdot)}(u)-h_2\mathcal{G}_{\nu_2'(\cdot)}(f_2)-2h_2\mathcal{G}_{\nu_2(\cdot)}(u)\\ &-h_3\mathcal{G}_{\nu_3'(\cdot)}(f_3)-2h_3\mathcal{G}_{\nu_3(\cdot)}(v)-h_4\mathcal{G}_{\nu_4'(\cdot)}(f_4)-2h_4\mathcal{G}_{\nu_4(\cdot)}(v),\end{aligned}$$

which holds for every $(u,v)\in\mathbb{R}^2$. Therefore, the inclusion

$$\Theta_2(\tau)\subset\Theta_1(\tau), \quad \text{for all } \tau>0,$$

is established. □

4 Main results

Definition 4.1. We say that $(\eta, \zeta) \in X$ is a weak solution of problem (1.1) if, for every $(\omega_1, \omega_2) \in X$, the following identity holds:

$$\begin{aligned}
&\int_Q a(\theta, \nabla\eta) \cdot \nabla\omega_1 \, d\theta + \int_Q a(\theta, \eta)\, \omega_1 \, d\theta + \int_Q b(\theta, \nabla\eta) \cdot \nabla\omega_1 \, d\theta + \int_Q b(\theta, \eta)\, \omega_1 \, d\theta \\
&+ \int_Q c(\theta, \nabla\zeta) \cdot \nabla\omega_2 \, d\theta + \int_Q c(\theta, \zeta)\, \omega_2 \, d\theta + \int_Q d(\theta, \nabla\zeta) \cdot \nabla\omega_2 \, d\theta + \int_Q d(\theta, \zeta)\, \omega_2 \, d\theta \\
&= \int_Q \left(\frac{\partial \Phi}{\partial \eta}(\theta, \eta, \zeta)\, \omega_1(\theta) + \frac{\partial \Phi}{\partial \zeta}(\theta, \eta, \zeta)\, \omega_2(\theta) \right) d\theta \\
&\quad + \int_{\partial Q} \psi_1(\theta, \eta)\, \omega_1 \, d\Gamma + \int_{\partial Q} \psi_2(\theta, \eta)\, \omega_1 \, d\Gamma + \int_{\partial Q} \psi_3(\theta, \zeta)\, \omega_2 \, d\Gamma + \int_{\partial Q} \psi_4(\theta, \zeta)\, \omega_2 \, d\Gamma.
\end{aligned}$$

The weak solutions of (1.1) are precisely the critical points of the functional $\Upsilon_1 + \Upsilon_2$.

One of our main results is given below.

Theorem 4.2. *Assume that* $\Upsilon_1(\cdot, \cdot)$ *and* $\Upsilon_2(\cdot, \cdot)$ *are as defined in* (3.9) *and* (3.10)*, and that conditions* (3.1)-(3.8) *and* (2.2) *are satisfied. Then:*

(a) *If there exist* $\vartheta_0 > 0$ *and* $(u_0, v_0) \in \mathbb{R}^2$ *such that* $(u_0, v_0) \in \operatorname{Int}(\Theta_2(\vartheta_0))$ *(where* $\operatorname{Int}(\Theta_2)$ *denotes the interior of* Θ_2*) and for every* $\theta \in Q$,

$$\max_{(u,v) \in \Theta_1(\vartheta_n)} \Phi(\theta, u, v) = \Phi(\theta, u_0, v_0), \quad \forall n > 0,$$

then problem (1.1) *admits a weak solution* $(\eta, \zeta) \in X$ *such that* $\Upsilon_1(\eta, \zeta) < \vartheta_0$.

(b) *If there exist sequences* $(\vartheta_n)_n \subset \mathbb{R}^+$ *with* $\vartheta_n \to \infty$ *as* $n \to +\infty$ *and* $(u_n)_n, (v_n)_n \subset \mathbb{R}$ *such that* $(u_n, v_n) \in \operatorname{Int}(\Theta_2(\vartheta_n))$ *and for all* $\theta \in Q$,

$$\max_{(u,v) \in \Theta_1(\vartheta_n)} \Phi(\theta, u, v) = \Phi(\theta, u_n, v_n), \quad \forall n > 0,$$

and if

$$\limsup_{(u,v) \to +\infty} \frac{\displaystyle\int_Q \Phi(\theta, u, v)\, d\theta + \sum_{i=1}^{4} \int_\Gamma \Psi_i(\theta, w_i)\, d\Gamma}{\displaystyle\int_Q [A(\theta, u) + B(\theta, u) + C(\theta, v) + D(\theta, v)]\, d\theta} > 1,$$

then problem (1.1) *admits an unbounded sequence of weak solutions in* X.

