

UNIQUENESS RESULTS ON MEROMORPHIC FUNCTIONS CONCERNING THEIR SHIFTS AND DERIVATIVES

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Communicated by: Sarika Verma

MSC 2020 Classifications: Primary 30D35.

Keywords and phrases: Meromorphic functions, Derivatives, Shifts, Small functions, Partial sharing.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract In this paper, we obtain uniqueness results of meromorphic functions when the j -th derivative of a meromorphic function $f(z)$ shares one value IM (respectively CM) and two values partially with a function involving its shifts and derivatives. The results of the paper extend and improve the results due to W.J. Chen and Z.G. Huang [J. Contem. Math. Anal., 57 (2022), 232-241]. We also pose an open problem for further research.

1 Introduction, Definitions and Results

In this paper, a meromorphic function always means meromorphic in the whole complex plane. We use the standard notations in value distribution theory, see [8, 15, 16]. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ possibly outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function $\alpha(z)$ is said to be a small function of $f(z)$ if it satisfies $T(r, \alpha) = S(r, f)$. We say that two meromorphic functions $f(z)$ and $g(z)$ share a finite value or a small function a CM (IM) if $f(z) - a$ and $g(z) - a$ have the same set of zeros counting multiplicities (ignoring multiplicities). The order $\rho(f)$ and the hyper-order $\rho_2(f)$ of a meromorphic function f are defined as follows:

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r},$$

$$\text{and } \rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ T(r, f)}{\log r}.$$

If $\rho(f) < +\infty$, then f is called a meromorphic function of finite order. For a nonzero complex number c , the shift $f_c(z)$ of $f(z)$ is defined as $f_c(z) = f(z + c)$.

In 1983, E. Mues and N. Steinmetz [11], G.G. Gunderson [4] proved the uniqueness of a meromorphic function concerning its derivative and their result is as follows.

Theorem 1.1. *Let $f(z)$ be a nonconstant meromorphic function. If $f(z)$ and $f'(z)$ share two distinct finite values a, b CM, then $f(z) \equiv f'(z)$.*

The uniqueness of entire and meromorphic functions sharing values or small functions with their shift has been investigated by many authors (see [2, 6, 7, 9, 10, 12, 13, 14]).

In 2011, J. Heittokangas et al. [9] proved the following result.

Theorem 1.2. *Let $f(z)$ be a nonconstant entire function of finite order, let c be a nonzero constant, and let a, b be two distinct finite values. If $f(z)$ and $f(z + c)$ share a, b CM, then $f(z) \equiv f(z + c)$.*

In 2018, X. Qi, N. Li and L. Yang [12] considered the uniqueness problem of $f'(z)$ and $f(z+c)$ of a meromorphic function $f(z)$ and obtained the following result.

Theorem 1.3. *Let $f(z)$ be a transcendental entire function of finite order and $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0, a CM, then $f'(z) \equiv f(z+c)$.*

In 2020, X. Qi and L. Yang [13] improved the above theorem as follows.

Theorem 1.4. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a CM and satisfy $f(z+c) = 0 \rightarrow f'(z) = 0$ and $f'(z) = \infty \rightarrow f(z+c) = \infty$, then $f'(z) \equiv f(z+c)$.*

The notation $f(z) = a \rightarrow g(z) = a$ in the above theorem means that any a -point of $f(z)$ with multiplicity p is also an a -point of $g(z)$ with multiplicity at least p , where $a \in \mathbb{C} \cup \{\infty\}$.

In 2022, S. Chen and A. Xu [1] proved the following result.

Theorem 1.5. *Let $f(z)$ be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$, c be a nonzero finite complex number, and k be a positive integer. If $f^{(k)}(z)$ and $f(z+c)$ share 0, ∞ CM and 1 IM, then $f^{(k)}(z) \equiv f(z+c)$.*

In the same year, W.J. Chen and Z.G. Huang [2] proved the following results where they used the notion of partial sharing of values which is defined as follows.

Definition 1.6. We denote by $E(a, f)$ the set of all zeros of $f - a$, where each zero with multiplicity m is counted m times. Similarly, we denote by $\bar{E}(a, f)$ the set of zeros of $f - a$, where each zero is counted only once. If $\bar{E}(a, f) \subseteq \bar{E}(a, g)$, then we say that $f(z)$ partially shares a with $g(z)$. If $E(a, f) \subseteq E(a, g)$, then we say that $f(z)$ and $g(z)$ partially share a CM.

