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SPECIAL SOLUTIONS TO SPACE-FRACTIONAL MAGNETOHYDRODYNAMICS EQUATIONS

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Abstract In this article, we explore the space-fractional magnetohydrodynamics (MHD) equations of order $\alpha \in (0, 1]$ to perform an extensive mathematical analysis. By applying fractional vector calculus based on the Grünwald–Letnikov operator, we reduce the governing system to six coupled fractional ordinary differential equations and construct a class of special solutions. A qualitative analysis reveals the presence of a degenerate saddle critical point, whose stability properties are examined in detail. We additionally prove the uniqueness of solutions to the corresponding initial value problem. The obtained solutions recover a special case of the ideal MHD equations when $\alpha = 1$. Illustrative examples are presented to highlight the results. This work provides an analytical benchmark that enhances the understanding and validation of space-fractional MHD models.

1 Introduction

The term magnetohydrodynamics (MHD), derived from the Lorentz force, is commonly employed in a variety of viscoelastic fluid flow problems, such as MHD generators, nuclear reactors, and medical investigations of biological fluids. In geophysical and astrophysical settings, MHD principles also govern large-scale fluid motion, complementing stratified Boussinesq models commonly used to describe oceanic and atmospheric flows. The mathematical properties of these equations have been thoroughly examined, notably by Majda [1]. In addition, Jiahong Wu [2] explored a generalized version of the MHD equations given by

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla P + \boldsymbol{b} \cdot \nabla \boldsymbol{b} - \nu(-\Delta)^{\gamma} \boldsymbol{u}, \\ \partial_t \boldsymbol{b} + \boldsymbol{u} \cdot \nabla \boldsymbol{b} &= \boldsymbol{b} \cdot \nabla \boldsymbol{u} - \eta(-\Delta)^{\beta} \boldsymbol{b}, \end{aligned} \tag{1.1}$$

where $\nu \geq 0, \eta \geq 0, \gamma > 0, \beta > 0$ are real numbers and the fluid velocity $\boldsymbol{u}$, magnetic field $\boldsymbol{b}$, and pressure P.

Wu [2] provided a rigorous analysis of the regularity and existence of solutions to these equations by varying the parameters ν, η, β, γ. Setting $\beta = \gamma = 1$ recovers the classical MHD equations. Significant analytical progress has been achieved for both stratified Boussinesq and MHD systems. In particular, Duvaut and Lions [3] established the existence of global weak solutions for the MHD equations in a manner consistent with the classical Leray–Hopf framework [4] for incompressible flows. Zujin Zhang [5] discussed the generalized MHD equations (1.1) for $\nu = \eta = 1, \gamma = \beta = 1$ and proved that if one of the directional derivatives of fluid velocity exists in $L^p(0, T, L^q(\mathbb{R}^3))$ where $\frac{2}{p} + \frac{3}{q} = 1, 3 < q \leq \infty$, then the solution of the MHD equations exists and is smooth. Desale and Shendage [6] obtained special solutions with $\nu = \eta = 0$.

In recent years, fractional differential equations have gained prominence because of their ability to incorporate memory and non-local effects. Although fractional-order models offer a more accurate representation of many complex physical processes, solving the corresponding fractional partial differential equations (FPDEs) remains challenging due to the presence of nonlocal operators and the absence of closed-form analytical solutions. Recent advancements have prompted the development of various numerical methods for fractional partial differential equations (FPDEs). Inça et al. [7] and Ahmad et al. [8] introduced meshless radial basis function and polynomial-based schemes for fractional financial models. Khan et al. [9] studied nonlinear fractional evolution equations, Kareem et al. [10] used polynomial collocation for inverse elliptic problems, and Mehnaz et al. [11] examined meshless approaches for fractional dispersive wave equations. Lie symmetry techniques have also been employed to derive exact solutions of nonlinear conformable fractional partial differential equations [12].

Parallel to these computing developments, fractional generalizations of numerous classical fluid models have been developed, most notably the Boussinesq equations [13, 14], and the Navier-Stokes equations [15, 16]. Motivated by these advances, we consider here a generalized formulation of the classical MHD system, i.e., the space-fractional MHD equations.

