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CHARACTERIZATION OF FERMATEAN FUZZY GK-REFLECTIVE BI-IDEALS OF GK-ALGEBRA.

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Abstract. In this paper, we define a new ideal called GK-reflective bi-ideal of GK-algebra and also we propose the concepts of Fermatean fuzzy GK-reflective bi-ideal of GK-algebra. Furthermore, we investigate properties of Fermatean $(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$- fuzzy GK-reflective bi-ideal of GK-algebra and also its homomorphic behaviour. In addition, we introduce τ-Fermatean $(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$- Fuzzy GK-reflective bi-ideal of GK-algebra.

1 Introduction

The fuzzy set concept was initiated by L.A. Zadeh in 1965 [17]. This fundamental idea was extended by Atanassov [5] with intuitionistic fuzzy sets (IFSs). Yagar [16] further developed this field with Pythagorean fuzzy sets(PFSs). The most recent development, Fermatean fuzzy sets (FFSs), was established by Senapati and Yager [15]. Fermatean fuzzy sets allow the sum of the cubes of membership and non-membership degrees to be ≤ 1, providing even more adaptability and accuracy. Muhammad et al. [14] introduced a fuzzy system known as the Fermatean fuzzy system. Balamurugan and Nagarajan [6] formulated a Fermatean fuzzy soft-covered generalized bi-ideal on a semigroup. Adak et al. [2] introduced the concept of Fermatean fuzzy semi-prime ideals and Fermatean fuzzy prime ideals of ordered semigroups. Also Kannirun Suayngam, Rukchart Prasertpong, Nareupanat Lekkoksung, Pongpun Jultha and Aiyared Iampan [12] established the concept of Fermatean fuzzy set theory applied to IUP-algebras.
A new type of fuzzy subgroup, that is, the $(\in, \in \vee q)$ fuzzy subgroup was founded by Bhakat and Das [7] using the composite notion of the belongingness and quasicoincidence with the fuzzy points and fuzzy sets. In [8] Davvaz established the concept of $(\in, \in \vee q)$ fuzzy subnear-rings of near-rings and explore its properties. Zhan and Yin [18] introduce $(\in, \in \vee q)$ - fuzzy subnear-rings (ideals) of a near-rings. Later M. Himaya Jaleela Begum and P. Ayesha Parveen [10] established the concept intuitionistic $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy prime ideals of near-rings. J.Kavitha and R. Gowri [13] introduced the notion of GK-algebra in 2018 and they studied some of its characteristics like Direct product of GK-algebra[13], fuzzy GK-algebra[13]etc., Motivated by these, in 2023 we establish fuzzy asterisk subalgebra and fuzzy asterisk ideals of GK-algebra[11]. Several construction models are available in [1, 3, 4].
In this paper we establish a new ideal called GK-reflective bi-ideal of GK-algebra and then we introduce the concept of Fermatean fuzzy GK-reflective bi-ideal of GK-algebra with an example. Also we introduce the notion of Fermatean $(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$- fuzzy GK-reflective bi-ideal of GK-algebra with an example and further we prove its equivalent conditions. The homomorphic behaviour of Fermatean $(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$- fuzzy GK-reflective bi-ideal of GK-algebra have been discussed. Futhermore, we establish the concept of τ-Fermatean $(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$- Fuzzy GK-reflective bi-ideal of GK-algebra with example and also we verify its equivalent conditions.

2 Preliminaries

In this section we locate the essential definitions that will be used in the continuation.

Definition 2.1. [9] A non-empty set X with fixed constant 1 and a binary operation $\circledast$ is called a ***GK-algebra*** if it satisfies the following axioms

(i) $i \circledast i = 1$

(ii) $i \circledast 1 = i$

(iii) $i \circledast j = 1$ and $j \circledast i = 1$ implies $i = j$

(iv) $(j \circledast k) \circledast (i \circledast k) = j \circledast i$

(v) $(i \circledast j) \circledast (1 \circledast j) = i \ \forall\, i, j, k \ \in X$

Throughout this paper X always means a GK-algebra.

Example 2.2. [13] Consider the set X={ $1, l, m, n$ }. The binary operation $\circledast$ is defined as follows:

$\circledast$	1	l	m	n
1	1	l	m	n
l	l	1	n	m
m	m	n	1	l
n	n	m	l	1

Hence $(X, \circledast, 1)$ is a GK-algebra.

Definition 2.3. [9] Let $(X, \circledast, 1)$ be a GK-algebra. A non-empty subset Y of X is called a GK-subalgebra of X if $i \circledast j \ \in Y$ for any $i, j \ \in Y$. A mapping $f : X \to Y$ of GK-algebra is called a homomorphism if $f(x \circledast y) = f(x) \circledast f(y)$ for all $x, y \ \in X$.
We now review some fuzzy concepts as follows: Let X be a non empty set. A function $\mu : X \to [0, 1]$ is called a **fuzzy set** on X. For any x $\in$ X, the number $\mu(x)$ is called the membership grade of x. The **complement** of μ, denoted by $\bar{\mu}$, is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x) \ \forall\, x \ \in X$.For any two fuzzy sets $A = \{ < x, \mu_A(x) >: x \ \in X \}$ and $B = \{ < x, \mu_B(x) >: x \ \in X \}$ in X, the following operations are defined,

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x) \ \forall\, x \ \in X,\ A \cap B\text{=min}\{\ \mu_A(x), \mu_B(x) \ \forall\, x \ \in X.\ \}$$

Definition 2.4. [12] Let X be universe of discourse. A Fermatean Fuzzy Set $\mathcal{F}$ (FFS) in X is an object having the form $\mathcal{F} = \{(x, \mu_{\mathcal{F}}(x), \eta_{\mathcal{F}}(x)) : x \in X\}$, where $\mu_{\mathcal{F}}(x) : X \to [0, 1]$ and $\eta_{\mathcal{F}}(x) : X \to [0, 1]$ including the following condition: $(\forall x \in X)(0 \leq (\mu_{\mathcal{F}}(x))^3 + (\eta_{\mathcal{F}}(x))^3 \leq 1)$ the number $\mu_{\mathcal{F}}(x)$ and $\eta_{\mathcal{F}}(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set $\mathcal{F}$.
For any FFS $\mathcal{F}$ and $x \in X$, $\Pi_{\mathcal{F}}(x) = \sqrt{1 - \mu_{\mathcal{F}}(x))^3 - (\eta_{\mathcal{F}}(x))^3}$ is identified as the degree of indeterminancy of x to $\mathcal{F}$.
For the sake of simplicity, we shall mention the symbol $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ for the FFS $\mathcal{F} = \{(x, \mu_{\mathcal{F}}(x), \eta_{\mathcal{F}}(x)) : x \in X\}$.
For a subset C of a non-empty set X, the characteristic functions $\mu_{\mathcal{F}_C}, \eta_{\mathcal{F}_C}$ are functions of X into {0,1} defined as follows:

$$\mu_{\mathcal{F}_C}(x) = \begin{cases} 1 & if\ x \in C \\ 0 & for\ all\ other\ elements \end{cases}$$

$$\eta_{\mathcal{F}_C}(x) = \begin{cases} 0 & if\ x \in C \\ 1 & for\ all\ other\ elements \end{cases}$$

By the definition of characteristic function, $\mu_{\mathcal{F}_C}$ and $\eta_{\mathcal{F}_C}$ are functions of X in $\{0, 1\} \subset [0, 1]$, Therefore, the FFS $\mathcal{F}_C = (\mu_{\mathcal{F}_C}, \eta_{\mathcal{F}_C})$ is defined as the characteristic FFS of C in X.