We now state the results of Chen and Huang [2].

Theorem 1.7. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let c be a nonzero finite complex number and j, k be integers with $0 \leq j < k$. Suppose that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a finite value $a(\neq 0)$ IM and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$. If $N\left(r, \frac{1}{f^{(j)}(z)}\right) + \bar{N}\left(r, \frac{1}{f(z)}\right) = S(r, f)$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

Theorem 1.8. *Let $f(z)$ be a nonconstant meromorphic function of finite order, c be a nonzero finite complex number and j, k be integers with $0 \leq j < k$. If $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share $a(\neq 0)$ CM and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

Though the standard definitions and notations of the value distribution theory are available in [8, 15, 16], we give the following definitions and notations which are used in this paper.

Definition 1.9. Let k be a positive integer. We use $N_{(k)}\left(r, \frac{1}{f-a}\right)$ to denote the counting function of a -points of f with multiplicity $\leq k$ and $N_{(k+1)}\left(r, \frac{1}{f-a}\right)$ to denote the counting function of a -points of f with multiplicity $> k$, where each a -point is counted on the basis of its multiplicity. Similarly, we define $\bar{N}_{(k)}\left(r, \frac{1}{f-a}\right)$ and $\bar{N}_{(k+1)}\left(r, \frac{1}{f-a}\right)$ where the a -points is counted with ignoring multiplicities.

Definition 1.10. We use $N_p\left(r, \frac{1}{f-a}\right)$ to denote the counting function of the zeros of $f - a$ where a p -fold zero is counted m times if $m \leq p$ and counted p times if $m > p$.

2 Main Results

Theorem 2.1. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let c_1, c_2, \dots, c_n be nonzero finite complex numbers and j, k_1, k_2, \dots, k_n be integers with $0 \leq j < \min\{k_1, k_2, \dots, k_n\}$. Let m_1, m_2, \dots, m_n be small functions of $f(z)$ such that*

$$F(z) = m_1 f^{(k_1)}(z + c_1) + m_2 f^{(k_2)}(z + c_2) + \dots + m_n f^{(k_n)}(z + c_n).$$

Suppose that $f^{(j)}(z)$ and $F(z)$ share a finite value $a(\neq 0)$ IM and satisfy $E(0, f^{(j)}(z)) \subset E(0, F(z))$ and $E(\infty, F(z)) \subset E(\infty, f^{(j)}(z))$. If $N\left(r, \frac{1}{f^{(j)}(z)}\right) + \bar{N}\left(r, \frac{1}{f(z)}\right) = S(r, f)$, then $f^{(j)}(z) \equiv F(z)$.

Theorem 2.2. Let $f(z)$ be a nonconstant meromorphic function of finite order, and let c_1, c_2, \dots, c_n be nonzero finite complex numbers and j, k_1, k_2, \dots, k_n be integers with $0 \leq j < \min\{k_1, k_2, \dots, k_n\}$. Let m_1, m_2, \dots, m_n be small functions of $f(z)$ such that

$$F(z) = m_1 f^{(k_1)}(z + c_1) + m_2 f^{(k_2)}(z + c_2) + \dots + m_n f^{(k_n)}(z + c_n).$$

If $f^{(j)}(z)$ and $F(z)$ share a finite value $a(\neq 0)$ CM and satisfy $E(0, f^{(j)}(z)) \subset E(0, F(z))$ and $E(\infty, F(z)) \subset E(\infty, f^{(j)}(z))$, then $f^{(j)}(z) \equiv F(z)$.

