

# ON SOLVABILITY OF $n$ -QUADRATIC ERDÉLYI-KÖBER FRACTIONAL INTEGRAL EQUATIONS IN ORLICZ SPACE

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**Abstract** We are studied the existence and uniqueness of monotonic solutions for an abstract class of Erdélyi-Köber fractional integral equations of  $n$ -quadratic type formulated within Orlicz spaces  $L_\Omega$ . The methodology integrates fixed-point theorem with a detailed analysis of the associated measure of noncompactness, grounded in the principles of fractional calculus.

## 1 Introduction

In recent decades, fractional and integral equations have attracted considerable interest from researchers. This growth in attention is driven by the development of various fractional integral and differential equations defined on bounded and unbounded intervals. Numerous theoretical results have been reported in this field, including the works of Miller and Ross [30], Agarwal et al. [2, 3], Benchohra et al. [7], Diethelm and Ford [16], Furati and Tatar [19], Delbosco and Rodino [15], Momani et al. [31], and the monograph by Kilbas et al. [21]. These contributions have found applications across a wide range of scientific and engineering disciplines, such as chemical and statistical physics, control theory, rheology, optics, viscoelasticity, robotics, acoustics, and both electrical and mechanical engineering. In 1965, O’Neill [33] laid the foundations for the study of fractional calculus in Orlicz spaces. Since then, many significant contributions have been made in this field [4, 32, 36]. The Orlicz spaces  $L_\Omega$  are an ideal setting for examining operators with highly nonlinear behavior—particularly those exhibiting exponential growth, in contrast to the milder polynomial growth characteristic of Lebesgue spaces  $L_p$ ,  $p \geq 1$  (see [11, 12]). Such spaces are of particular interest in problems from statistical and mathematical physics (see [14, 22]). For example, the thermodynamic equation

$$k(s) + \int_I G(s, t)e^{k(t)} dt = 0,$$

naturally involves exponential nonlinearities (cf. [37]). Quadratic integral equations play a role in astrophysics, radiative transfer theory, and neutron transport [13]. They have been studied in both  $L_p$  spaces [1, 25] and  $L_\Omega$ -spaces [11, 12, 29] using measure of noncompactness techniques with the Darbo’s fixed-point theorem. Also many researchers have investigated fractional integral, differential, and other related integral problems through the use of fixed point theory and the measure of noncompactness see [20, 34, 35]. While Medved and Brestovanská [8, 9] investigated operator products in Banach algebras of continuous functions, their approach differs from ours. Because Orlicz spaces do not possess the structure of Banach algebras, we utilize the techniques of [27, 28]. The work in [28] introduced fixed-point theorems and used them to

investigate the solvability of

$$k(s) = \prod_{i=1}^n \left[ h_i(s) + \int_a^s C_i(s, \mu, k(\mu)) d\mu \right],$$

within  $L_p$  where  $p > 1$  and the Orlicz spaces  $L_\Omega(I)$ , with  $I = [a, b]$ , and  $\Omega$  satisfying the  $\Delta_2$ -condition. Later [27], extended this investigation to the existence of solutions for systems of  $n$ -integral equations based on  $n$  different Orlicz spaces, for  $n \geq 2$ , assuming that  $\Omega$  meets the  $\Delta_0, \Delta_3$ , and  $\Delta_2$ -conditions. Finally, [26] studied the fractional integral operator of Riemann-Liouville type, establishing fundamental theorems and demonstrating existence results for the equation

$$k(s) = k(s) + \mathcal{H}(k)(s) \int_0^s \frac{(s-\mu)^{\gamma-1}}{\Gamma(\gamma)} f(\mu, k(\mu)) d\mu, \quad 0 < \gamma < 1,$$

within  $L_\Omega$ -spaces over  $[0, d]$ .

In 1940 [18] Erdélyi-Köber fractional integral operators is a fractional integration operation introduced by Erdélyi and Köber. In [10] several fundamental theorems concerning the Erdélyi-Köber fractional-order integral operator were presented and proved, and further explored the existence result related to the equation

$$k(s) = \mathcal{H}_2 k(s) + \frac{\mathcal{H}_1(k)(s)}{\Gamma(\gamma)} \beta \int_0^t \frac{\mu^{\sigma-1}}{(\eta^\sigma - \mu^\sigma)^{1-\gamma}} \mathcal{H}_3 k(\mu) d\mu, \quad t \in [0, 1]$$

$t \in [0, 1], 0 < \gamma < 1, \beta > 0$  in Orlicz space  $L_\Omega$ , where  $\mathcal{H}_i, i = 1, 2, 3$  are general operator. Similarly, in [24], Metwali and Alsallami derived important results concerning the Erdélyi-Köber fractional integral operator and investigated existence theorems for

$$k(s) = p(s) + t_1(s, k(s)) + t_2 \left( s, \frac{\sigma h_1(s, k(s))}{\Gamma(\gamma)} \cdot \int_0^s \frac{\mu^{\sigma-1} h_2(\mu, k(\mu))}{(s^\sigma - \mu^\sigma)^{1-\gamma}} d\mu \right), \quad s \in [0, d]$$

where  $\sigma > 0$  and  $0 < \gamma < 1$  considered in  $L_p$  and Orlicz spaces  $L_\Omega$ .

This paper focuses on a broad and abstract framework involving the product of  $n$ -quadratic fractional integral equations, and provides proofs for the existence and uniqueness of their solutions. The problem under investigation is Erdélyi-Köber-type equation posed in Orlicz spaces  $L_\Omega$

$$k(s) = \prod_{i=1}^n \left[ p_i(s) + \mathcal{H}_{2_i}(k)(s) + \frac{\mathcal{H}_{1_i} k(s)}{\Gamma(\gamma_i)} \sigma \int_0^s \frac{\mu^{\sigma-1}}{(\eta^\sigma - \mu^\sigma)^{1-\gamma_i}} \mathcal{H}_{3_i}(\mu) \cdot k(\mu) \cdot d\mu \right] \quad (1.1)$$

for  $\gamma_i \in (0, 1), s \in [0, 1]$ .