Definition 2.5. [12] Let $\mathcal{F}$ be FFS in a non-empty set X. Then the FFS $\overline{\mathcal{F}} = (\overline{\mu_{\mathcal{F}}}, \overline{\eta_{\mathcal{F}}})$ is called the complement of $\mathcal{F}$ in X.

Definition 2.6. [13] A fuzzy set μ in X is called a fuzzy GK-subalgebra if it satisfies the inequality $\mu(x \circledast y) \geq \min\{\ \mu(x), \mu(y)\ \}\ \forall\ x, y\ \in X.$

Example 2.7. Consider the set X={ 1,a,b,c } is a GK-algebra

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Define a mapping $\mu : X \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0.9 & if\ x = 1, a \\ 0.5 & for\ all\ other\ elements \end{cases}$$

Then μ is a fuzzy GK-subalgebra of X.

Definition 2.8. [13] A fuzzy set μ in X is called a fuzzy GK-ideal of X if it satisfies the inequality (i) $\mu(1) \geq \mu(x)$ and (ii) $\mu(x \circledast z) \geq \min\{\ \mu(y \circledast z), \mu(y \circledast x)\ \}\ \forall\ x, y, z\ \in X.$

Example 2.9. Refer example(2.7), it can be easily verify that it is a fuzzy GK-ideal.

Definition 2.10. [10] A fuzzy set μ of X of the form

$$\mu(y) = \begin{cases} t(\neq 0) & if\ y = x \\ 0 & if\ y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t is said to belongs to (respectively be quasi-coincident with) a fuzzy set μ, written as $x_t \vdash \mu$ (respectively $x_t \psi \mu$) if $\mu(x) \geq t$ (respectively $\mu(x) + t > 1$). If $x_t \vdash \mu$ (or) $x_t \psi \mu$, then we write $x_t \vdash \vee \psi \mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$) then, we call $x_t \overline{\vdash} \mu$ (resp., $x_t \overline{\psi} \mu$). We note that the symbol $(\overline{\vdash \vee \psi})$ means that $\vdash \vee \psi$ does not hold.

Remark 2.11. [10] Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For a fuzzy point x_r and a fuzzy set μ of X, we say that
1. $x_r \vdash_\gamma \mu$ if $\mu(x) \geq r > \gamma$
2. $x_r \psi_\delta \mu$ if $\mu(x) + r > 2\delta$
3. $x_r \vdash_\gamma \vee \psi_\delta \mu$ if $x_r \vdash_\gamma \mu$ or $x_r \psi_\delta \mu$.

3 Fermatean $(\vdash_\gamma, \vdash_\gamma \vee \psi_\delta)$- fuzzy GK-reflective bi-ideal of GK-algebra

Definition 3.1. A Fermatean fuzzy set(FFS) $\mathcal{F}$ in X is called a Fermatean fuzzy GK-ideal (FFGKI) of X if it satisfies the following properties:
1. $\mu_{\mathcal{F}}(1) \geq \mu_{\mathcal{F}}(i)$
2. $\eta_{\mathcal{F}}(1) \leq \eta_{\mathcal{F}}(i)$
3. $\mu_{\mathcal{F}}(i \circledast k) \geq \min\{\mu_{\mathcal{F}}(j \circledast k), \mu_{\mathcal{F}}(j \circledast i)\}$
4. $\eta_{\mathcal{F}}(i \circledast k) \leq \max\{\eta_{\mathcal{F}}(j \circledast k), \eta_{\mathcal{F}}(j \circledast i)\}\ \forall\ i, j, k\ \in\ X.$

Definition 3.2. A non-empty subset D of X is called GK-reflective bi-ideal (GKRBiI) of X if it satisfies the following conditions:
1. $1 \in D$
2. $j \circledast k \in D, j \circledast (i \circledast l) \in D \Rightarrow i \circledast (k \circledast l) \in D\ \forall\ i, j, k, l \in X.$

Definition 3.3. A FFS $\mathcal{F}$ in X is called a Fermatean fuzzy GK-reflective bi-ideal(FFGKRBiI) of X if it satisfies the following properties:
1. $\mu_{\mathcal{F}}(1) \geq \mu_{\mathcal{F}}(i)$
2. $\eta_{\mathcal{F}}(1) \leq \eta_{\mathcal{F}}(i)$
3. $\mu_{\mathcal{F}}(i \circledast (k \circledast l) \geq \min\{\mu_{\mathcal{F}}(j \circledast k), \mu_{\mathcal{F}}(j \circledast (i \circledast l)\}$
4. $\eta_{\mathcal{F}}(i \circledast (k \circledast l)) \leq \max\{\eta_{\mathcal{F}}(j \circledast k), \eta_{\mathcal{F}}(j \circledast (i \circledast l)\} \ \forall \ i, j, k, l \ \in \ X.$

Example 3.4. Let X=1,2,3,4 with the following cayley table

$\circledast$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Then X is a GK-algebra. We define a FFS $\mathcal{F}$ in X as follows:

$$\mu_{\mathcal{F}}(i) = \begin{cases} 0.9 & if \ i = 1, 2 \\ 0.2 & for \ all \ other \ elements \end{cases}$$

and

$$\eta_{\mathcal{F}}(i) = \begin{cases} 0.1 & if \ i = 1, 2 \\ 0.8 & for \ all \ other \ elements \end{cases}$$

Then $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ is FFGKRBiI of X.

Definition 3.5. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fermatean fuzzy set $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ written as $x_t \vdash \mu$ (resp., $x_t \psi \mu$) if $\mu_{\mathcal{F}}(x) \geq t$ (resp.,$\mu_{\mathcal{F}}(x) + t > 1$) and $x_t \vdash \eta$ (resp., $x_t \psi \eta$) $\eta_{\mathcal{F}}(x) < t$ (resp.,$\eta_{\mathcal{F}}(x) + t \leq 1$). If $x_t \vdash \mu$ or $x_t \psi \mu$, then we write $x_t \vdash \vee \psi \mu$

Note: $x_t \vdash \mathcal{F} \Rightarrow x_t \vdash \mu_{\mathcal{F}}$ and $x_t \overline{\vdash} \eta_{\mathcal{F}}$ and $x_t \psi \mathcal{F} \Rightarrow x_t \psi \mu_{\mathcal{F}}$ and $x_t \overline{\psi} \eta_{\mathcal{F}}$

Remark 3.6. Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For a fuzzy point i_t and a membership function $\mu_{\mathcal{F}}$ of X, we say that
1. $i_t \vdash_{\gamma} \mu_{\mathcal{F}}$ if $\mu_{\mathcal{F}}(i) \geq t > \gamma$
2. $i_t \psi_{\delta} \mu_{\mathcal{F}}$ if $\mu_{\mathcal{F}}(i) + t > 2\delta$
3. $i_t \vdash_{\gamma} \vee \psi_{\delta} \mu_{\mathcal{F}}$ if $i_t \vdash_{\gamma} \mu_{\mathcal{F}}$ or $i_t \psi_{\delta} \mu_{\mathcal{F}}$

Remark 3.7. Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For a fuzzy point i_t and a non-membership function $\eta_{\mathcal{F}}$ of X, we say that
1. $i_t \vdash_{\delta} \eta_{\mathcal{F}}$ if $\eta_{\mathcal{F}}(i) < t < \delta$
2. $i_t \psi_{\gamma} \eta_{\mathcal{F}}$ if $\eta_{\mathcal{F}}(i) + t \leq 2\gamma$
3. $i_t \vdash_{\delta} \vee \psi_{\gamma} \eta_{\mathcal{F}}$ if $i_t \vdash_{\delta} \eta_{\mathcal{F}}$ or $i_t \psi_{\gamma} \eta_{\mathcal{F}}$.