3 Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 3.1. [15] Let $f(z)$ be an entire function and k be a positive integer. Then

$$m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = S(r, f).$$

Lemma 3.2. [3, 5] Let $f(z)$ be a meromorphic function of finite order. Then for any $c \in \mathbb{C} - \{0\}$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 3.3. [17] Let f be a nonconstant transcendental meromorphic function. If $f^{(k)} \neq 0$, then

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f).$$

Lemma 3.4. [15] Let $f(z)$ be a nonconstant meromorphic function in the complex plane and k be a positive integer. Set

$$\psi(z) = \sum_{k=0}^n a_k(z) f^{(k)}(z),$$

where $a_k(z) (k = 0, 1, \dots, n)$ are all small functions of $f(z)$. Then

$$T(r, \psi) \leq T(r, f) + k\bar{N}(r, f) + S(r, f) \leq (k+1)T(r, f) + S(r, f),$$

$$N\left(r, \frac{1}{\psi}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f).$$

Lemma 3.5. [3] Let $f(z)$ be a nonconstant meromorphic function of finite order and $c \in \mathbb{C}$. Then

$$T(r, f(z+c)) = T(r, f) + S(r, f),$$

$$N(r, f(z+c)) = N(r, f) + S(r, f), \quad N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f),$$

$$\bar{N}(r, f(z+c)) = \bar{N}(r, f) + S(r, f), \quad \bar{N}\left(r, \frac{1}{f(z+c)}\right) = \bar{N}\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

Lemma 3.6. Suppose that $f(z)$ is a nonconstant meromorphic function of finite order in $|z| < R$ and $a_i (i = 1, 2, \dots, q)$ are $q (\geq 2)$ distinct finite complex numbers. Let j, k_1, k_2, \dots, k_n be integers with $0 \leq j < \min\{k_1, k_2, \dots, k_n\}$. Then for $0 < r < R$, we have

$$m(r, f^{(j)}(z)) + \sum_{i=1}^q m\left(r, \frac{1}{f^{(j)}(z) - a_i}\right) \leq 2T(r, f^{(j)}(z)) - N_{pair}(r, f) + S(r, f),$$

where

$$N_{\text{pair}}(r, f) = 2N(r, f^{(j)}(z)) - N(r, F) + N\left(r, \frac{1}{F}\right) + S(r, f),$$

where F is defined as in Theorem 2.1.

Proof. Let $A(z) = \sum_{i=1}^q \frac{1}{f^{(j)}(z) - a_i}$. Then

$$B(z) = A(z)F(z) = \sum_{i=1}^q \frac{F(z)}{f^{(j)}(z) - a_i}.$$

Using Lemmas 3.1 and 3.2, we get

$$\begin{aligned} m(r, B(z)) &= m\left(r, \sum_{i=1}^q \frac{F(z)}{f^{(j)}(z) - a_i}\right) \\ &\leq \sum_{i=1}^q m\left(r, \frac{F(z)}{f^{(j)}(z) - a_i}\right) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Therefore

$$\begin{aligned} m(r, A(z)) &= m\left(r, \frac{B(z)}{F(z)}\right) \\ &\leq m(r, B(z)) + m\left(r, \frac{1}{F}\right) \\ &= m\left(r, \frac{1}{F}\right) + S(r, f). \end{aligned}$$

Now using the first fundamental theorem of Nevanlinna and the above inequality, we obtain

$$\begin{aligned} T(r, F(z)) &= T\left(r, \frac{1}{F}\right) + O(1) \\ &= m\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F}\right) + S(r, f) \\ &\geq m(r, A(z)) + N\left(r, \frac{1}{F}\right) + S(r, f) \\ &= m\left(r, \sum_{i=1}^q \frac{1}{f^{(j)}(z) - a_i}\right) + N\left(r, \frac{1}{F}\right) + S(r, f). \end{aligned} \quad (3.1)$$

Therefore from (3.1), we have

$$\begin{aligned} \sum_{i=1}^q m\left(r, \frac{1}{f^{(j)}(z) - a_i}\right) &= m\left(r, \sum_{i=1}^q \frac{1}{f^{(j)}(z) - a_i}\right) + O(1) \\ &\leq T(r, F) - N\left(r, \frac{1}{F}\right) + S(r, f). \end{aligned} \quad (3.2)$$