(c) *If there exist sequences* $(\vartheta_n)_n \subset \mathbb{R}^+$ *with* $\vartheta_n \to 0$ *as* $n \to +\infty$ *and* $(u_n)_n, (v_n)_n \subset \mathbb{R}$ *such that* $(u_n, v_n) \in \operatorname{Int}(\Theta_2(\vartheta_n))$ *and for all* $\theta \in Q$,

$$\max_{(u,v) \in \Theta_1(\vartheta_n)} \Phi(\theta, u, v) = \Phi(\theta, u_n, v_n), \quad \forall n > 0,$$

and if

$$\limsup_{(u,v) \to (0,0)} \frac{\displaystyle\int_Q \Phi(\theta, u, v)\, d\theta + \sum_{i=1}^{4} \int_\Gamma \Psi_i(\theta, w_i)\, d\Gamma}{\displaystyle\int_Q [A(\theta, u) + B(\theta, u) + C(\theta, v) + D(\theta, v)]\, d\theta} > 1,$$

then problem (1.1) *admits a sequence of nontrivial weak solutions that strongly converges to* (η, ζ) *in* X.

Proof. **Step 1: Proof of assertion (a).** We apply part (a) of Theorem 1.1 to show that $\Lambda(\vartheta_0) = 0$, where Λ is the function defined in Theorem 1.1, and we assume $\mu = 1$.

First, for all $(\eta,\zeta)\in\Upsilon_1^{-1}(]-\infty,\vartheta_0[)$, we have

$$\begin{aligned} 0\le\Lambda(\vartheta_0) &= \inf_{\Upsilon_1^{-1}(]-\infty,\vartheta_0[)} \frac{\Upsilon_2(\eta,\zeta)-\inf_{\overline{(\Upsilon_1^{-1}(]-\infty,\vartheta_0[))}^{w}}\Upsilon_2(\eta,\zeta)}{\vartheta_0-\Upsilon_1(\eta,\zeta)} \\ &\le \frac{\Upsilon_2(\eta,\zeta)-\inf_{\overline{\Upsilon_1^{-1}(]-\infty,\vartheta_0[)}^{w}}\Upsilon_2(\eta,\zeta)}{\vartheta_0-\Upsilon_1(\eta,\zeta)}. \end{aligned} \tag{4.1}$$

Let $\eta_0(\theta)=u_0$ and $\zeta_0(\theta)=v_0$ for all $\theta\in\mathcal{Q}$. Then $\nabla\eta_0=\nabla\zeta_0=0$. Since $(u_0,v_0)\in\operatorname{Int}(\Theta_2(\vartheta_0))$, we obtain

$$\begin{aligned} \Upsilon_1(\eta_0,\zeta_0) &= \int_{\mathcal{Q}}A(\theta,u_0)\,d\theta+\int_{\mathcal{Q}}B(\theta,u_0)\,d\theta+\int_{\mathcal{Q}}C(\theta,v_0)\,d\theta+\int_{\mathcal{Q}}D(\theta,v_0)\,d\theta \\ &\quad -\int_{\Gamma}\Psi_1(\theta,u_0)\,d\Gamma-\int_{\Gamma}\Psi_2(\theta,u_0)\,d\Gamma-\int_{\Gamma}\Psi_3(\theta,v_0)\,d\Gamma-\int_{\Gamma}\Psi_4(\theta,v_0)\,d\Gamma \\ &< \vartheta_0. \end{aligned}$$