**Motivation:** The study of fractional integral equations in Orlicz spaces is motivated by their ability to handle nonlinearities with exponential growth, which cannot be extended beyond the limitations of classical  $L_p$ -spaces. Erdélyi-Köber type operators, in particular, provide a rich framework for modeling complex dynamical systems with memory and hereditary properties. This paper investigates the solvability of  $n$ -quadratic Erdélyi-Köber equations, establishing the existence and uniqueness of monotonic solutions via Darbo’s fixed-point theorem and the measure of noncompactness. The results not only generalize previous work on linear and quadratic cases but also open new directions for higher-order fractional systems. In the future, these techniques can be applied to coupled fractional models, systems with variable exponents, and nonlinear problems in physics and engineering. Such extensions would broaden the applicability of fractional calculus to real-world processes involving nonlocal and hereditary effects.

## 2 Preliminaries

Let  $\mathbb{R} = (-\infty, \infty)$  and  $I = [0, \psi] \subset [0, \infty] = \mathbb{R}^+$ . For the Young function  $\Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$\Upsilon(v_1) = \int_0^{v_1} z(\mu) d\mu$$

for  $v_1 \geq 0$ , where  $u : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a function that increases on the left continuous and is neither equal to zero nor exactly infinite in  $\mathbb{R}^+$ . If  $\mathfrak{B}(x') = \sup_{y' \geq 0} \{x'y' - \Upsilon(x')\}$ , then the functions  $\Upsilon$  and  $\mathfrak{B}$  are said to be complementary Young functions. Specifically, if  $\Upsilon$  has finite values, where  $\lim_{v_1 \rightarrow \infty} \frac{\Upsilon(v_1)}{v_1} = 0$ ,  $\lim_{v_1 \rightarrow \infty} \frac{\Upsilon(v_1)}{v_1} = \infty$  and  $\Upsilon(v_1) > 0$  if  $v_1 > 0$  ( $\Upsilon(v_1) = 0 \Leftrightarrow v_1 = 0$ ). Then  $\Upsilon$  is called a complementary function of  $\mathfrak{B}$ . The Orlicz space denoted by  $\mathfrak{L}_\Upsilon = \mathfrak{L}_\Upsilon(I)$  consists of every bounded function  $s : I \rightarrow \mathbb{R}$  using the Luxemburg norm

$$\|s\|_\Upsilon = \inf_{\zeta > 0} \left\{ \zeta > 0 : \int \Upsilon \left( \frac{|s(\mu)|}{\zeta} \right) d\mu \leq 1 \right\}.$$

Assume that  $\mathcal{D}_\Upsilon(I) = \mathfrak{L}_\Upsilon(I)$  absolutely continuous norms and contains the set of all bounded functions of  $\mathfrak{L}_\Upsilon$ .

**Proposition 2.1.** [12] *Suppose that  $\Upsilon$  is a Young function*

(i) *for  $\gamma_1 \in (0, 1)$  and  $\int_0^s \Upsilon(\mu^{-\gamma_1})d\mu$  is finite for every  $s > 0$ , if  $\gamma_2 < \gamma_1$ , then the integral*

$$\int_0^s \Upsilon(\mu^{-\gamma_2})d\mu$$

*is finite as well.*

(ii) *for every  $s \in \mathbb{R}^+$  and  $\gamma \in (0, 1)$ ,*

$$\Upsilon(s) = \left\{ \zeta > 0 : \frac{1}{\zeta^{\frac{1}{\gamma-1}}} \int_0^{s\zeta^{\frac{1}{\gamma-1}}} \Upsilon(\mu^{1-\gamma})d\mu \leq 1 \right\}$$

*is a set of continuous functions and  $\Upsilon(s) = 0$  with increasing property .*

**Definition 2.2.** [5] For a continuous function  $f$ , the Erdélyi-Köber fractional integral of order  $\beta > 0$  is defined as

$$\mathfrak{R}_\sigma^\gamma k(s) = \frac{\sigma}{\Gamma(\gamma)} \int_0^s \frac{\mu^{\sigma-1}}{(\eta^\sigma - \mu^\sigma)^{1-\gamma}} k(\mu)d\mu, \sigma > 0, 0 < \gamma < 1,$$

where  $\Gamma(\gamma) = \int_0^\infty \mu^{\gamma-1} e^{-s} ds, \gamma > 0$ .

**Proposition 2.3.** [6] *The operator  $\mathfrak{R}_\sigma^\gamma$  transforms functions that are nonnegative and almost everywhere nondecreasing into functions of the same nature.*

**Lemma 2.4** ([23] Theorem 1). *Suppose  $n$  is greater than or equal to 2. If  $\Omega$  and  $\Omega_i$  are Young functions (for each  $i = 1, 2, 3, \dots, n$ ), then the following equivalences hold:*

(I)  $\exists$  a constant  $\mathcal{K} > 0$  such that

$$\left\| \prod_{i=1}^n z_i \right\|_\Omega \leq \mathcal{K} \prod_{i=1}^n \|z_i\|_{\Omega_i},$$

for each  $z_i \in \mathfrak{L}_{\Omega_i}, i = 1, 2, 3, \dots, n$

(II)  $\exists$  a constant  $\mathcal{T} > 0$  such that

$$\prod_{i=1}^n \Omega_i^{-1}(s) \leq \mathcal{T} \Omega^{-1}(s), \text{ for each } s \geq 0.$$