Thus from (3.2), we obtain

$$\begin{aligned}
 & m(r, f^{(j)}(z)) + \sum_{i=1}^q m\left(r, \frac{1}{f^{(j)}(z) - a_i}\right) \\
 & \leq m(r, f^{(j)}(z)) + T(r, F) - N\left(r, \frac{1}{F}\right) + S(r, f) \\
 & = T(r, f^{(j)}(z)) - N(r, f^{(j)}(z)) + m(r, F) + N(r, F) - N\left(r, \frac{1}{F}\right) + S(r, f) \\
 & \leq T(r, f^{(j)}(z)) - N(r, f^{(j)}(z)) + m(r, f^{(j)}(z)) + m\left(r, \frac{F}{f^{(j)}(z)}\right) \\
 & + N(r, F) - N\left(r, \frac{1}{F}\right) + S(r, f) \\
 & = 2T(r, f^{(j)}(z)) - 2N(r, f^{(j)}(z)) + N(r, F) - N\left(r, \frac{1}{F}\right) + S(r, f) \\
 & = 2T(r, f^{(j)}(z)) - \left[2N(r, f^{(j)}(z)) - N(r, F) + N\left(r, \frac{1}{F}\right)\right] + S(r, f).
 \end{aligned}$$

This proves the lemma. □

4 Proof of Main Results

Proof of Theorem 2.1. If possible, we suppose that $f^{(j)}(z) \neq F(z)$. Without loss of generality, we may assume that $f^{(j)}(z)$ and $F(z)$ share $a = 1$ IM. Since $E(0, f^{(j)}(z)) \subset E(0, F(z))$ and $E(\infty, F(z)) \subset E(\infty, f^{(j)}(z))$, we have

$$\frac{F(z)}{f^{(j)}(z)} = \phi(z), \tag{4.1}$$

where $\phi(z)$ is an entire function. Using Lemmas 3.1 and 3.2, from (4.1), we have

$$\begin{aligned}
 m(r, \phi(z)) & = m\left(r, \frac{F(z)}{f^{(j)}(z)}\right) \\
 & \leq m\left(r, \frac{f^{(k_1)}(z + c_1)}{f^{(j)}(z)}\right) + \dots + m\left(r, \frac{f^{(k_n)}(z + c_n)}{f^{(j)}(z)}\right) \\
 & \leq m\left(r, \frac{f^{(k_1)}(z + c_1)}{f^{(k_1)}(z)}\right) + m\left(r, \frac{f^{(k_1)}(z)}{f^{(j)}(z)}\right) + \dots \\
 & + m\left(r, \frac{f^{(k_n)}(z + c_n)}{f^{(k_n)}(z)}\right) + m\left(r, \frac{f^{(k_n)}(z)}{f^{(j)}(z)}\right) \\
 & = S(r, f).
 \end{aligned} \tag{4.2}$$

Also we have

$$N(r, \phi(z)) = 0. \tag{4.3}$$

From (4.2) and (4.3), we get

$$T(r, \phi(z)) = S(r, f). \tag{4.4}$$

Let

$$\begin{aligned}
 \psi(z) & = \frac{1}{f^{(j)}(z)} \left(\frac{F'}{F-1} - \frac{f^{(j+1)}(z)}{f^{(j)}(z)-1} \right) \\
 & = \frac{F}{f^{(j)}(z)} \left(\frac{F'}{F-1} - \frac{F'}{F} \right) - \left(\frac{f^{(j+1)}(z)}{f^{(j)}(z)-1} - \frac{f^{(j+1)}(z)}{f^{(j)}(z)} \right).
 \end{aligned} \tag{4.5}$$

Using Lemma 3.1 and (4.2), we obtain

$$m(r, \psi) = S(r, f). \tag{4.6}$$

Now the possible poles of $\psi(z)$ may occur at the zeros of $f^{(j)}(z)$, the 1-points of $f^{(j)}(z)$ and F , and the poles of $f^{(j)}(z)$ and F . It can easily be shown that 1-points of $f^{(j)}(z)$ and 1-points of F are the simple poles of $\psi(z)$. Again since $E(\infty, F(z)) \subset E(\infty, f^{(j)}(z))$, it can be shown that the poles of $f^{(j)}(z)$ and F are not the poles of $\psi(z)$. Therefore the poles of $\psi(z)$ can occur at the 1-points of $f^{(j)}(z)$ and F , and the zeros of $f^{(j)}(z)$.