Definition 3.8. A Fermatean fuzzy set FFS $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ of a GK-algebra X, is called a Fermatean $(\vdash_{\gamma}, \vdash_{\gamma} \vee \psi_{\delta})$- fuzzy GK-ideal (F$(\vdash_{\gamma}, \vdash_{\gamma} \vee \psi_{\delta})$FGKI) of X if $\forall i, j, k, l \in X$
1. $i_t \vdash_{\gamma} \mu_{\mathcal{F}} \Rightarrow 1_t \vdash_{\gamma} \vee \psi_{\delta} \mu_{\mathcal{F}}$
2. $(j \circledast k)_t \vdash_{\gamma} \mu_{\mathcal{F}}, (j \circledast i)_s \vdash_{\gamma} \mu_{\mathcal{F}} \Rightarrow (i \circledast k)_{\min(t,s)} \vdash_{\gamma} \vee \psi_{\delta} \mu_{\mathcal{F}}$
3. $i_t \vdash_{\delta} \eta_{\mathcal{F}} \Rightarrow 1_t \vdash_{\delta} \vee \psi_{\gamma} \eta_{\mathcal{F}}$
4. $(j \circledast k)_t \vdash_{\delta} \eta_{\mathcal{F}}, (j \circledast i)_s \vdash_{\delta} \eta_{\mathcal{F}} \Rightarrow (i \circledast k)_{\max(t,s)} \vdash_{\delta} \vee \psi_{\gamma} \eta_{\mathcal{F}}$

Example 3.9. Consider the example (3.4) in that we define a fermatean fuzzy set $\mathcal{F}$ in X as follows:

$$\mu_{\mathcal{F}}(i) = \begin{cases} 0.9 & if \ i = 1, 2 \\ 0.7 & for \ all \ other \ elements \end{cases}$$

and

$$\eta_{\mathcal{F}}(i) = \begin{cases} 0.1 & if \ i = 1, 2 \\ 0.2 & for \ all \ other \ elements \end{cases}$$

Then $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a $(F(\vdash_{0.2}, \vdash_{0.2} \vee \psi_{0.4}) FGKI)$ of X.

Definition 3.10. A Fermatean fuzzy set FFS $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ of a GK-algebra X, is called a Fermatean $(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$- fuzzy GK-reflective bi-ideal (F$(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$FGKRBiI) of X if $\forall i, j, k, l \in X$
1. $i_t \vdash_\gamma \mu_{\mathcal{F}} \Rightarrow 1_t \vdash_\gamma \vee\psi_\delta\mu_{\mathcal{F}}$
2. $(j\circledast k)_t \vdash_\gamma \mu_{\mathcal{F}}, (j\circledast(i\circledast l))_s \vdash_\gamma \mu_{\mathcal{F}} \Rightarrow (i\circledast(k\circledast l))_{\min(t,s)} \vdash_\gamma \vee\psi_\delta\mu_{\mathcal{F}}$
3. $i_t \vdash_\delta \eta_{\mathcal{F}} \Rightarrow 1_t \vdash_\delta \vee\psi_\gamma\eta_{\mathcal{F}}$
4. $(j\circledast k)_t \vdash_\delta \eta_{\mathcal{F}}, (j\circledast(i\circledast l))_s \vdash_\delta \eta_{\mathcal{F}} \Rightarrow (i\circledast(k\circledast l))_{\max(t,s)} \vdash_\delta \vee\psi_\gamma\eta_{\mathcal{F}}$

Example 3.11. consider the example (3.4) in that we define a fermatean fuzzy set $\mathcal{F}$ in X as follows:

$$\mu_{\mathcal{F}}(i) = \begin{cases} 0.8 & if\ i = 1, 2 \\ 0.6 & for\ all\ other\ elements \end{cases}$$

and

$$\eta_{\mathcal{F}}(i) = \begin{cases} 0.2 & if\ i = 1, 2 \\ 0.1 & for\ all\ other\ elements \end{cases}$$

Then $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a $(F(\vdash_{0.1}, \vdash_{0.1} \vee\psi_{0.4})FGKRBiI)$ of X.

Theorem 3.12. *A FFS $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ in X is a $(F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI)$ of X if and only if for any $i, j, k, l \in X$ the following conditions satisfies:*
(F1) $\max\{\mu_{\mathcal{F}}(1), \gamma\} \geq \min\{\mu_{\mathcal{F}}(i), \delta\}$
(F2) $\min\{\eta_{\mathcal{F}}(1), \delta\} \leq \max\{\eta_{\mathcal{F}}(i), \gamma\}$
(F3) $\max\{\mu_{\mathcal{F}}(i\circledast(k\circledast l)), \gamma\} \geq \min\{\mu_{\mathcal{F}}(j\circledast k), \mu_{\mathcal{F}}(j\circledast(i\circledast l)), \delta\}$
(F4) $\min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\} \leq \max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l)), \gamma\}$