(III) For every  $z_i \in \mathfrak{L}_{\Omega_i}, \prod_{i=1}^n z_i \in \mathfrak{L}_\Omega$

(IV)  $\exists$  a constant  $\mathcal{T} \geq 0$  s.t.

$$\Omega \left( \frac{\prod_{i=1}^n s_i}{\mathcal{T}} \right) \leq \sum_{i=1}^n \Omega_i(s_i).$$

for every  $s_i \geq 0, i = 1, 2, \dots, n$

**Lemma 2.5.** [36] Consider a bounded set  $\mathcal{X} \subset \mathfrak{L}_\Upsilon$  and let a family  $(\mathcal{N}_c)_{0 \leq c \leq \psi} \subset I$  exists. Then  $\mathcal{N}_c = c$  for all  $c \in [0, \psi]$  and for each

$$k(s_1) \geq k(s_2), (s_1 \in \mathcal{N}_c, s_2 \notin \mathcal{N}_c),$$

here  $c$  denotes the measure of the subset of the interval  $I$  and is denoted by  $\mathcal{N}_c$ . Thus,  $X$  is a set that is compact in measure in the space  $\mathfrak{L}_\Upsilon$ .

**Definition 2.6.** [36] Let  $\emptyset \neq \mathcal{X} \subset \mathfrak{L}_\Upsilon$  be bounded. Then the Hausdorff measure of noncompactness (MNC)  $\alpha(X)$  is defined by

$$\alpha_H(\mathcal{X}) = \inf \{ \tau > 0 : \text{there exists a finite subset } \mathcal{Y} \text{ of } \mathfrak{L}_\Upsilon \text{ s.t. } \mathcal{X} \subset \mathcal{Y} + \mathcal{T}_\tau \},$$

where  $\mathcal{T}_\tau = \{x \in \mathfrak{L}_\Upsilon : \|x\|_\Upsilon \leq \tau\}$ . Given any  $\zeta > 0$ , define  $c$  as the equi-integrability measure of  $\mathcal{X} \in \mathfrak{L}_\Upsilon$  defined as

$$c(\mathcal{X}) = \lim_{\zeta \rightarrow 0} \sup_{m \in \mathcal{S}} \sup_{\mathcal{F} \leq \zeta} \sup_{k \in \mathcal{X}} \|k \cdot \chi_{\mathcal{F}}\|_{\mathfrak{L}_\Upsilon},$$

where  $\chi_{\mathcal{F}}$  denotes the characteristic function of the set  $\mathcal{F} \subset I$ .

**Lemma 2.7.** [17, 36] Let  $\mathcal{X} \subset \mathcal{D}_\Upsilon$  be a compact, nonempty, bounded measurable set. Then

$$\alpha_H(\mathcal{X}) = c(\mathcal{X}).$$

### 3 New results

To solve Eq (1.1), we required the following lemma.

**Lemma 3.1.** Suppose that  $\Upsilon$  and  $\mathfrak{B}$  are complementary Young functions satisfying  $\int_0^s \Upsilon(1-\gamma) d\mu < \infty$  for some  $\gamma \in (0, 1)$ . Furthermore, let  $\Omega$  be an  $\Upsilon$ -function, where

$$k(s) = \frac{1}{\zeta^{\frac{1}{\gamma-1}}} \int_0^{\zeta^{\frac{1}{\gamma-1}}} \Upsilon(\mu^{1-\gamma}) d\mu \in \mathcal{D}_\Omega$$

for  $\zeta > 0$  and almost every  $\mu \in I$ , operator  $\mathfrak{R}_\sigma^\gamma : \mathfrak{L}_\mathfrak{B} \rightarrow \mathfrak{L}_\Omega$  remains continuous.

*Proof.* Suppose that

$$K(\eta, \mu) = \begin{cases} \frac{\mu^{\gamma-1}}{(\eta^\sigma - \mu^\sigma)^{1-\gamma}} & \text{if } \mu \in [0, \eta], \eta > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Thus, Hölder inequality and for  $k \in \mathfrak{L}_\mathfrak{B}$  we have

$$\|\mathfrak{R}_\sigma^\gamma k(s)\|_\Omega \leq \frac{2}{\Gamma(\gamma)} \|k\|_\Omega \|s\|_\mathfrak{B}.$$

Hence, using Proposition 2.1 and [22, Lemma 16.3], we see that the operator  $\mathfrak{R}_\sigma^\gamma : \mathfrak{L}_\mathfrak{B} \rightarrow \mathfrak{L}_\Omega$  is continuous. □

Now, we rewrite Eq (1.1) as

$$k(s) = \mathcal{A}(s) = \prod_{i=1}^n \mathcal{A}_i(s) = \prod_{i=1}^n (p_i + \mathcal{H}_{2_i}(s) + \mathcal{W}_i(s)),$$

where

$$\mathcal{W}_i(s) = \mathcal{H}_{1_i}(s) \cdot \mathcal{M}_i(s), \quad \mathcal{M}_i(s) = \mathfrak{R}_{\sigma_i}^{\gamma_i} \mathcal{H}_{3_i}(s),$$

such that  $\mathfrak{R}_\sigma^\gamma$  is in Definition 2.2 and  $\mathcal{H}_{f_i}(s)$  are operators in general that mappings acting on various Orlicz spaces, with  $f = 1, 2, 3$  and  $i = 1, 2, 3, \dots, n$

### 3.1 The existence of solutions of Eq (1.1) in Orlicz space $\mathfrak{L}_\Omega$

Suppose that  $\Upsilon_i$  and  $\mathfrak{B}_i$  are complementary  $\mathfrak{B}$  functions and  $\Omega, \Omega_i, \Omega_1, \Omega_2$  are  $\mathfrak{B}$  functions for  $i = 1, 2, 3, \dots, n$  and that  $\int_0^s \Upsilon_i(\mu^{\sigma_i-1}) d\mu < \infty$ , where  $\sigma_i \in (0, 1)$ ,

(F1)  $\exists$  a constant  $\hat{\mathcal{K}} > 0$  such that for each  $z_i \in \mathfrak{B}_i$  and we get  $\|\prod_{i=1}^n z_i\|_\Omega \leq \hat{\mathcal{K}} \prod_{i=1}^n \|z_i\|_{\Omega_i}$ .