Since $f^{(j)}(z)$ and $F(z)$ share 1 IM, by the given hypothesis that $N\left(r, \frac{1}{f^{(j)}(z)}\right) = S(r, f)$, (4.1) and (4.4), we have

$$\begin{aligned} N(r, \psi) &\leq \bar{N}\left(r, \frac{1}{f^{(j)}(z) - 1}\right) + \bar{N}\left(r, \frac{1}{F - 1}\right) + N\left(r, \frac{1}{f^{(j)}(z)}\right) \\ &\leq N\left(r, \frac{1}{\phi - 1}\right) + N\left(r, \frac{1}{\phi - 1}\right) + N\left(r, \frac{1}{f^{(j)}(z)}\right) \\ &\leq 2T(r, \phi) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$T(r, \psi) = S(r, f). \tag{4.8}$$

If $\psi \equiv 0$, then from (4.5), we see that $F - 1 = c(f^{(j)}(z) - 1)$, where c is a nonzero constant. We assert that $c = 1$. Since if $c \neq 1$, then we have

$$m\left(r, \frac{1}{f^{(j)}(z)}\right) = \frac{1}{1 - c} m\left(r, \frac{F(z)}{f^{(j)}(z)} - c\right) = S(r, f).$$

So from the given hypothesis, we can obtain $T\left(r, \frac{1}{f^{(j)}(z)}\right) = S(r, f)$, and using Nevanlinna's first fundamental theorem, we get $T(r, f^{(j)}(z)) = S(r, f)$. Again by Lemma 3.3, we have

$$\begin{aligned} T(r, f) &\leq T(r, f^{(j)}(z)) - N_p\left(r, \frac{1}{f^{(j)}}\right) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq T(r, f^{(j)}(z)) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq T(r, f^{(j)}(z)) + S(r, f), \end{aligned}$$

and hence by Lemma 3.4, we obtain

$$T(r, f^{(j)}(z)) \leq (j + 1)T(r, f) + S(r, f).$$

Thus $S(r, f^{(j)}) = S(r, f)$ and hence $T(r, f^{(j)}(z)) = S(r, f^{(j)}(z))$, a contradiction.

Let $\psi \not\equiv 0$. Then from Lemma 3.1, (4.5) and (4.8), we have

$$\begin{aligned} m(r, f^{(j)}(z)) &\leq m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{F'}{F - 1} - \frac{f^{(j+1)}(z)}{f^{(j)}(z) - 1}\right) \\ &\leq T(r, \psi) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{4.9}$$

Let

$$H(z) = \frac{F'(f^{(j)}(z) - 1)}{f^{(j+1)}(z)(F - 1)} = \left(\frac{F'}{F - 1} - \frac{F'}{F}\right) \frac{F}{f^{(j+1)}(z)} (f^{(j)}(z) - 1). \tag{4.10}$$

Using Lemma 3.1, (4.2) and (4.9), we obtain from (4.10) that

$$\begin{aligned} m(r, H) &\leq m\left(r, \frac{F'}{F-1} - \frac{F'}{F}\right) + m\left(r, \frac{F}{f^{(j+1)}(z)}\right) + m(r, f^{(j)}(z) - 1) \\ &= S(r, f). \end{aligned} \tag{4.11}$$

Now the possible poles of $H(z)$ can occur at the 1-points of F , the poles of F' and $f^{(j)}(z)$, and the zeros of $f^{(j+1)}(z)$. Since $f^{(j)}(z)$ and $F(z)$ share 1 IM, we can conclude that $H(z)$ is analytic at the 1-points of F . Again from (4.10) and the relation $E(\infty, F(z)) \subset E(\infty, f^{(j)}(z))$, it can be shown that the poles of F' and $f^{(j)}(z)$ are not the poles of $H(z)$. Let z_0 be zero of $f^{(j+1)}(z)$ with multiplicity p . Now if z_0 is also a zero of $f^{(j)}(z)$ (respectively $f^{(j)}(z) - 1$) with multiplicity $p + 1$, then z_0 is a pole of $H(z)$ with multiplicity at most p . Therefore

$$N(r, H) \leq N_0\left(r, \frac{1}{f^{(j+1)}(z)}\right) + S(r, f), \tag{4.12}$$

where $N_0\left(r, \frac{1}{f^{(j+1)}(z)}\right)$ denotes all those zeros of $f^{(j+1)}(z)$ which are not the zeros of $f^{(j)}(z) - 1$. Combining (4.11) and (4.12), we obtain

$$T(r, H) \leq N_0\left(r, \frac{1}{f^{(j+1)}(z)}\right) + S(r, f). \tag{4.13}$$

Now, let z_0 be a simple pole of $f^{(j)}(z)$. Since $E(\infty, F(z)) \subset E(\infty, f^{(j)}(z))$, the following two cases arises.