Proof. (F3) Let us assume that $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ in X is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X and $i, j, k, l \in X$.
Choose p such that $\max\{\mu_{\mathcal{F}}(i\circledast(k\circledast l)), \gamma\} < p < \min\{\mu_{\mathcal{F}}(j\circledast k), \mu_{\mathcal{F}}(j\circledast(i\circledast l)), \delta\}$
$\Rightarrow \mu_{\mathcal{F}}(j\circledast k) > p, \mu_{\mathcal{F}}(j\circledast(i\circledast l)) > p$ and $\mu_{\mathcal{F}}(i\circledast(k\circledast l)) < p$ and also $\mu_{\mathcal{F}}(i\circledast(k\circledast l)) + p < 2p < 2\delta \Rightarrow (j\circledast k)_p \vdash_\gamma \mu_{\mathcal{F}}, (j\circledast(i\circledast l))_p \vdash_\gamma \mu_{\mathcal{F}}$ but $(i\circledast(k\circledast l))_p \overline{\vdash_\gamma \vee\psi_\delta}\mu_{\mathcal{F}}$, which is a contradiction to our assumption. Hence condition (F3) is true.
Conversly, suppose that $\max\{\mu_{\mathcal{F}}(i\circledast(k\circledast l)), \gamma\} \geq \min\{\mu_{\mathcal{F}}(j\circledast k), \mu_{\mathcal{F}}(j\circledast(i\circledast l)), \delta\}$.
If suppose there exists $i, j, k, l \in X$ and $s, t \in (\gamma, 1]$ such that $(j\circledast k)_t \vdash_\gamma \mu_{\mathcal{F}}, (j\circledast(i\circledast l))_s \vdash_\gamma \mu_{\mathcal{F}}$ but $(i\circledast(k\circledast l))_{\min(t,s)} \overline{\vdash_\gamma \vee\psi_\delta}\mu_{\mathcal{F}} \Rightarrow \mu_{\mathcal{F}}(j\circledast k) \geq t, \mu_{\mathcal{F}}(j\circledast(i\circledast l)) \geq s, \mu_{\mathcal{F}}(i\circledast(k\circledast l)) < \min(t, s)$ and $\mu_{\mathcal{F}}(i\circledast(k\circledast l)) + \min(t, s) \leq 2\delta$ and so $\max\{\mu_{\mathcal{F}}(i\circledast(k\circledast l)), \gamma\} < \min\{t, s, \delta\} \leq \min\{\mu_{\mathcal{F}}(j\circledast k), \mu_{\mathcal{F}}(j\circledast(i\circledast l)), \delta\}$, which is a contradiction to our assumption.
Therefore $(j\circledast k)_t \vdash_\gamma \mu_{\mathcal{F}}, (j\circledast(i\circledast l))_s \vdash_\gamma \mu_{\mathcal{F}} \Rightarrow (i\circledast(k\circledast l))_{\min(t,s)} \vdash_\gamma \vee\psi_\delta\mu_{\mathcal{F}}$.
(F4)Assume that $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ in X is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X and $i.j, k, l \in X$.
Choose t such that $\min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\} > t > \max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l)), \gamma\}$
$\Rightarrow \eta_{\mathcal{F}}(j\circledast k) < t, \eta_{\mathcal{F}}(j\circledast(i\circledast l)) < t$ and $\eta_{\mathcal{F}}(i\circledast(k\circledast l)) > t$ and also $\eta_{\mathcal{F}}(i\circledast(k\circledast l)) + t > 2t > 2\gamma \Rightarrow (j\circledast k)_t \vdash_\delta \eta_{\mathcal{F}}, (j\circledast(i\circledast l))_t \vdash_\delta \eta_{\mathcal{F}}$ but $(i\circledast(k\circledast l))_t \overline{\vdash_\delta \vee\psi_\gamma}\eta_{\mathcal{F}}$, which is a contradiction to our assumption. Hence condition (F4) is true.
Conversly, suppose that $\min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\} \leq \max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l)), \gamma\}$.
If suppose there exists $i, j, k, l \in X$ and $s, t \in (\gamma, 1]$ such that $(j\circledast k)_t \vdash_\delta \eta_{\mathcal{F}}, (j\circledast(i\circledast l))_s \vdash_\delta \eta_{\mathcal{F}}$ but $(i\circledast(k\circledast l))_{\max(t,s)} \overline{\vdash_\delta \vee\psi_\gamma}\eta_{\mathcal{F}} \Rightarrow \eta_{\mathcal{F}}(j\circledast k) < t, \eta_{\mathcal{F}}(j\circledast(i\circledast l)) < s, \eta_{\mathcal{F}}(i\circledast(k\circledast l)) \geq \max(t, s)$ and $\eta_{\mathcal{F}}(i\circledast(k\circledast l)) + \max(t, s) > 2\gamma$ and so $\min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\} > \max\{t, s, \gamma\} > \max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l)), \gamma\}$, which is a contradiction to our assumption.
Therefore $(j\circledast k)_t \vdash_\delta \eta_{\mathcal{F}}, (j\circledast(i\circledast l))_s \vdash_\delta \eta_{\mathcal{F}} \Rightarrow (i\circledast(k\circledast l)) \vdash_\delta \vee\psi_\gamma\eta_{\mathcal{F}}$.
Likewise we can prove (F1) and (F2). □

Definition 3.13. Let $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ be a FFS of X and $\forall\ t \in (\gamma, 1]$, we define $(\mu_{\mathcal{F}})_t^\gamma = \{i \in X/i_t \vdash_\gamma \mu_{\mathcal{F}}\}, (\mu_{\mathcal{F}})_t^\delta = \{i \in X/i_t\psi_\delta\mu_{\mathcal{F}}\}, [\mu_{\mathcal{F}}]_t^\delta = \{i \in X/i_t \vdash_\gamma \vee\psi_\delta\mu_{\mathcal{F}}\}$. It is clear that $[\mu_{\mathcal{F}}]_t^\delta = (\mu_{\mathcal{F}})_t^\gamma \cup (\mu_{\mathcal{F}})_t^\delta$ where $(\mu_{\mathcal{F}})_t^\gamma, (\mu_{\mathcal{F}})_t^\delta$ and $[\mu_{\mathcal{F}}]_t^\delta$ are $\vdash_\gamma$-level set, ψ_δ-level set and $\vdash_\gamma \vee\psi_\delta$-level set of $\mu_{\mathcal{F}}$ respectively.
Also, $(\eta_{\mathcal{F}})_t^\delta = \{i \in X/i_t \vdash_\delta \eta_{\mathcal{F}}\}, (\eta_{\mathcal{F}})_t^\gamma = \{i \in X/i_t\psi_\gamma\eta_{\mathcal{F}}\}, [\eta_{\mathcal{F}}]_t^\gamma = \{i \in X/i_t \vdash_\delta \vee\psi_\gamma\eta_{\mathcal{F}}\}$. It is clear that $[\eta_{\mathcal{F}}]_t^\gamma = (\eta_{\mathcal{F}})_t^\delta \cup (\eta_{\mathcal{F}})_t^\gamma$ where $(\eta_{\mathcal{F}})_t^\delta, (\eta_{\mathcal{F}})_t^\gamma$ and $[\eta_{\mathcal{F}}]_t^\gamma$ are $\vdash_\delta$-level set, ψ_γ-level set and $\vdash_\delta \vee\psi_\gamma$-level set of $\eta_{\mathcal{F}}$ respectively.

Theorem 3.14. *A FFS $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ in X is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X if and only if all the $(\mu_{\mathcal{F}})^\gamma_t \neq \emptyset$ and $(\eta_{\mathcal{F}})^\delta_t \neq \emptyset$ are GK-reflective bi-ideals of X for all $t, s \in (\gamma, \delta)$*