(F2)  $\exists$  a constant  $k_1 > 0$  such that for each  $z_1 \in \mathfrak{L}_{\Omega_1}$  and  $z_2 \in \mathfrak{L}_{\Omega_2}$ , we get

$$\|z_1 z_2\|_{\Omega_i} \leq k_1 \|z_1\|_{\Omega_1} \|z_2\|_{\Omega_2}$$

(F3) Each function  $p_i \in D_{\Omega_i}$  then  $p_i$  is nondecreasing almost everywhere on  $I$ .

(F4)  $\mathcal{H}_{1_i} : \mathfrak{L}_{\Omega_1}(I) \rightarrow \mathfrak{L}_{\Omega_1}(I)$ , taking continuously  $\mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\Omega_1}(I)$  and the operator  $\mathcal{H}_{2_i} : \mathfrak{L}_\Omega(I) \rightarrow \mathfrak{L}_{\Omega_i}(I)$ , takes continuously  $\mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\Omega_i}(I)$  into itself and the operator  $\mathcal{H}_{3_i} : \mathfrak{L}_\Omega(I) \rightarrow \mathfrak{L}_{\mathfrak{B}_i}(I)$ , takes continuously  $\mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\mathfrak{B}_i}(I)$ .

(F5)  $\exists$  positive functions  $e_{1_i} \in L_{\Omega_1}, e_{2_i} \in L_{\Omega_i}, e_{3_i} \in L_{\mathfrak{B}_i}$  such that for  $s \in I$ ,

$$|\mathcal{H}_{f_i}(k)(s)| \leq e_{f_i}(s) \|k\|_\Omega,$$

and  $\mathcal{H}_{f_i}, f = 1, 2, 3$ , is the family of functions that are nondecreasing almost everywhere and take values in functions of the same type. Additionally, suppose that for any  $k \in D_\Omega$ , we have  $\mathcal{H}_{1_i}(k) \in \mathcal{D}_{\Omega_1}, \mathcal{H}_{2_i}(k) \in \mathcal{D}_{\Omega_i}$ , and  $\mathcal{H}_{3_i}(y) \in \mathcal{D}_{\mathfrak{B}_i}$

(F6) Suppose that

$$k_i(s) = \frac{1}{\zeta^{\frac{1}{\gamma_i-1}}} \int_0^{\zeta^{\frac{1}{\gamma_i-1}}} \Upsilon_i(\mu^{1-\gamma_i}) d\mu \in \mathcal{D}_{\Omega_{2_i}} \quad \text{for } \zeta > 0 \text{ and } s \in I.$$

(F7) Assume that  $\exists \tau > 0$  and  $T_i > 0$  such that

$$\prod_{i=1}^n T_i = \hat{\mathcal{K}} \prod_{i=1}^n \left( \|p_i\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \cdot \tau + \frac{2k_{1_i} \|k_{1_i}\|_{\Omega_{2_i}}}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_1} \|e_{3_i}\|_{\mathfrak{B}_i} \cdot \tau^2 \right) \leq \tau \quad (3.1)$$

and

$$\prod_{i=1}^n \left( \|e_{2_i}\|_{\Omega_i} + \frac{2k_{1_i} \|k_{1_i}\|_{\Omega_{2_i}} \cdot \tau}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_1} \|e_{3_i}\|_{\mathfrak{B}_i} \right) < \frac{1}{\tau^n \hat{\mathcal{K}}}.$$

**Theorem 3.2.** *There exist a solution  $s \in \mathcal{D}_\Omega$  of Eq. (1.1), which is almost everywhere nondecreasing on  $I$ , if the assumption (F1) to (F7) are satisfied.*

*Proof. Step (I):* First of all, Lemma 3.1 implies that  $\mathfrak{R}_\sigma^\gamma : \mathfrak{L}_{\mathfrak{B}_i}(I) \rightarrow \mathfrak{L}_{\Omega_{2_i}}(I)$  is continuous and by (F4) the operator  $\mathcal{H}_{1_i} : \mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\Omega_1}(I), \mathcal{H}_{2_i} : \mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\Omega_i}(I)$ , and  $\mathcal{H}_{3_i} : \mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\mathfrak{B}_i}(I)$  are continuous, then the operator  $\mathcal{M}_i = \mathfrak{R}_\sigma^\gamma \mathcal{H}_{3_i} : \mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\Omega_{2_i}}(I)$  is continuous. By suppositions (F2) and the Hölder inequality, we obtain that the operator  $\mathcal{W}_i = \mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\Omega_i}(I)$  are continuous. Also using supposition (F3) we get the operator  $\mathcal{A}_i : \mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_{\Omega_i}(I)$ , where we put  $i = 1, 2, \dots, n$ . Lastly, by applying supposition (F1) and along with the Hölder inequality ensure that the operator  $\mathcal{A} = \prod_{i=1}^n \mathcal{A}_i : \mathcal{D}_\Omega(I) \rightarrow \mathcal{D}_\Omega(I)$  is continuous.