Case 1. Assume that z_0 is not a simple pole of F . We put

$$f^{(j)}(z) = \frac{\alpha_{-1}}{z - z_0} + \alpha_0 + \alpha_1(z - z_0) + \alpha_2(z - z_0)^2 + \dots$$

and

$$F(z) = \beta_0 + \beta_1(z - z_0) + \beta_2(z - z_0)^2 + \dots,$$

where α_j ($j = -1, 0, 1, 2, \dots$) and β_j ($j = 0, 1, 2, \dots$) are the coefficients of the Laurent series expansion of $f^{(j)}(z)$ and $F(z)$ respectively. Now from (4.10), by a simple calculation we can deduce that $H(z_0) = 0$. If $H(z) \equiv 0$, then $F'(z)(f^{(j)}(z) - 1) \equiv 0$. Noting that F is nonconstant, by integration, we get $f(z)$ is a nonconstant polynomial, a contradiction with the fact that $f(z)$ is a transcendental function. Thus $H(z) \not\equiv 0$, and hence

$$N_{1)}(r, f^{(j)}(z)) \leq N\left(r, \frac{1}{H}\right). \tag{4.14}$$

Case 2. Suppose that z_0 is a simple pole of F . We put

$$f^{(j)}(z) = \frac{\alpha_{-1}}{z - z_0} + \alpha_0 + \alpha_1(z - z_0) + \alpha_2(z - z_0)^2 + \dots$$

and

$$F(z) = \frac{\beta_{-1}}{z - z_0} + \beta_0 + \beta_1(z - z_0) + \beta_2(z - z_0)^2 + \dots,$$

where α_j ($j = -1, 0, 1, 2, \dots$) and β_j ($j = -1, 0, 1, 2, \dots$) are the coefficients of the Laurent series expansion of $f^{(j)}(z)$ and $F(z)$ respectively. Now from (4.5), by a simple calculation we can deduce that $\psi(z_0) = 0$. If $\psi(z) \equiv 0$, then we have $F - 1 = t(f^{(j)}(z) - 1)$ with $t \neq 0$, a constant. Similarly, we can assert that $t = 1$, then $F \equiv f^{(j)}(z)$. This gives a contradiction. Thus $\psi(z) \not\equiv 0$, and hence

$$N_{1)}(r, f^{(j)}(z)) \leq N\left(r, \frac{1}{\psi}\right). \tag{4.15}$$

From (4.8), (4.13), (4.14) and (4.15), we obtain

$$\begin{aligned} N_1(r, f^{(j)}(z)) &\leq N\left(r, \frac{1}{\psi}\right) + N\left(r, \frac{1}{H}\right) \\ &\leq T(r, \psi) + T(r, H) \\ &\leq N_0\left(r, \frac{1}{f^{(j+1)}(z)}\right) + S(r, f). \end{aligned} \tag{4.16}$$

Since $f^{(j)}(z)$ and $F(z)$ share 1 IM, it follows from (4.1) and (4.4) that

$$\bar{N}\left(r, \frac{1}{f^{(j)}(z) - 1}\right) \leq N\left(r, \frac{1}{\phi - 1}\right) \leq T(r, \phi) = S(r, f). \tag{4.17}$$

From the given hypothesis $N\left(r, \frac{1}{f^{(j)}(z)}\right) = S(r, f)$, (4.16), (4.17) and $S(r, f^{(j)}) = S(r, f)$, we get

$$\begin{aligned} T(r, f^{(j)}(z)) &\leq N\left(r, \frac{1}{f^{(j)}(z)}\right) + \bar{N}(r, f^{(j)}(z)) + \bar{N}\left(r, \frac{1}{f^{(j)}(z) - 1}\right) \\ &\quad - N_0\left(r, \frac{1}{f^{(j+1)}(z)}\right) + S(r, f^{(j)}(z)) \\ &= S(r, f^{(j)}(z)), \end{aligned}$$

which is not possible. Therefore, $f^{(j)}(z) \equiv F(z)$. This proves Theorem 2.1. □