Proof. Let $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ in X is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X and $(\mu_{\mathcal{F}})^\gamma_t \neq \emptyset, (\eta_{\mathcal{F}})^\delta_t \neq \emptyset$
(1) Let $i \in (\mu_{\mathcal{F}})^\gamma_t \Rightarrow i_t \vdash_\gamma \mu_{\mathcal{F}} \Rightarrow \mu_{\mathcal{F}}(i) \geq t > \gamma$.
$\max\{\mu_{\mathcal{F}}(1), \gamma\} \geq \min\{\mu_{\mathcal{F}}(i), \delta\} \geq t \Rightarrow 1_t \vdash_\gamma \mu_{\mathcal{F}} \Rightarrow 1 \in (\mu_{\mathcal{F}})^\gamma_t$
(2) Let $(j\circledast k) \in (\mu_{\mathcal{F}})^\gamma_t, (j\circledast(i\circledast l)) \in (\mu_{\mathcal{F}})^\gamma_t \Rightarrow (j\circledast k) \vdash_\gamma \mu_{\mathcal{F}}, (j\circledast(i\circledast l)) \vdash_\gamma \mu_{\mathcal{F}} \Rightarrow \mu_{\mathcal{F}}(j\circledast k) \geq t > \gamma, \mu_{\mathcal{F}}(j\circledast(i\circledast l)) \geq t > \gamma$.
Now, $\max\{\mu_{\mathcal{F}}(i\circledast(k\circledast l)), \gamma\} \geq \min\{\mu_{\mathcal{F}}(j\circledast k), \mu_{\mathcal{F}}(j\circledast(i\circledast l)), \delta\} \geq t$
$\Rightarrow \max\{\mu_{\mathcal{F}}(i\circledast(k\circledast l)), \gamma\} \geq t \Rightarrow (i\circledast(k\circledast l)) \vdash_\gamma \mu_{\mathcal{F}} \Rightarrow (i\circledast(k\circledast l)) \in (\mu_{\mathcal{F}})^\gamma_t$.
(3) Let $i \in (\eta_{\mathcal{F}})^\delta_t \Rightarrow i_t \vdash_\delta \eta_{\mathcal{F}} \Rightarrow \eta_{\mathcal{F}}(i) < t < \delta$.
Now, $\min\{\eta_{\mathcal{F}}(1), \delta\} \leq \max\{\eta_{\mathcal{F}}(i), \gamma\} < t \Rightarrow 1_t \vdash_\delta \eta_{\mathcal{F}} \Rightarrow 1 \in (\eta_{\mathcal{F}})^\delta_t$
(4) Let $(j\circledast k) \in (\eta_{\mathcal{F}})^\delta_t, (j\circledast(i\circledast l)) \in (\eta_{\mathcal{F}})^\delta_t \Rightarrow (j\circledast k) \vdash_\delta \eta_{\mathcal{F}}, (j\circledast(i\circledast l)) \vdash_\delta \eta_{\mathcal{F}} \Rightarrow \eta_{\mathcal{F}}(j\circledast k) < t < \delta, \eta_{\mathcal{F}}(j\circledast(i\circledast l)) < t < \delta$.
Now, $\min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\} \leq \max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l)), \gamma\} < t$
$\Rightarrow \min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\} < t \Rightarrow (i\circledast(k\circledast l)) \vdash_\delta \eta_{\mathcal{F}} \Rightarrow (i\circledast(k\circledast l)) \in (\eta_{\mathcal{F}})^\delta_t$.
Therefore $(\mu_{\mathcal{F}})^\gamma_t, (\eta_{\mathcal{F}})^\delta_t$ are GKRBiI of X.
Conversly, suppose that $(\mu_{\mathcal{F}})^\gamma_t, (\eta_{\mathcal{F}})^\delta_t$ are GKRBiI of X for all $t, s \in (\gamma, \delta]$
(F1) Let $i \in X$ such that $\max\{\mu_{\mathcal{F}}(1), \gamma\} < t < \min\{\mu_{\mathcal{F}}(i), \delta\} \Rightarrow \mu_{\mathcal{F}}(i) > t$ but $\mu_{\mathcal{F}}(1) < t$ and $\mu_{\mathcal{F}}(1) + t < 2t \leq 2\delta \Rightarrow i_t \vdash_\gamma \mu_{\mathcal{F}}$ but $\overline{\vdash_\gamma \vee\psi_\delta}(\mu_{\mathcal{F}})^\gamma_t$ that is $i \in (\mu_{\mathcal{F}})^\gamma_t$ but $1 \notin (\mu_{\mathcal{F}})^\gamma_t$, which is a contradiction to our assumption. Therefore $\max\{\mu_{\mathcal{F}}(1), \gamma\} \geq \min\{\mu_{\mathcal{F}}(i), \delta\}$. Similarly we can prove (F2).
(F4) Suppose $(j\circledast k) \in (\eta_{\mathcal{F}})^\delta_t, (j\circledast(i\circledast l)) \in (\eta_{\mathcal{F}})^\delta_t \Rightarrow (i\circledast(k\circledast l)) \in (\eta_{\mathcal{F}})^\delta_t$.
Let $i, j, k, l \in X$ such that $\min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\} > t > \max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l)), \gamma\} \Rightarrow \eta_{\mathcal{F}}(j\circledast k) < t, \eta_{\mathcal{F}}(j\circledast(i\circledast l)) < t$ but $\eta_{\mathcal{F}}(i\circledast(k\circledast l)) > t$ and $\eta_{\mathcal{F}}(i\circledast(k\circledast l)) + t > 2t > 2\gamma \Rightarrow (j\circledast k)_t \vdash_\gamma \eta_{\mathcal{F}}, (j\circledast(i\circledast l))_t \vdash_\gamma \eta_{\mathcal{F}}$ but $(i\circledast(k\circledast l))\overline{\vdash_\gamma \vee\psi_\delta}\eta_{\mathcal{F}} \Rightarrow (j\circledast k)_t \in (\eta_{\mathcal{F}})^\delta_t, (j\circledast(i\circledast l))_t \in (\eta_{\mathcal{F}})^\delta_t$ but $(i\circledast(k\circledast l)) \notin (\eta_{\mathcal{F}})^\delta_t$, which is a contradiction to our assumption.
Therefore $\min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\} \leq \max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l)), \gamma\}$. Similarly we can prove (F3). Therefore $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ in X is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X. □

Theorem 3.15. *Let $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ be $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X and if $i \leq j$ then* $\max\{\mu_{\mathcal{F}}(i), \gamma\} \geq \min\{\mu_{\mathcal{F}}(j), \delta\}$ *and* $\min\{\eta_{\mathcal{F}}(i), \delta\} \leq \max\{\eta_{\mathcal{F}}(j), \gamma\}$

Proof. Let $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ be $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X and $i \leq j \Rightarrow i\circledast j = j\circledast i = 1$ also let $i, j, k, l \in X$
Now,

$$\begin{aligned}
\max\{\mu_{\mathcal{F}}(i), \gamma\} &= \max\{\mu_{\mathcal{F}}(i\circledast 1), \gamma\} \\
&= \max\{\mu_{\mathcal{F}}(i\circledast(1\circledast 1)), \gamma\} \\
&\geq \min\{\mu_{\mathcal{F}}(j\circledast 1), \mu_{\mathcal{F}}(j\circledast(i\circledast 1)), \delta\} \\
&= \min\{\mu_{\mathcal{F}}(j), \mu_{\mathcal{F}}(j\circledast i), \delta\} \\
&= \min\{\mu_{\mathcal{F}}(j), \mu_{\mathcal{F}}(1), \delta\} \\
&\geq \min\{\mu_{\mathcal{F}}(j), \max\{\mu_{\mathcal{F}}(1), \gamma\}, \delta\} \\
&\geq \min\{\mu_{\mathcal{F}}(j), \min\{\mu_{\mathcal{F}}(j), \delta\}, \delta\} \\
&= \min\{\mu_{\mathcal{F}}(j), \delta\} \\
\text{Therefore } \max\{\mu_{\mathcal{F}}(i), \gamma\} &\geq \min\{\mu_{\mathcal{F}}(j), \delta\}.
\end{aligned}$$