**Step (II):** We aim to define a ball  $\mathcal{A}_\tau(\mathcal{D}_\Omega) = \{s \in \mathfrak{L}_\Omega : \|s\|_\Omega \leq \tau\}$ , where  $\tau$  is as given in assumption (F7). Assume that  $s \in \mathcal{A}_\tau(\mathcal{D}_\Omega)$ . Then by Lemma 3.1 we obtain

$$\begin{aligned}
 \|\mathcal{A}_i(s)\|_{\Omega_i} &\leq \|p_i\|_{\Omega_i} + \|\mathcal{H}_{2_i}(s)\|_{\Omega_i} + \|\mathcal{W}_i(s)\|_{\Omega_i} \\
 &\leq \|p_i\|_{\Omega_i} + \|e_{2_i} \cdot \|s\|_{\Omega}\|_{\Omega_i} + \|\mathcal{H}_{1_i}(s) \cdot \mathcal{M}_i(s)\|_{\Omega_i} \\
 &\leq \|p_i\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s\|_{\Omega} + k_{1_i} \|H_{1_i}(s)\|_{\Omega_{1_i}} \cdot \|\mathcal{M}_i(s)\|_{\Omega_{2_i}} \\
 &\leq \|p_i\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s\|_{\Omega} + k_{1_i} \|e_{1_i} \cdot \|s\|_{\Omega}\|_{\Omega_{1_i}} \cdot \|\mathfrak{A}_{\sigma_i}^{\gamma_i} \mathcal{H}_{3_i}(s)\|_{\Omega_{2_i}} \\
 &\leq \|p_i\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s\|_{\Omega} + k_{1_i} \|e_{1_i}\|_{\Omega_{1_i}} \|s\|_{\Omega} \cdot \frac{2}{\Gamma(\gamma_i)} \|k_i\|_{\Omega_{2_i}} \|e_{3_i} \cdot \|y\|_{\Omega}\|_{\mathfrak{B}_i} \\
 &\leq \|p_i\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s\|_{\Omega} + k_{1_i} \|e_{1_i}\|_{\Omega_{1_i}} \|s\|_{\Omega} \cdot \frac{2}{\Gamma(\gamma_i)} \|k_i\|_{\Omega_{2_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \cdot \|s\|_{\Omega} \\
 &\leq \|p_i\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s\|_{\Omega} + \frac{2k_{1_i} \|k_i\|_{\Omega_{2_i}}}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \|s\|_{\Omega}^2 \\
 &\leq \|p_i\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \cdot \mathfrak{r} + \frac{2k_{1_i} \|k_i\|_{\Omega_{2_i}}}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \cdot \mathfrak{r}^2
 \end{aligned}$$

and from supposition (F1), we have

$$\begin{aligned}
 \|\mathcal{A}(s)\|_{\Omega} &\leq \mathcal{K} \prod_{i=1}^n \|\mathcal{A}_i(s)\|_{\Omega_i} \\
 &\leq \mathcal{K} \prod_{i=1}^n \left( \|p_i\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \cdot \mathfrak{r} + \frac{2k_{1_i} \|k_i\|_{\Omega_{2_i}}}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \cdot \mathfrak{r}^2 \right) \leq \mathfrak{r}.
 \end{aligned}$$

Hence, according to the supposition (F7),  $\mathcal{A} : \mathcal{A}_{\mathfrak{r}}(\mathcal{D}_{\Omega}) \rightarrow \mathcal{D}_{\Omega}$  is continuous.

**Step (III):** Consider the set  $\mathcal{V}_{\mathfrak{r}} \subset \mathcal{A}_{\mathfrak{r}}(\mathcal{D}_{\mathfrak{r}})$ , which consists of all functions that are almost everywhere nondecreasing on  $I$ . According to [ref [36]], this set is convex, bounded, nonempty, and closed in  $\mathcal{L}_{\Omega}$ . Moreover, by Lemma 2.5, the set  $\mathcal{V}_{\mathfrak{r}}$  is compact with respect to the convergence in the measure.

**Step (IV):** Again, we demonstrate that the operator  $\mathcal{A}$  maintains the monotonicity of the function. Suppose  $s \in \mathcal{V}_{\mathfrak{r}}$ , which implies that  $s$  is almost everywhere nondecreasing on  $I$ . Based on supposition (F5), the operator  $\mathcal{H}_{j_i}$  for  $j = 1, 2, 3$  is nondecreasing almost everywhere on  $I$  and Proposition 2.3 shows that the operator  $\mathcal{M}_i$  also satisfies this property. Consequently, the operator  $\mathcal{W}_i(s) = \mathcal{H}_{1_i}(s) \cdot \mathcal{M}_i(s)$  is almost everywhere nondecreasing on  $I$ . Furthermore, supposition (F3), it follows that the operator  $\mathcal{A} : \mathcal{V}_{\mathfrak{r}} \rightarrow \mathcal{V}_{\mathfrak{r}}$  is continuous.

**Step (V):** Here we aim to show that the operator  $\mathcal{A}$  is a contraction with respect to the measure of noncompactness (MNC). Let  $S \in \mathcal{V}_{\mathfrak{r}}$  be a nonempty set. For any  $s \in S$  and for any measurable set  $\mathcal{F} \subset I$  with  $\zeta > 0$  such that  $meas(\mathcal{F}) \leq \zeta$ . Hence, supposition (F4), gives that

$$\|\mathcal{H}_{1_i}(s) \cdot \chi_{\mathcal{F}}\|_{\Omega_{1_i}} \leq \|H_{1_i}(s \cdot \chi_{\mathcal{F}})\|_{\Omega_{1_i}} \leq \|e_{1_i} \cdot \|s \cdot \chi_{\mathcal{F}}\|_{\Omega}\|_{\Omega_{1_i}} \leq \|e_{1_i}\|_{\Omega_{1_i}} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega}.$$