This work focuses on deriving special solutions of the space-fractional MHD equations and investigating their associated critical points. To this end, we consider the fractional MHD equations in the following form:

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u} &= -\nabla^{\bar{\alpha}} P + \boldsymbol{b} \cdot \nabla^{\bar{\alpha}} \boldsymbol{b}, \\ \partial_t \boldsymbol{b} + \boldsymbol{u} \cdot \nabla^{\bar{\alpha}} \boldsymbol{b} &= \boldsymbol{b} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u}, \\ \nabla^{\bar{\alpha}} \cdot \boldsymbol{u} &= 0, \\ \nabla^{\bar{\alpha}} \cdot \boldsymbol{b} &= 0, \end{aligned} \tag{1.2}$$

where $\bar{\alpha} = (\alpha, \alpha, \alpha) \in \mathbb{R}^3$ with $0 < \alpha \leq 1$, the spatial variable is $\boldsymbol{x} = (x_1, x_2, x_3)$, the fluid velocity is $\boldsymbol{u} = (u_1, u_2, u_3)$, the magnetic field is $\boldsymbol{b} = (b_1, b_2, b_3)$, and the pressure is given by $P = P(x_1, x_2, x_3)$. Now we assume the initial conditions that are compatible with (1.2) as follows:

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{b}(\boldsymbol{x}, 0) = \boldsymbol{b}_0(\boldsymbol{x}). \tag{1.3}$$

The fractional derivative that has been used in the Grünwald-Letnikov sense [17]. The properties of fractional vectorial operators are considered within the framework established by Ortigueira and Machado [18].

The paper is organized as follows. In Section 2, we outline the fundamentals of fractional calculus, define the notation, and provide a convenient reference. These details, while not exhaustive, are sufficient for the aims of this paper. Section 3 extends the Majda procedure to derive special solutions to (1.2). In Section 4, we perform a qualitative analysis of the solutions. In Section 5, we present two examples, followed by the conclusion in Section 6.

2 Preliminaries and Notations

In this section, we first review the essential concepts of fractional calculus. This field concerns the theory of integrals and derivatives of arbitrary order, extending and unifying the classical ideas of integer-order differentiation. Numerous references [19, 20, 21, 22] explore fractional calculus in depth and describe multiple definitions of fractional integration and differentiation, such as those proposed by Grünwald-Letnikov, Riemann-Liouville, Caputo, etc. In this work, we employ Grünwald-Letnikov's definition of fractional differentiation, as it allows the initial conditions for fractional differential equations to retain the familiar form used in integer order cases. For further background and theoretical justification, one may refer to Ortigueira and Machado [18] and Atici et al. [23]. We shall recall the formal definition of the Grünwald-Letnikov fractional derivatives (GL-FD), the partial derivatives based on GL-FD, the fractional vectorial differential operators, and some of their properties.

2.1 The Grünwald-Letnikov fractional derivative

In this subsection, we shall state some basic definitions and properties of the Grünwald-Letnikov fractional derivative.

Definition 2.1. Let $f(x)$ be a real valued function. For $\alpha \in \mathbb{R}$, $\alpha > 0$, the Grünwald Letnikov fractional derivative(GL-FD) is defined in [24] as follows:

$$D_x^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} f(x-nh). \tag{2.1}$$

We will be using the GL-FD uniformly throughout the paper. Hereby, we list some important properties of the GL-FD below:

(i) It is a linear operator.

(ii) It is additive and commutative, i.e.

$$D^\alpha[D^\beta f(t)] = D^\beta[D^\alpha f(t)] = D^{\alpha+\beta} f(t). \tag{2.2}$$

(iii) It has a neutral element. For $\beta = -\alpha$ we get

$$D^\alpha[D^{-\alpha} f(t)] = D^{-\alpha}[D^\alpha f(t)] = f(t). \tag{2.3}$$

(iv) Using the definition stated above and the Riemann integral definition as in [17], we obtain the backward compatibility as follows

$$D^{-\alpha} f(t) = \int_0^\infty f(t-\tau)\tau^{\alpha-1} d\tau.$$

For the above properties, one may refer to [17, 24].

(v) The fractional Leibniz rule for GL-FD, defined in [17, 25], is as follows. Let $f(x)$, and $g(x)$ be real valued functions. If $f(x)$ is continuous on $[a,t]$ and $g(x)$ has $n+1$ continuous derivatives in $[a,t]$ then the fractional derivative of the product $g(t)f(t)$ is given by

$$D^\alpha(g(t)f(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k} f(t). \tag{2.4}$$

2.2 Fractional vectorial operators

In this subsection, we define the fractional-order vectorial operators, specifically the fractional gradient, curl, and divergence, as formulated in [18]. We will assume that we are working on $\mathbb{R}^3$ and that the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ constitutes its standard orthonormal basis. We define the vector

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3.$$

A vectorial function is represented by

$$\boldsymbol{f}(x_1,x_2,x_3) = f_1(x_1,x_2,x_3)\boldsymbol{e}_1 + f_2(x_1,x_2,x_3)\boldsymbol{e}_2 + f_3(x_1,x_2,x_3)\boldsymbol{e}_3.$$