Similarly we can prove $\min\{\eta_{\mathcal{F}}(i), \delta\} \leq \max\{\eta_{\mathcal{F}}(j), \gamma\}$. □

Theorem 3.16. *Let $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ be $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X and if the inequality $j\circledast i \leq k$ carry in X, then* $\max\{\mu_{\mathcal{F}}(i), \gamma\} \geq \min\{\mu_{\mathcal{F}}(j), \mu_{\mathcal{F}}(k), \delta\}$ *and* $\min\{\eta_{\mathcal{F}}(i), \delta\} \leq \max\{\eta_{\mathcal{F}}(j), \eta_{\mathcal{F}}(k), \gamma\}$

Proof. Let $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ be $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X and $j\circledast i \leq k$.
Now, $\max\{\mu_{\mathcal{F}}(i\circledast(k\circledast l)), \gamma\} \geq \min\{\mu_{\mathcal{F}}(j\circledast k), \mu_{\mathcal{F}}(j\circledast(i\circledast l)), \delta\}$. Put $k, l = 1$ we get,

$$\begin{aligned}\max\{\mu_{\mathcal{F}}(i\circledast(1\circledast 1)),\gamma\} &= \max\{\mu_{\mathcal{F}}(i),\gamma\}\\ &\geq \min\{\mu_{\mathcal{F}}(j\circledast 1),\mu_{\mathcal{F}}(j\circledast(i\circledast 1)),\delta\}\\ &= \min\{\mu_{\mathcal{F}}(j),\mu_{\mathcal{F}}(j\circledast i),\delta\}\\ &\geq \min\{\mu_{\mathcal{F}}(j),\max\{\mu_{\mathcal{F}}(j\circledast i),\gamma\},\delta\}\\ &\geq \min\{\mu_{\mathcal{F}}(j),\min\{\mu_{\mathcal{F}}(k),\delta\},\delta\} \text{ (by theorem(3.15))}\\ &= \min\{\mu_{\mathcal{F}}(j),\mu_{\mathcal{F}}(k),\delta\}\end{aligned}$$

Therefore $\max\{\mu_{\mathcal{F}}(i),\gamma\} \geq \min\{\mu_{\mathcal{F}}(j),\mu_{\mathcal{F}}(k),\delta\}$.

Similarly we can prove $\min\{\eta_{\mathcal{F}}(i),\delta\} \leq \max\{\eta_{\mathcal{F}}(j),\eta_{\mathcal{F}}(k),\gamma\}$. □

4 τ-Fermatean $(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$- Fuzzy GK-reflective bi-ideal of GK-algebra

Definition 4.1. Let $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ be a Fermatean fuzzy set in a non-empty set X and $\tau \in [0,1]$. Then the τ- Fermatean Fuzzy Set (τ-FFS) $\mathcal{F}_\tau$ in a non-empty set X is an object having the form $\mathcal{F}_\tau = \{(x, \mu_{\mathcal{F}_\tau}(x), \eta_{\mathcal{F}_\tau}(x))|x \in X\}$, where the function $\mu_{\mathcal{F}_\tau} : X \to [0,1]$ and $\eta_{\mathcal{F}_\tau} : X \to [0,1]$ denote the degree of membership and degree of non-membership respectively such that $\mu_{\mathcal{F}_\tau}(x) = \min\{\mu_{\mathcal{F}}(x),\tau\}$ and $\eta_{\mathcal{F}_\tau}(x) = \max\{\eta_{\mathcal{F}}(x), 1-\tau\}$ satisfying the condition $0 \leq (\mu_{\mathcal{F}_\tau}(x))^3 + (\eta_{\mathcal{F}_\tau}(x))^3 \leq 1 \;\forall\, x \in X$.
For the sake of simplicity , we shall use the symbol $\mathcal{F}_\tau = (\mu_{\mathcal{F}_\tau}, \eta_{\mathcal{F}_\tau})$ for τ-FFS

Definition 4.2. A τ-Fermatean fuzzy set $(\tau - FFS)$ $\mathcal{F}_\tau = (\mu_{\mathcal{F}_\tau}, \eta_{\mathcal{F}_\tau})$ of a GK-algebra X, is called a τ-Fermatean $(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$- fuzzy GK-reflective bi-ideal (τ-F$(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$FGKRBiI) of X if $\forall i,j,k,l \in X$ and $\tau \in [\gamma,\delta]$
1. $i_t \vdash_\gamma \mu_{\mathcal{F}_\tau} \Rightarrow 1_t \vdash_\gamma \vee\psi_\delta \mu_{\mathcal{F}_\tau}$
2. $(j\circledast k)_t \vdash_\gamma \mu_{\mathcal{F}_\tau}, (j\circledast(i\circledast l))_s \vdash_\gamma \mu_{\mathcal{F}_\tau} \Rightarrow (i\circledast(k\circledast l))_{\min(t,s)} \vdash_\gamma \vee\psi_\delta \mu_{\mathcal{F}_\tau}$
3. $i_t \vdash_\delta \eta_{\mathcal{F}_\tau} \Rightarrow 1_t \vdash_\delta \vee\psi_\gamma \eta_{\mathcal{F}_\tau}$
4. $(j\circledast k)_t \vdash_\delta \eta_{\mathcal{F}_\tau}, (j\circledast(i\circledast l))_s \vdash_\delta \eta_{\mathcal{F}_\tau} \Rightarrow (i\circledast(k\circledast l))_{\max(t,s)} \vdash_\delta \vee\psi_\gamma \eta_{\mathcal{F}_\tau}$

Theorem 4.3. *A τ-FFS $\mathcal{F}_\tau = (\mu_{\mathcal{F}_\tau}, \eta_{\mathcal{F}_\tau})$ in X is a $\tau - (F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI)$ of X if and only if for any $i,j,k,l \in X$ and $\tau \in [\gamma,\delta]$ the following conditions satisfies:*
(F'1) $\max\{\mu_{\mathcal{F}_\tau}(1),\gamma\} \geq \min\{\mu_{\mathcal{F}_\tau}(i),\delta\}$
(F'2) $\min\{\eta_{\mathcal{F}_\tau}(1),\delta\} \leq \max\{\eta_{\mathcal{F}_\tau}(i),\gamma\}$
(F'3) $\max\{\mu_{\mathcal{F}_\tau}(i\circledast(k\circledast l)),\gamma\} \geq \min\{\mu_{\mathcal{F}_\tau}(j\circledast k), \mu_{\mathcal{F}_\tau}(j\circledast(i\circledast l)),\delta\}$
(F'4) $\min\{\eta_{\mathcal{F}_\tau}(i\circledast(k\circledast l)),\delta\} \leq \max\{\eta_{\mathcal{F}_\tau}(j\circledast k), \eta_{\mathcal{F}_\tau}(j\circledast(i\circledast l)),\gamma\}$

Proof. Proof is similar to the proof of the theorem (3.12) □

Example 4.4. consider the example (3.4) in that we define a fermatean fuzzy set $\mathcal{F}$ in X as follows:

$$\mu_{\mathcal{F}}(i) = \begin{cases} 0.4 & if\ i = 1 \\ 0.3 & for\ all\ other\ elements \end{cases}$$

and

$$\eta_{\mathcal{F}}(i) = \begin{cases} 0.5 & if\ i = 1 \\ 0.6 & for\ all\ other\ elements \end{cases}$$

If we take $\tau = 0.2, \gamma = 0.1, \delta = 0.9$ then we have $\mu_{\mathcal{F}_\tau}(i) = \min\{\mu_{\mathcal{F}}(i),\tau\} = 0.2$ and $\eta_{\mathcal{F}_\tau}(i) = \max\{\eta_{\mathcal{F}}(i), 1-\tau\} = 0.8 \;\forall\; i \in X$. Then by routine calculation, we can easily verify that $\mathcal{F}_\tau = (\mu_{\mathcal{F}_\tau}, \eta_{\mathcal{F}_\tau})$ is a τ-F$(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$FGKRBiI of X.