Similarly,

$$\|\mathcal{H}_{2_i}(s) \cdot \chi_{\mathcal{F}}\|_{\Omega_{1_i}} \leq \|e_{2_i}\|_{\Omega_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega}.$$

Then we have

$$\begin{aligned}
 \|\mathcal{A}_i(s) \cdot \chi_{\mathcal{F}}\|_{\Omega_i} &\leq \|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|\mathcal{H}_{2_i}(s) \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|\mathcal{W}_i(s) \cdot \chi_{\mathcal{F}}\|_{\Omega_i} \\
 &\leq \|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|\mathcal{H}_{2_i}(y \cdot \chi_{\mathcal{F}})\|_{\Omega_i} + \|\mathcal{H}_{1_i}(y) \cdot \mathcal{M}_i(y) \cdot \chi_{\mathcal{F}}\|_{\Omega_i} \\
 &\leq \|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} + k_{1_i} \|\mathcal{H}_{1_i}(s)\|_{\Omega_{1_i}} \|\chi_{\mathcal{F}}\|_{\Omega_{1_i}} \|\mathcal{M}_i(s) \cdot \chi_{\mathcal{F}}\|_{\Omega_{2_i}}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} + k_{1_i} \|\mathcal{H}_{1_i}(s \cdot \chi_{\mathcal{F}})\|_{\Omega_{1_i}} \cdot \|\mathcal{M}_i(s)\|_{\Omega_{2_i}} \\
 &\leq \|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} + \frac{2k_{1_i}}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} \|k_i\|_{\Omega_{2_i}} \|\mathcal{H}_{3_i}(s)\|_{\mathfrak{B}_i} \\
 &\leq \|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} + \frac{2k_{1_i}}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} \|k_i\|_{\Omega_{2_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \|s\|_{\Omega} \\
 &\leq \|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} + \frac{2k_{1_i} \|k_i\|_{\Omega_{2_i}} \cdot \tau}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \|\mathcal{A}(s) \cdot \chi_{\mathcal{F}}\|_{\Omega} &\leq \mathcal{K} \prod_{i=1}^n \|\mathcal{A}_i(s) \cdot \chi_{\mathcal{F}}\|_{\Omega_i} \\
 &\leq \mathcal{K} \prod_{i=1}^n \left( \|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i} + \|e_{2_i}\|_{\Omega_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} + \frac{2k_{1_i} \|k_i\|_{\Omega_{2_i}} \cdot \tau}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \|s \cdot \chi_{\mathcal{F}}\|_{\Omega} \right).
 \end{aligned}$$

Since  $p_i \in \mathcal{D}_{\Omega_i}$ , we get

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup_{\text{meas } \mathcal{D} \leq \zeta} \left[ \sup_{s \in \mathcal{S}} (\|p_i \cdot \chi_{\mathcal{F}}\|_{\Omega_i}) \right] \right\} = 0.$$

Then, by definition  $c(\mathcal{S})$ , we obtain

$$c(\mathcal{A}(\mathcal{S})) \leq \tau^n \mathcal{K} \prod_{i=1}^n \left( \|e_{2_i}\|_{\Omega_i} + \frac{2k_{1_i} \|k_i\|_{\Omega_{2_i}} \cdot \tau}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \right) c(\mathcal{S}),$$

Since  $\mathcal{S} \subset \mathcal{V}_{\tau}$  is a nonempty, bounded subset of  $\mathcal{D}_{\Omega}$  that is compact in measure, we may use the Lemma 2.7 to obtain

$$\beta_{\mathcal{H}}(\mathcal{A}(\mathcal{S})) \leq \tau^n \mathcal{K} \prod_{i=1}^n \left( \|e_{2_i}\|_{\Omega_i} + \frac{2k_{1_i} \|k_i\|_{\Omega_{2_i}} \cdot \tau}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \right) \cdot \beta_{\mathcal{H}}(\mathcal{S}).$$

Since

$$\prod_{i=1}^n \left( \|e_{2_i}\|_{\Omega_i} + \frac{2k_{1_i} \|k_i\|_{\Omega_{2_i}} \cdot \tau}{\Gamma(\gamma_i)} \|e_{1_i}\|_{\Omega_{1_i}} \|e_{3_i}\|_{\mathfrak{B}_i} \right) < \frac{1}{\tau^n \mathcal{K}}.$$

Thus, the result is established (see [28]). □

### 3.2 On the Uniqueness of Solutions:

Now, we proceed to study the uniqueness for the solution of (1.1).

**Theorem 3.3.** *We assume that suppositions (F1) to (F7) are satisfied. If*

$$\mathcal{P} = \sum_{f=1}^n \left[ \mathcal{K} \left( \|e_{2_f}\|_{\Omega_f} + \frac{4k_{1_f} \cdot \tau \|k_f\|_{\Omega_{2_f}}}{\Gamma(\gamma_f)} \|e_{1_f}\|_{\Omega_{1_f}} \|e_{3_f}\|_{\mathfrak{B}_f} \right) \cdot \prod_{i=1, i \neq f}^n T_i \right] < 1,$$

where supposition (F7) gives the value of  $\tau$  and  $T_i$ . Then  $s \in \mathcal{L}_{\Omega}$  contained in the set  $\mathcal{V}_{\tau}$  has a solution for (1.1)