Then, for almost every $\theta\in\overline{\mathcal{Q}}$ and for all $(\eta,\zeta)\in\overline{\Upsilon_1^{-1}(]-\infty,\vartheta_0[)}^{w}$, one has

$$\begin{aligned} &\frac{1}{\ell_1^+}\mathcal{F}_{\ell_1(\cdot)}(u)+\frac{1}{\kappa_1^+}\mathcal{F}_{\kappa_1(\cdot)}(u)+\frac{1}{\ell_2^+}\mathcal{F}_{\ell_2(\cdot)}(v)+\frac{1}{\kappa_2^+}\mathcal{F}_{\kappa_2(\cdot)}(v) \\ &-h_1\mathcal{G}_{\nu_1'(\cdot)}(f_1)-2h_1\mathcal{G}_{\nu_1(\cdot)}(u)-h_2\mathcal{G}_{\nu_2'(\cdot)}(f_2)-2h_2\mathcal{G}_{\nu_2(\cdot)}(u) \\ &-h_3\mathcal{G}_{\nu_3'(\cdot)}(f_3)-2h_3\mathcal{G}_{\nu_3(\cdot)}(v)-h_4\mathcal{G}_{\nu_4'(\cdot)}(f_4)-2h_4\mathcal{G}_{\nu_4(\cdot)}(v) \\ &\le\Upsilon_1(\eta,\zeta)\le\vartheta_0. \end{aligned} \tag{4.2}$$

The second inequality in (4.2) follows from the fact that $\overline{\Upsilon_1^{-1}(]-\infty,\vartheta_0[)}^{w}=\Upsilon_1^{-1}(]-\infty,\vartheta_0])$.

Since $(u,v)\in\Theta_1(\vartheta_0)$ and $\Phi(\theta,u,v)\le\Phi(\theta,u_0,v_0)$ for all $\theta\in\overline{\mathcal{Q}}$, we deduce that

$$-\Upsilon_2(\eta,\zeta)\le-\Upsilon_2(\eta_0,\zeta_0),\ \text{ for all } (\eta,\zeta)\in\overline{\Upsilon_1^{-1}(]-\infty,\vartheta_0[)}^{w}.$$

Thus,

$$-\Upsilon_2(\eta_0,\zeta_0)=\sup_{\overline{\Upsilon_1^{-1}(]-\infty,\vartheta_0[)}^{w}}(-\Upsilon_2(\eta,\zeta))=-\inf_{\overline{\Upsilon_1^{-1}(]-\infty,\vartheta_0[)}^{w}}\Upsilon_2(\eta,\zeta).$$

Since $\Upsilon_1(\eta_0,\zeta_0)<\vartheta_0$, we get

$$\Upsilon_2(\eta_0,\zeta_0)-\inf_{\overline{\Upsilon_1^{-1}(]-\infty,\vartheta_0[)}^{w}}\Upsilon_2(\eta,\zeta)=0.$$

Hence, by substituting $(\eta,\zeta)=(\eta_0,\zeta_0)$ in (4.1), we obtain $\Lambda(\vartheta_0)=0$.

By conclusion (a) of Theorem 1.1, there exists a critical point of $\Upsilon_1+\Upsilon_2$.

Step 2: Proof of assertion (b). We now apply part (b) of Theorem 1.1. From Step 1, we already know that $\Lambda(\vartheta_n)=0$ for all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\vartheta_n=+\infty$, it follows that

$$\liminf_{\vartheta\to\infty}\Lambda(\vartheta)\le\liminf_{n\to\infty}\Lambda(\vartheta_n)=0<1=\mu.$$

Fix $\lambda>0$ such that

$$\limsup_{(u,v)\to+\infty}\frac{\displaystyle\int_{\mathcal{Q}}\Phi(\theta,u,v)\,d\theta+\sum_{i=1}^{4}\int_{\Gamma}\Psi_i(\theta,\cdot)\,d\Gamma}{\displaystyle\int_{\mathcal{Q}}[A(\theta,u)+B(\theta,u)+C(\theta,v)+D(\theta,v)]\,d\theta}>\lambda>1.$$

Choose a sequence $(r_n,t_n)\subset\mathbb{R}^2$ such that $\sqrt{r_n^2+t_n^2}\ge n$ and

$$\int_{\mathcal{Q}}\Phi(\theta,r_n,t_n)\,d\theta+\sum_{i=1}^{4}\int_{\Gamma}\Psi_i(\theta,\cdot)\,d\Gamma>\lambda\int_{\mathcal{Q}}[A(\theta,r_n)+B(\theta,r_n)+C(\theta,t_n)+D(\theta,t_n)]\,d\theta.$$