Theorem 4.5. *If $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ is a F$(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$FGKRBiI of X and $\mu_{\mathcal{F}_\tau}(i) = \min\{\mu_{\mathcal{F}}(i),\tau\}$ and $\eta_{\mathcal{F}_\tau}(i) = \max\{\eta_{\mathcal{F}}(i), 1-\tau\} \;\forall\; i \in X$ and $\tau \in [\gamma,\delta]$ then $F_\tau = (\mu_{\mathcal{F}_\tau}, \eta_{\mathcal{F}_\tau})$ is also a $\tau - F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$*

Proof. Let $\mathcal{F} = (\mu_{\mathcal{F}}, \eta_{\mathcal{F}})$ be a F$(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$FGKRBiI of X and $i \in X, \tau \in [\gamma, \delta]$.
Now,

$$\begin{aligned}
\max\{\mu_{\mathcal{F}_\tau}(1), \gamma\} &= \max\{\min\{\tau, \mu_{\mathcal{F}}(1)\}, \gamma\} \\
&= \min\{\tau, \max\{\mu_{\mathcal{F}}(1), \gamma\}\} \\
&\geq \min\{\tau, \min\{\mu_{\mathcal{F}}(i), \delta\}\} \\
&= \min\{\min\{\mu_{\mathcal{F}}(i), \tau\}, \delta\} \\
&= \min\{\mu_{\mathcal{F}_\tau}(i), \delta\}
\end{aligned}$$

Therefore $\max\{\mu_{\mathcal{F}_\tau}(1), \gamma\} \geq \min\{\mu_{\mathcal{F}_\tau}(i), \delta\}$.

Similarly we can prove $\min\{\eta_{\mathcal{F}_\tau}(1), \delta\} \leq \max\{\eta_{\mathcal{F}_\tau}(i), \gamma\}$
Now, Let $i, j, k, l \in X$

$$\begin{aligned}
\min\{\eta_{\mathcal{F}_\tau}(i\circledast(k\circledast l)), \delta\} &= \min\{\max\{1-\tau, \eta_{\mathcal{F}}(i\circledast(k\circledast l))\}, \delta\} \\
&= \max\{1-\tau, \min\{\eta_{\mathcal{F}}(i\circledast(k\circledast l)), \delta\}\} \\
&\leq \max\{1-\tau, \max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l))\gamma\}\} \\
&= \max\{\max\{\eta_{\mathcal{F}}(j\circledast k), \eta_{\mathcal{F}}(j\circledast(i\circledast l)), 1-\tau\}, \gamma\} \\
&= \max\{\max\{\eta_{\mathcal{F}}(j\circledast k), 1-\tau\}, \max\{\eta_{\mathcal{F}}(j\circledast(i\circledast l)), 1-\tau\}, \gamma\} \\
&= \max\{\eta_{\mathcal{F}_\tau}(j\circledast k), \eta_{\mathcal{F}_\tau}(j\circledast(i\circledast l)), \gamma\}
\end{aligned}$$

Therefore $\min\{\eta_{\mathcal{F}_\tau}(i\circledast(k\circledast l)), \delta\} \leq \max\{\eta_{\mathcal{F}_\tau}(j\circledast k), \eta_{\mathcal{F}_\tau}(j\circledast(i\circledast l)), \gamma\}$.

Similarly we can prove $\max\{\mu_{\mathcal{F}_\tau}(i\circledast(k\circledast l)), \gamma\} \geq \min\{\mu_{\mathcal{F}_\tau}(j\circledast k), \mu_{\mathcal{F}_\tau}(j\circledast(i\circledast l)), \delta\}$.
Hence $\mathcal{F}_\tau$ is $\tau - F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$.
The converse of this theorem need not be true as shown in the following example. □

Example 4.6. consider the example (3.4) in that we define a fermatean fuzzy set $\mathcal{F}$ in X as follows:

$$\mu_{\mathcal{F}}(i) = \begin{cases} 0.5 & if\ i = 1 \\ 0.4 & for\ all\ other\ elements \end{cases}$$

and

$$\eta_{\mathcal{F}}(i) = \begin{cases} 0.3 & if\ i = 1 \\ 0.1 & for\ all\ other\ elements \end{cases}$$

If we take $\gamma = 0.2, \delta = 0.9$ Since $\max\{\mu_{\mathcal{F}}(1), 0.2\} = \max\{0.5, 0.2\} \geq \min\{0.4, 0.9\}$ and $\min\{\eta_{\mathcal{F}}(1), \delta\} = \min\{0.3, 0.9\} \geq \max\{0.1, 0.2\}$, it follows that $\mathcal{F}$ is not a F$(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$FGKRBiI of X. But if we take $\tau = 0.6$ then

$$\mu_{\mathcal{F}_\tau}(i) = \begin{cases} 0.5 & if\ i = 1 \\ 0.4 & for\ all\ other\ elements \end{cases}$$

and $\eta_{\mathcal{F}_\tau}(i) = 0.4\ \forall\ i = 1, 2, 3, 4$. By routine calculation, we can easily verify that $\mathcal{F}_\tau = (\mu_{\mathcal{F}_\tau}, \eta_{\mathcal{F}_\tau})$ is a τ-F$(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)$FGKRBiI of X.

Theorem 4.7. *Let* $\mathcal{F}_\tau = (\mu_{\mathcal{F}_\tau}, \eta_{\mathcal{F}_\tau})$ *be* $\tau - F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ *of X and if* $i \leq j$ *then* $\max\{\mu_{\mathcal{F}_\tau}(i), \gamma\} \geq \min\{\mu_{\mathcal{F}_\tau}(j), \delta\}$ *and* $\min\{\eta_{\mathcal{F}_\tau}(i), \delta\} \leq \max\{\eta_{\mathcal{F}_\tau}(j), \gamma\}$.

Proof. Proof is straightforward. □

Theorem 4.8. *Let* $\mathcal{F}_\tau = (\mu_{\mathcal{F}_\tau}, \eta_{\mathcal{F}_\tau})$ *be* $\tau - F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ *of X and if the inequality* $j\circledast i \leq k$ *carry in X, then* $\max\{\mu_{\mathcal{F}_\tau}(i), \gamma\} \geq \min\{\mu_{\mathcal{F}_\tau}(j), \mu_{\mathcal{F}_\tau}(k), \delta\}$ *and* $\min\{\eta_{\mathcal{F}_\tau}(i), \delta\} \leq \max\{\eta_{\mathcal{F}_\tau}(j), \eta_{\mathcal{F}_\tau}(k), \gamma\}$.