*Proof.* Suppose  $s$  and  $t$  are two possible solutions of Eq. (1.1), then we have

$$\begin{aligned}
 |s - t| &= \left| \prod_{i=1}^n \mathcal{A}_i(s) - \prod_{i=1}^n \mathcal{A}_i(t) \right| \\
 &\leq \left| \prod_{i=1}^n \mathcal{A}_i(s) - \mathcal{A}_1(t) \prod_{i=2}^n \mathcal{A}_i(s) \right| + \left| \mathcal{A}_1(t) \prod_{i=2}^n \mathcal{A}_i(s) - \mathcal{A}_1(t) \mathcal{A}_2(t) \prod_{i=3}^n \mathcal{A}_i(s) \right| \\
 &\quad + \dots + \left| \mathcal{A}_n(s) \prod_{i=1}^{n-1} \mathcal{A}_i(t) - \prod_{i=1}^n \mathcal{A}_i(t) \right| \\
 &\leq \left| \mathcal{A}_1(s) \prod_{i=2}^n \mathcal{A}_i(s) - \mathcal{A}_1(t) \prod_{i=2}^n \mathcal{A}_i(s) \right| + \left| \mathcal{A}_1(t) \mathcal{A}_2(s) \prod_{i=3}^n \mathcal{A}_i(s) - \mathcal{A}_1(t) \mathcal{A}_2(t) \prod_{i=3}^n \mathcal{A}_i(s) \right| \\
 &\quad + \dots + \left| \mathcal{A}_n(s) \prod_{i=1}^{n-1} \mathcal{A}_i(t) - \mathcal{A}_n(t) \prod_{i=1}^{n-1} \mathcal{A}_i(t) \right| \\
 &\leq |\mathcal{A}_1(s) - \mathcal{A}_1(t)| \cdot \prod_{i=2}^n |\mathcal{A}_i(s)| + |\mathcal{A}_1(t)| \cdot |\mathcal{A}_2(s) - \mathcal{A}_2(t)| \cdot \prod_{i=3}^n |\mathcal{A}_i(s)| \\
 &\quad + \dots + |\mathcal{A}_n(s) - \mathcal{A}_n(t)| \cdot \prod_{i=1}^{n-1} |\mathcal{A}_i(t)|.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \|s - t\|_{\Omega} &\leq \mathcal{K} \|\mathcal{A}_1(s) - \mathcal{A}_1(t)\|_{\Omega_1} \prod_{i=2}^n \|\mathcal{A}_i(s)\|_{\Omega_i} + \mathcal{K} \|\mathcal{A}_1(t)\|_{\Omega_1} \|\mathcal{A}_2(s) - \mathcal{A}_2(t)\|_{\Omega_2} \prod_{i=3}^n \|\mathcal{A}_i(s)\|_{\Omega_i} \\
 &\quad + \dots + \mathcal{K} \|\mathcal{A}_n(s) - \mathcal{A}_n(t)\|_{\Omega_n} \prod_{i=1}^{n-1} \|\mathcal{A}_i(t)\|_{\Omega_i}.
 \end{aligned} \tag{3.2}$$

To proceed with evaluating the above inequality, we apply Lemma 3.1, which provides the bound required for each  $j = 1, 2, \dots, n$ .

$$\begin{aligned}
 \|\mathcal{A}_f(s) - \mathcal{A}_f(t)\|_{\Omega_f} &\leq \|\mathcal{H}_{2_f}(s) - \mathcal{H}_{2_f}(t)\|_{\Omega_f} + \|\mathcal{H}_{1_f}(s)\mathcal{M}_f(s) - \mathcal{H}_{1_f}(t)\mathcal{M}_f(t)\|_{\Omega_f} \\
 &\leq \|e_{2_f} \cdot \|s\|_{\Omega} - e_{2_f} \cdot \|t\|_{\Omega}\|_{\Omega_f} + \|\mathcal{H}_{1_f}(s)\mathcal{M}_f(s) - \mathcal{H}_{1_f}(t)\mathcal{M}_f(s)\|_{\Omega_f} \\
 &\quad + \|\mathcal{H}_{1_f}(t)\mathcal{M}_f(s) - \mathcal{H}_{1_f}(t)\mathcal{M}_f(t)\|_{\Omega_f} \\
 &\leq \|e_{2_f} \cdot (\|s\|_{\Omega} - \|t\|_{\Omega})\|_{\Omega_f} + k_{1_f} \|\mathcal{H}_{1_f}(s) - \mathcal{H}_{1_f}(t)\|_{\Omega_{1_f}} \|\mathcal{M}_f(s)\|_{\Omega_{2_f}} \\
 &\quad + k_{1_f} \|\mathcal{H}_{1_f}(t)\|_{\Omega_{1_f}} \|\mathcal{M}_f(s) - \mathcal{M}_f(t)\|_{\Omega_{2_f}} \\
 &\leq \|e_{2_f}\|_{\Omega_f} \|s - t\|_{\Omega} + k_{1_f} \|e_{1_f} \cdot (\|s\|_{\Omega} - \|t\|_{\Omega})\|_{\Omega_f} \|\mathfrak{R}_{\sigma_f}^{\gamma_f} \mathcal{H}_{3_f}(s)\|_{\Omega_{2_f}} \\
 &\quad + k_{1_f} \|e_{1_f} \cdot \|t\|_{\Omega}\|_{\Omega_{1_f}} \|\mathfrak{R}_{\sigma_f}^{\gamma_f} \mathcal{H}_{3_f}(s) - \mathfrak{R}_{\sigma_f}^{\gamma_f} \mathcal{H}_{3_f}(t)\|_{\Omega_{2_f}} \\
 &\leq \|e_{2_f}\|_{\Omega_f} \|s - t\|_{\Omega} + k_{1_f} \|e_{1_f}\|_{\Omega_{1_f}} \|s - t\|_{\Omega} \frac{2}{\Gamma(\gamma_f)} \|k_f\|_{\Omega_{2_f}} \|e_{3_f}\|_{\mathfrak{B}_f} \|s\|_{\Omega} \\
 &\quad + k_{1_f} \|e_{1_f}\|_{\Omega_{1_f}} \|t\|_{\Omega} \frac{2}{\Gamma(\gamma_f)} \|k_f\|_{\Omega_{2_f}} \|e_{3_f}\|_{\mathfrak{B}_f} \|s - t\|_{\Omega} \\
 &\leq \left( \|e_{2_f}\|_{\Omega_f} + \frac{4k_{1_f} \cdot \mathfrak{r} \|k_f\|_{\Omega_{2_f}}}{\Gamma(\gamma_f)} \|e_{1_f}\|_{\Omega_{1_f}} \|e_{3_f}\|_{\mathfrak{B}_f} \right) \|s - t\|_{\Omega}.
 \end{aligned} \tag{3.3}$$