Let η_n and ζ_n be the constant functions on $\mathcal{Q}$ taking the values r_n and t_n, respectively. Then,

$$\begin{aligned}\Upsilon_2(\eta_n,\zeta_n)+\Upsilon_1(\eta_n,\zeta_n) &\leq (1-\lambda)\int_{\mathcal{Q}}[A(\theta,r_n)+B(\theta,r_n)+C(\theta,t_n)+D(\theta,t_n)]\,d\theta\\ &< 0, \quad \text{for all } n\in\mathbb{N}.\end{aligned}$$

Since $(\sqrt{r_n^2+t_n^2})_n$ is unbounded, at least one of $(r_n)_n$ or $(t_n)_n$ diverges. Hence, the functional $\Upsilon_1+\Upsilon_2$ is unbounded from below. By conclusion (b) of Theorem 1.1, there exists a sequence (θ_n,z_n) of critical points of $\Upsilon_1+\Upsilon_2$ such that $\Upsilon_1(\theta_n,z_n)\to+\infty$. Moreover, since Υ_1 is bounded on every bounded subset of X, the sequence $(\theta_n,z_n)_n$ must be unbounded in X.

Step 3: Proof of assertion (c). We now apply part (c) of Theorem 1.1. From Step 1, we have $\Lambda(\vartheta_n)=0$ for all $n\in\mathbb{N}$.

Since $\inf_X\Upsilon_1=\Upsilon_1(\eta,\zeta)=0$ and $\lim_{n\to\infty}\vartheta_n=0$, we obtain

$$\chi=\liminf_{\vartheta\to0^+}\Lambda(\vartheta)\leq\liminf_{n\to\infty}\Lambda(\vartheta_n)=0<1=\mu.$$

Fix $\lambda>0$ such that

$$\limsup_{(u,v)\to(0,0)}\frac{\displaystyle\int_{\mathcal{Q}}\Phi(\theta,u,v)\,d\theta+\sum_{i=1}^{4}\int_{\Gamma}\Psi_i(\theta,\cdot)\,d\Gamma}{\displaystyle\int_{\mathcal{Q}}[A(\theta,u)+B(\theta,u)+C(\theta,v)+D(\theta,v)]\,d\theta}>\lambda>1.$$

Choose $(r_n,t_n)\in\mathbb{R}^2\setminus\{(0,0)\}$ such that $\sqrt{r_n^2+t_n^2}\leq\frac{1}{n}$ and the above inequality holds. Let η_n and ζ_n be the constant functions equal to r_n and t_n, respectively. By Proposition 2.1, $(\eta_n,\zeta_n)\to(\eta,\zeta)$ strongly in X, and

$$\Upsilon_2(\eta_n,\zeta_n)+\Upsilon_1(\eta_n,\zeta_n)<0,\quad \text{for all } n\in\mathbb{N}.$$

Since $\Upsilon_1(\eta,\zeta)+\Upsilon_2(\eta,\zeta)=0$, (η,ζ) cannot be a local minimum of $\Upsilon_1+\Upsilon_2$. As (η,ζ) is the unique global minimum of Υ_1, the conclusion (c) of Theorem 1.1 ensures the existence of a sequence of pairwise distinct critical points (θ_n,z_n) of $\Upsilon_1+\Upsilon_2$ such that $\Upsilon_1(\theta_n,z_n)\to0$ and $(\theta_n,z_n)\rightharpoonup 0$. Therefore, the sequence $(\theta_n,z_n)_n$ is infinitesimal in norm. □

The following theorem provides a more practical formulation of Theorem 4.2(b).

Theorem 4.3. *Let $(\alpha_n)_n$ and $(\beta_n)_n$ be two sequences in $\mathbb{R}^+$ satisfying*

$$\beta_n<\alpha_n \quad \text{for all } n\in\mathbb{N},\qquad \lim_{n\to+\infty}\beta_n=+\infty,\qquad \text{and}\qquad \lim_{n\to+\infty}\frac{\alpha_n}{\beta_n}=+\infty.$$