Proof of Theorem 2.2. Set

$$\frac{F(z)}{f^{(j)}(z)} = \eta(z). \tag{4.18}$$

From the given assumption $E(0, f^{(j)}(z)) \subset E(0, F(z))$ and $E(\infty, F(z)) \subset E(\infty, f^{(j)}(z))$, we can conclude that $\eta(z)$ is an entire function. Hence

$$N(r, \eta(z)) = 0. \tag{4.19}$$

Now from Lemma 3.4, we have

$$\begin{aligned} T(r, f^{(j)}(z)) &\leq T(r, f(z)) + j\bar{N}(r, f(z)) + S(r, f(z)) \\ &\leq (j + 1)T(r, f(z)) + S(r, f(z)). \end{aligned} \tag{4.20}$$

Again by Lemmas 3.4 and 3.5, we get

$$\begin{aligned} T(r, f^{(k_1)}(z + c_1)) &= T(r, f^{(k_1)}(z)) + S(r, f) \\ &\leq T(r, f(z)) + k_1\bar{N}(r, f(z)) + S(r, f(z)) \\ &\leq (k_1 + 1)T(r, f(z)) + S(r, f(z)). \end{aligned} \tag{4.21}$$

Combining (4.20) and (4.21), we obtain $S(r, f^{(j)}(z)) = S(r, f^{(k_1)}(z + c_1)) = S(r, f)$. Similarly, we can prove that $S(r, f^{(j)}(z)) = S(r, f^{(k_2)}(z + c_2)) = S(r, f), \dots, S(r, f^{(j)}(z)) = S(r, f^{(k_n)}(z + c_n)) = S(r, f)$. Now we consider the following two cases.

Case 1. Consider $\eta(z) \equiv 1$. Then $f^{(j)}(z) \equiv F(z)$.

Case 2. Let $\eta(z) \neq 1$. From Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
 m(r, \eta(z)) &= m\left(r, \frac{F(z)}{f^{(j)}(z)}\right) \\
 &\leq m\left(r, \frac{f^{(k_1)}(z + c_1)}{f^{(j)}(z)}\right) + \dots + m\left(r, \frac{f^{(k_n)}(z + c_n)}{f^{(j)}(z)}\right) \\
 &\leq m\left(r, \frac{f^{(k_1)}(z + c_1)}{f^{(k_1)}(z)}\right) + m\left(r, \frac{f^{(k_1)}(z)}{f^{(j)}(z)}\right) + \dots \\
 &+ m\left(r, \frac{f^{(k_n)}(z + c_n)}{f^{(k_n)}(z)}\right) + m\left(r, \frac{f^{(k_n)}(z)}{f^{(j)}(z)}\right) \\
 &= S(r, f).
 \end{aligned}
 \tag{4.22}$$

Combining (4.19) and (4.22), we obtain

$$T(r, \eta(z)) = S(r, f). \tag{4.23}$$

Without loss of generality, we may assume that $f^{(j)}(z)$ and $F(z)$ share $a = 1$ CM. Then using (4.23), we have

$$\bar{N}\left(r, \frac{1}{f^{(j)}(z) - 1}\right) \leq N\left(r, \frac{1}{\eta(z) - 1}\right) \leq T(r, \eta(z)) = S(r, f). \tag{4.24}$$

Now let z_0 be a 1-point of $f^{(j)}(z)$ with multiplicity $p \geq 2$. Then by the given assumption, z_0 is also a 1-point of $F(z)$ with multiplicity $p \geq 2$. Again differentiating (4.18), we get

$$\eta'(z) = \frac{f^{(j)}(z)F'(z) - F(z)f^{(j+1)}(z)}{[f^{(j)}(z)]^2}.$$

From (4.18) and the above equality, we get

$$\begin{aligned}
 \frac{\eta'(z)}{\eta(z)} &= \frac{f^{(j)}(z)F'(z) - F(z)f^{(j+1)}(z)}{[f^{(j)}(z)]^2} \frac{f^{(j)}(z)}{F(z)} \\
 &= \frac{f^{(j)}(z)F'(z) - F(z)f^{(j+1)}(z)}{f^{(j)}(z)F(z)} \\
 &= \frac{F'(z)}{F(z)} - \frac{f^{(j+1)}(z)}{f^{(j)}(z)}.
 \end{aligned}
 \tag{4.25}$$