Now, putting (3.1) and (3.3) in (3.2) we have

$$\begin{aligned} \|s - t\|_{\Omega} &\leq \left[ \mathcal{K} \left( \|e_{2_1}\|_{\Omega_1} + \frac{4k_{1_1} \cdot \tau \|k_1\|_{\Omega_{2_1}}}{\Gamma(\gamma_1)} \|e_{1_1}\|_{\Omega_1} \|e_{3_1}\|_{\mathfrak{B}_1} \right) \prod_{i=2}^n T_i \right. \\ &\quad + \mathcal{K} \mathcal{L}_1 \left( \|e_{2_2}\|_{\Omega_2} + \frac{4k_{1_2} \cdot \tau \|k_2\|_{\Omega_{2_2}}}{\Gamma(\gamma_2)} \|e_{1_2}\|_{\varphi_{1_2}} \|e_{3_2}\|_{\mathfrak{B}_2} \right) \prod_{i=3}^n T_i \\ &\quad + \dots + \mathcal{K} \left( \|e_{2_n}\|_{\Omega_n} + \frac{4k_{1_n} \cdot \tau \|k_n\|_{\Omega_{2_n}}}{\Gamma(\gamma_n)} \|e_{1_n}\|_{\Omega_n} \|e_{3_n}\|_{\mathfrak{B}_n} \right) \prod_{i=1}^{n-1} T_i \left. \right] \|s - t\|_{\Omega} \\ &= \mathcal{P} \cdot \|s - t\|_{\Omega}. \end{aligned}$$

Since  $\mathcal{P} < 1$ , we have  $s = t$  (almost everywhere). This completes the proof. □

To illustrate our findings, we present the following examples.

**Example 3.4.** Consider the  $\mathfrak{B}$ -functions  $\Upsilon_i(y) = \mathfrak{B}_i(y) = 2y^3$  and  $\Omega_{2_i}(y) = y^2 + 3y + 2$ . We need to show that the operators  $\mathfrak{R}_{\sigma_i}^{\gamma_i} : \mathfrak{L}_{\mathfrak{B}_i} \rightarrow \mathfrak{L}_{\Omega_{2_i}}$  is continuous, for  $i = 1, 2, 3, \dots, n$  and Lemma 3.1 is fulfilled.

$$k(t) = \int_0^t \Upsilon_i(q^{1-\gamma_i}) dq = \int_0^t (2q)^{3-3\gamma_i} dq = 2^{1-\gamma_i} \left( \frac{t^{4-3\gamma_i}}{4-3\gamma_i} \right)$$

which confirms that Proposition 2.1 has been varified. Furthermore,

$$\int_0^t \Omega_{2_i}(k_i(t)) dt = \int_0^t \left[ \left( 2^{1-\gamma_i} \frac{t^{4-3\gamma_i}}{4-3\gamma_i} \right)^2 + 3 \left( 2^{1-\gamma_i} \frac{t^{4-3\gamma_i}}{4-3\gamma_i} \right) + 2 \right] dt < \infty.$$

Therefore, if  $s \in \mathfrak{L}_{\mathfrak{B}_i}$  then the operator  $\mathfrak{R}_{\sigma_i}^{\gamma_i} : \mathfrak{L}_{\mathfrak{B}_i} \rightarrow \mathfrak{L}_{\Omega_{2_i}}$  is continuous.

**Remark 3.5.** Further details on the operational behavior and continuity conditions of the operator  $\mathcal{H}_i(s) = e_i(s).k(s)$  are given in [22, Theorem 18.2] (see supposition (F5)).

**Example 3.6.** Let,  $\mathcal{H}_f(k)(s) = e_i(s).k(s)$ ,  $f = 1, 2, 3$  and  $i = 1, 2, \dots, n$ . Then we get

$$k(s) = \prod_{i=1}^n \left[ p_i(s) + e_{2_i}(s) k(s) + \frac{e_{1_i}(s) k(s)}{\Gamma(\gamma_i)} \sigma \int_0^s \frac{\mu^{\sigma-1}}{(\eta^{\sigma} - \mu^{\sigma})^{1-\gamma_i}} e_{3_i}(\mu) \cdot k(\mu) \cdot d\mu \right],$$

for  $\gamma_i \in (0, 1)$ ,  $s \in [0, 1]$ , which represent a special type of the Eq. (1.1).

### 4 Conclusion remarks

In this paper, we establish the existence and uniqueness of monotonic solutions to n-quadratic Erdelyi-Köber fractional integral equations within Orlicz spaces, proving the existence and uniqueness of monotonic solutions using advanced mathematical techniques such as the measure of noncompactness and the fixed-point theorem. These findings extend previously established results for quadratic and higher-order fractional systems. The framework demonstrates how Orlicz spaces provide a natural setting for fractional operators of nonconvolution type. The imposed conditions ensure the continuity and boundedness of the involved operators. In addition, conditions that ensure both existence and uniqueness were clearly formulated and verified with the help of examples. Hence, this study contributes to the advancement of fractional integral theory in generalized function spaces. Future work may focus on numerical approximations and extensions to systems of such equations and similar problems in diverse function spaces.

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