Define

$$\begin{aligned}S_n \;=\; \Bigg\{(u,v)\in\mathbb{R}^2 \;:\; &\frac{1}{\ell_1^+}\mathcal{F}_{\ell_1(\cdot)}(u)+\frac{1}{\kappa_1^+}\mathcal{F}_{\kappa_1(\cdot)}(u)+\frac{1}{\ell_2^+}\mathcal{F}_{\ell_2(\cdot)}(v)+\frac{1}{\kappa_2^+}\mathcal{F}_{\kappa_2(\cdot)}(v)\\ &-h_1\mathcal{G}_{\nu_1'(\cdot)}(f_1)-2h_1\mathcal{G}_{\nu_1(\cdot)}(u)-h_2\mathcal{G}_{\nu_2'(\cdot)}(f_2)-2h_2\mathcal{G}_{\nu_2(\cdot)}(u)\\ &-h_3\mathcal{G}_{\nu_3'(\cdot)}(f_3)-2h_3\mathcal{G}_{\nu_3(\cdot)}(v)-h_4\mathcal{G}_{\nu_4'(\cdot)}(f_4)-2h_4\mathcal{G}_{\nu_4(\cdot)}(v)\leq\alpha_n\Bigg\},\end{aligned}$$

and

$$\begin{aligned}W_n \;=\; \Bigg\{(u,v)\in\mathbb{R}^2 \;:\; &\int_{\mathcal{Q}}A(\theta,u)\,d\theta+\int_{\mathcal{Q}}B(\theta,u)\,d\theta+\int_{\mathcal{Q}}C(\theta,v)\,d\theta+\int_{\mathcal{Q}}D(\theta,v)\,d\theta\\ &-\int_{\Gamma}\Psi_1(\theta,u)\,d\Gamma-\int_{\Gamma}\Psi_2(\theta,u)\,d\Gamma-\int_{\Gamma}\Psi_3(\theta,v)\,d\Gamma-\int_{\Gamma}\Psi_4(\theta,v)\,d\Gamma\leq\beta_n\Bigg\}.\end{aligned}$$

Assume further that

$$\sup_{(u,v)\in S_n\setminus \mathrm{Int}(W_n)} \Phi(\theta,u,v) \le 0, \quad \textit{for all } \theta\in\mathcal{Q} \textit{ and } n\in\mathbb{N}.$$

Finally, suppose that

$$\limsup_{(u,v)\to+\infty} \frac{\displaystyle\int_{\mathcal{Q}} \Phi(\theta,u,v)\,d\theta + \int_\Gamma \Psi_1(\theta,u)\,d\Gamma + \int_\Gamma \Psi_2(\theta,u)\,d\Gamma + \int_\Gamma \Psi_3(\theta,v)\,d\Gamma + \int_\Gamma \Psi_4(\theta,v)\,d\Gamma}{\displaystyle\int_{\mathcal{Q}} A(\theta,u)\,d\theta + \int_{\mathcal{Q}} B(\theta,u)\,d\theta + \int_{\mathcal{Q}} C(\theta,v)\,d\theta + \int_{\mathcal{Q}} D(\theta,v)\,d\theta} > 1.$$

Then the problem (1.1) *admits an unbounded sequence of weak solutions in X.*

Proof. Since $\beta_n < \alpha_n$, it follows directly that $W_n \subseteq S_n$. Let

$$\gamma' = \min\left\{\frac{1}{\ell_1^+}, \frac{1}{\ell_2^+}, \max\{h_1,h_2,h_3,h_4\}\right\} > 0.$$

In view of $\lim_{n\to+\infty}\frac{\alpha_n}{\beta_n} = +\infty$, we can find n large enough such that

$$\frac{1}{\gamma'} < \frac{\alpha_n}{\beta_n}.$$

Define $\vartheta_n = \gamma'\alpha_n$. Then $\{\vartheta_n\}_n \subset \mathbb{R}^+$ is a divergent sequence, and for sufficiently large n, the following inclusions hold:

$$\mathrm{Int}\, W_n \subseteq W_n \subseteq \Theta_2(\vartheta_n) \subseteq \Theta_1(\vartheta_n) \subseteq S_n.$$

Since Φ is nonpositive on $S_n \setminus \mathrm{Int}\, W_n$ for all $n\in\mathbb{N}$, we obtain

$$\max_{(u,v)\in \mathrm{Int}\, W_n} \Phi(\theta,u,v) = \max_{(u,v)\in S_n} \Phi(\theta,u,v), \quad \forall\,\theta\in\mathcal{Q}.$$