Therefore by Laurent series expansion of $F(z)$ and $f^{(j)}(z)$, and from (4.25), we can prove that z_0 is also a zero of $\frac{\eta'(z)}{\eta(z)}$ with multiplicity at least $p - 1$. Hence by Lemma 3.4 and (4.23), we get

$$\begin{aligned}
 N_{(2)}\left(r, \frac{1}{f^{(j)}(z) - 1}\right) &\leq 2N\left(r, \frac{\eta(z)}{\eta'(z)}\right) \\
 &\leq 2N(r, \eta) + 2N\left(r, \frac{1}{\eta'}\right) \\
 &\leq 2N(r, \eta) + 2\left[N\left(r, \frac{1}{\eta}\right) + \bar{N}(r, \eta) + S(r, f)\right] \\
 &\leq 6T(r, \eta) + S(r, f) \\
 &= S(r, f).
 \end{aligned}
 \tag{4.26}$$

Combining (4.24) and (4.26), we obtain

$$\begin{aligned}
 N\left(r, \frac{1}{f^{(j)}(z) - 1}\right) &= \bar{N}\left(r, \frac{1}{f^{(j)}(z) - 1}\right) + N_{(2)}\left(r, \frac{1}{f^{(j)}(z) - 1}\right) \\
 &= S(r, f).
 \end{aligned}
 \tag{4.27}$$

Again by the given hypothesis, we can deduce that

$$N\left(r, \frac{1}{f^{(j)}(z)}\right) \leq N\left(r, \frac{1}{F}\right), \quad N(r, F) \leq N(r, f^{(j)}(z)). \quad (4.28)$$

From Lemma 3.6, we have

$$\begin{aligned} m(r, f^{(j)}(z)) + m\left(r, \frac{1}{f^{(j)}(z)}\right) + m\left(r, \frac{1}{f^{(j)}(z) - 1}\right) \\ \leq 2T(r, f^{(j)}(z)) - N_{pair}(r, f) + S(r, f), \end{aligned} \quad (4.29)$$

where $N_{pair}(r, f) = 2N(r, f^{(j)}(z)) - N(r, F) + N\left(r, \frac{1}{F}\right) + S(r, f)$.

Now adding $N(r, f^{(j)}(z)) + N\left(r, \frac{1}{f^{(j)}(z)}\right) + N\left(r, \frac{1}{f^{(j)}(z) - 1}\right)$ on both sides of (4.29) and using (4.27) and (4.28), we obtain

$$\begin{aligned} T(r, f^{(j)}(z)) &\leq N(r, f^{(j)}(z)) + N\left(r, \frac{1}{f^{(j)}(z)}\right) + N\left(r, \frac{1}{f^{(j)}(z) - 1}\right) - N_{pair}(r, f) + S(r, f) \\ &= N\left(r, \frac{1}{f^{(j)}(z) - 1}\right) + \left[N\left(r, \frac{1}{f^{(j)}(z)}\right) - N\left(r, \frac{1}{F}\right)\right] \\ &\quad + [N(r, F) - N(r, f^{(j)}(z))] + S(r, f) \\ &\leq N\left(r, \frac{1}{f^{(j)}(z) - 1}\right) + S(r, f) \\ &= S(r, f). \end{aligned}$$

This gives a contradiction. So, $\eta(z) \equiv 1$. Thus we have $f^{(j)}(z) \equiv F(z)$. This proves Theorem 2.2. \square

5 Open Problems

Is it possible in any way to remove the condition $N\left(\frac{1}{f^{(j)}(z)}\right) + \bar{N}\left(r, \frac{1}{f(z)}\right) = S(r, f)$ in Theorem 2.1 keeping the other conditions intact?

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Received: 2025-08-25

Accepted: 2025-11-24