Proof. Proof is straightforward. □

5 Homomorphism of $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of GK-algebra.

Theorem 5.1. *A mapping $\sigma : X \to X'$ be a homomorphism of GK-algebra. If a FFS $\mathcal{F} = (\mu_\mathcal{F}, \eta_\mathcal{F})$ is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X' then a FFS $\mathcal{F}^\sigma = (\mu^\sigma_\mathcal{F}, \eta^\sigma_\mathcal{F})$ is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X.*

Proof. Let $i, j, k, l \in X$. Assume that $\mathcal{F} = (\mu_\mathcal{F}, \eta_\mathcal{F})$ is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X'.
Now, (F1) $\max\{\mu^\sigma_\mathcal{F}(1), \gamma\} = \max\{\mu_\mathcal{F}(\sigma(1)), \gamma\} \geq \min\{\mu_\mathcal{F}(\sigma(i)), \delta\} = \min\{\mu^\sigma_\mathcal{F}(i), \delta\}$. Therefore $\max\{\mu^\sigma_\mathcal{F}(1), \gamma\} \geq \min\{\mu^\sigma_\mathcal{F}(i), \delta\}$.
Similarly we can prove $\min\{\eta^\sigma_\mathcal{F}(1), \delta\} \leq \max\{\eta^\sigma_\mathcal{F}(i), \gamma\}$.

$$\begin{aligned}(F4)\ \min\{\eta^\sigma_\mathcal{F}(i\circledast(k\circledast l)), \delta\} &= \min\{\eta_\mathcal{F}\sigma(i\circledast(k\circledast l)), \delta\}\\ &= \min\{\eta_\mathcal{F}(\sigma(i)\circledast\sigma(k\circledast l)), \delta\}\\ &= \min\{\eta_\mathcal{F}(\sigma(i)\circledast(\sigma(k)\circledast\sigma(l))), \delta\}\\ &\leq \max\{\eta_\mathcal{F}(\sigma(j)\circledast\sigma(k)), \eta_\mathcal{F}(\sigma(j)\circledast(\sigma(i)\circledast\sigma(l))), \gamma\}\\ &= \max\{\eta_\mathcal{F}(\sigma(j\circledast k)), \eta_\mathcal{F}(\sigma(j\circledast(i\circledast l))), \gamma\}\\ &= \max\{\eta^\sigma_\mathcal{F}(j\circledast k), \eta^\sigma_\mathcal{F}(j\circledast(i\circledast l)), \gamma\}\end{aligned}$$

Therefore $\min\{\eta^\sigma_\mathcal{F}(i\circledast(k\circledast l)), \delta\} \leq \max\{\eta^\sigma_\mathcal{F}(j\circledast k), \eta^\sigma_\mathcal{F}(j\circledast(i\circledast l)), \gamma\}$.

Similarly we can prove $\max\{\mu^\sigma_\mathcal{F}(i\circledast(k\circledast l)), \gamma\} \geq \min\{\mu^\sigma_\mathcal{F}(j\circledast k), \mu^\sigma_\mathcal{F}(j\circledast(i\circledast l)), \delta\}$.
Hence $\mathcal{F}^\sigma = (\mu^\sigma_\mathcal{F}, \eta^\sigma_\mathcal{F})$ is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X. □

Theorem 5.2. *Let $\sigma : X \to X'$ be an epimorphism of GK-algebra. Let $\mathcal{F} = (\mu_\mathcal{F}, \eta_\mathcal{F})$ be a FFS of X'. If $\mathcal{F}^\sigma = (\mu^\sigma_\mathcal{F}, \eta^\sigma_\mathcal{F})$ is a $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X.*

Proof. Let $i, j, k, l \in X$. Since σ is an epimorphism there exists $a, b, c, d \in X'$ such that $\sigma(i) = a, \sigma(j) = b, \sigma(k) = c, \sigma(l) = d$.
Assume that $\mathcal{F}^\sigma$ is $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$.
Now, (F1) $\max\{\mu_\mathcal{F}(1'), \gamma\} = \max\{\mu_\mathcal{F}(\sigma(1)), \gamma\}$ (where $1' \in X'$)= $\max\{\mu^\sigma_\mathcal{F}(1), \gamma\} \geq \min\{\mu^\sigma_\mathcal{F}(i), \delta\} = \min\{\mu_\mathcal{F}(\sigma(i)), \delta\} = \min\{\mu_\mathcal{F}(a), \delta\}$
Therefore $\max\{\mu_\mathcal{F}(1'), \gamma\} \geq \min\{\mu_\mathcal{F}(a), \delta\}$.
Similarly we can prove $\min\{\eta_\mathcal{F}(1'), \delta\} \leq \max\{\eta_\mathcal{F}(a), \gamma\}$.
Now,

$$\begin{aligned}\min\{\eta_\mathcal{F}(a\circledast(c\circledast d)), \delta\} &= \min\{\eta_\mathcal{F}(\sigma(i)\circledast(\sigma(k)\circledast\sigma(l))), \delta\}\\ &= \min\{\eta^\sigma_\mathcal{F}(i\circledast(k\circledast l)), \delta\}\\ &\leq \max\{\eta^\sigma_\mathcal{F}(j\circledast k), \eta^\sigma_\mathcal{F}(j\circledast(i\circledast l)), \gamma\}\\ &= \max\{\eta_\mathcal{F}(\sigma(j)\circledast\sigma(k)), \eta_\mathcal{F}(\sigma(j)\circledast(\sigma(i)\circledast\sigma(l))), \gamma\}\\ &= \max\{\eta_\mathcal{F}(b\circledast c), \eta_\mathcal{F}(b\circledast(a\circledast d)), \gamma\}\end{aligned}$$

Therefore $\min\{\eta_\mathcal{F}(a\circledast(c\circledast d)), \delta\} \leq \max\{\eta_\mathcal{F}(b\circledast c), \eta_\mathcal{F}(b\circledast(a\circledast d)), \gamma\}$.

Similarly we can prove $\max\{\mu_\mathcal{F}(a\circledast(c\circledast d)), \gamma\} \geq \min\{\mu_\mathcal{F}(b\circledast c), \mu_\mathcal{F}(b\circledast(a\circledast d)), \delta\}$.
Therefore $F(\vdash_\gamma, \vdash_\gamma \vee\psi_\delta)FGKRBiI$ of X. □

6 Conclusion remarks

This paper indroduced novel concepts like F($\vdash_\gamma, \vdash_\gamma \vee\psi_\delta$)FGKI, F($\vdash_\gamma, \vdash_\gamma \vee\psi_\delta$)FGKRBiI and τ-F($\vdash_\gamma, \vdash_\gamma \vee\psi_\delta$)FGKRBiI within the framework of Fermatean Fuzzy Set (FFS) and GK-algebra. The study delves into their fundamental properties, ultimately revealing intricate structures within these algebraic systems. The investigation into the interplay between FFSs and GK-algebra is presented as a dynamic evolution in fuzzy set theory, promising significant advancements and practical applications.

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