In particular, for large n, we have

$$\max_{\mathrm{Int}\, W_n} \Phi = \max_{\Theta_1(\vartheta_n)} \Phi,$$

which implies the existence of at least one sequence $(\xi_n,\eta_n)_n \subset \mathrm{Int}\, W_n$ such that, for all sufficiently large n,

$$\max_{\Theta_1(\vartheta_n)} \Phi(\theta,u,v) = \Phi(\theta,\xi_n,\eta_n), \quad \forall\,\theta\in\mathcal{Q}.$$

Thus, the sequences $(\xi_n)_n$, $(\eta_n)_n$, and $(\vartheta_n)_n$ satisfy the assumptions of Theorem 4.2 (b). This completes the proof. □

The next result provides a practicable version of Theorem 4.2 (c).

Theorem 4.4. *Let $(\alpha_n)_n$ and $(\beta_n)_n$ be two sequences in $\mathbb{R}^+$ such that*

$$\beta_n < \alpha_n \quad \forall\, n\in\mathbb{N}, \qquad \lim_{n\to+\infty} \alpha_n = 0, \qquad \lim_{n\to+\infty} \frac{\alpha_n}{\beta_n} = +\infty.$$

Define

$$\begin{aligned} A_n \;=\; \Bigg\{&(u,v)\in\mathbb{R}^2 :\ \frac{1}{\ell_1^+}\mathcal{F}_{\ell_1(\cdot)}(u) + \frac{1}{\kappa_1^+}\mathcal{F}_{\kappa_1(\cdot)}(u) + \frac{1}{\ell_2^+}\mathcal{F}_{\ell_2(\cdot)}(v) + \frac{1}{\kappa_2^+}\mathcal{F}_{\kappa_2(\cdot)}(v) \\ &-h_1\mathcal{G}_{\nu_1'(\cdot)}(f_1) - 2h_1\mathcal{G}_{\nu_1(\cdot)}(u) - h_2\mathcal{G}_{\nu_2'(\cdot)}(f_2) - 2h_2\mathcal{G}_{\nu_2(\cdot)}(u) \\ &-h_3\mathcal{G}_{\nu_3'(\cdot)}(f_3) - 2h_3\mathcal{G}_{\nu_3(\cdot)}(v) - h_4\mathcal{G}_{\nu_4'(\cdot)}(f_4) - 2h_4\mathcal{G}_{\nu_4(\cdot)}(v) \le \alpha_n\Bigg\}, \end{aligned}$$

and

$$B_n = \left\{(u,v) \in \mathbb{R}^2 : \int_{\mathcal{Q}} [A(\theta,u) + B(\theta,u) + C(\theta,v) + D(\theta,v)]\, d\theta \right. \\ \left. - \int_{\Gamma} [\Psi_1(\theta,u) + \Psi_2(\theta,u) + \Psi_3(\theta,v) + \Psi_4(\theta,v)]\, d\Gamma \leq \beta_n \right\}.$$

Assume that

$$\sup_{(u,v)\in A_n \setminus \operatorname{Int} B_n} \Phi(\theta,u,v) \leq 0 \quad \forall\, \theta \in \mathcal{Q},\ \forall\, n \in \mathbb{N}.$$

Finally, suppose that

$$\limsup_{(u,v)\to+\infty} \frac{\displaystyle\int_{\mathcal{Q}} \Phi(\theta,u,v)\, d\theta + \sum_{i=1}^{4} \int_{\Gamma} \Psi_i(\theta,\cdot)\, d\Gamma}{\displaystyle\int_{\mathcal{Q}} [A(\theta,u) + B(\theta,u) + C(\theta,v) + D(\theta,v)]\, d\theta} > 1.$$

Then, problem (1.1) *admits a sequence of nontrivial weak solutions that strongly converges to* (η,ζ) *in* X.

Proof. By an argument analogous to that of Theorem 4.2 (c), the above result follows immediately. The proof is therefore omitted. □

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Author information

A. Ahmed, Department of Mathematics, Faculty of Science and Technology
Research Unit Geometry, Algebra, Analysis and Applications (G3A), Mauritania.
E-mail: `ahmedmath2001@gmail.com`

G. Lorenzini, Department of Industrial Systems and Technologies Engineering, University of Parma, Italy.
E-mail: `giulio.lorenzini@unipr.it`

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