Definition 2.2. For $\alpha \in \mathbb{R}$, $\alpha > 0$, the Grünwald-Letnikov fractional partial derivative is defined as

$$D_{x_1}^\alpha \boldsymbol{f}(x_1,x_2,x_3) = \lim_{h\to 0} h^{-\alpha} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \boldsymbol{f}(x_1-nh,x_2,x_3), \tag{2.5}$$

where $x = (x_1,x_2,x_3) \in \mathbb{R}^3$

Definition 2.3. For any $\bar{\alpha} = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3$, the fractional gradient operator is defined in [18] by

$$\nabla^{\bar{\alpha}}(.) := D_{x_1}^{\alpha_1}(.)e_1 + D_{x_2}^{\alpha_2}(.)e_2 + D_{x_3}^{\alpha_3}(.)e_3. \tag{2.6}$$

Definition 2.4. We define fractional divergence and curl operator, for any $\bar{\alpha} = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3$, in [18] as follows:

$$div^{\bar{\alpha}}(\boldsymbol{f}) = \nabla^{\bar{\alpha}} \cdot \boldsymbol{f} := D_{x_1}^{\alpha_1} f_1 + D_{x_2}^{\alpha_2} f_2 + D_{x_3}^{\alpha_3} f_3, \tag{2.7}$$

$$curl^{\bar{\alpha}}(\boldsymbol{f}) = \nabla^{\bar{\alpha}} \times \boldsymbol{f} := \begin{vmatrix} D_{x_1}^{\alpha_1} & D_{x_2}^{\alpha_2} & D_{x_3}^{\alpha_3} \\ f_1 & f_2 & f_3 \\ e_1 & e_2 & e_3 \end{vmatrix} \tag{2.8}$$

Throughout this study, we adopt a uniform fractional order by setting $\bar{\alpha} = (\alpha, \alpha, \alpha)$, such that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. where $\alpha \in \mathbb{R}$. For $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\bar{\alpha} = (\alpha, \alpha, \alpha)$ we get

$$\nabla^{\bar{\alpha}} \boldsymbol{u} = \begin{bmatrix} D^{\alpha}_{x_1} u_1 & D^{\alpha}_{x_2} u_1 & D^{\alpha}_{x_3} u_1 \\ D^{\alpha}_{x_1} u_2 & D^{\alpha}_{x_2} u_2 & D^{\alpha}_{x_3} u_2 \\ D^{\alpha}_{x_1} u_3 & D^{\alpha}_{x_2} u_3 & D^{\alpha}_{x_3} u_3 \end{bmatrix}$$

2.3 Fractional Multivariate Taylor Series Expansion

For a multivariate function $\boldsymbol{u}(\boldsymbol{x}, t) = (u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t), u_3(\boldsymbol{x}, t))$ the Taylor series expansion is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}_0, t) + \nabla \boldsymbol{u}|_{(\boldsymbol{x}_0, t)} (\boldsymbol{x} - \boldsymbol{x}_0) + O(|\boldsymbol{x} - \boldsymbol{x}_0|^2). \tag{2.9}$$

where $\boldsymbol{x}_0 = (a, b, c)$ and the term $\nabla \boldsymbol{u}|_{(\boldsymbol{x}_0, t)} (\boldsymbol{x} - \boldsymbol{x}_0)$ can be rewritten as

$$\nabla \boldsymbol{u} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) = \nabla \boldsymbol{u} \begin{pmatrix} x_1 - a \\ x_2 - b \\ x_3 - c \end{pmatrix} \tag{2.10}$$

As in literature, [26, 27, 28, 29], the fractional form of the Taylor series expansion in equation (2.9) about the point $\boldsymbol{x_0}$ is given below:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}_0, t) + \frac{\nabla^{\bar{\alpha}} \boldsymbol{u}|_{(\boldsymbol{x}_0, t)} (\boldsymbol{x} - \boldsymbol{x}_0)_{\alpha}}{\Gamma(\alpha + 1)} + O(|\boldsymbol{x} - \boldsymbol{x}_0|_{2\alpha}), \tag{2.11}$$

where we denote

$$(\boldsymbol{x} - \boldsymbol{x_0})_{\alpha} = \begin{pmatrix} (x_1 - a)^{\alpha} \\ (x_2 - b)^{\alpha} \\ (x_3 - c)^{\alpha} \end{pmatrix} \tag{2.12}$$

3 Special Solutions of the Fractional MHD Equations

We aim to obtain special solutions of the fractional MHD equations with respect to space variables (1.2), following an approach analogous to that employed in [1]. Consider the fractional Taylor series expansion of smooth velocity and magnetic field about some point $\boldsymbol{x}_0$ as stated in (2.11).

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \boldsymbol{u}(\boldsymbol{x}_0, t) + \frac{\nabla^{\bar{\alpha}} \boldsymbol{u}|_{(\boldsymbol{x}_0, t)} (\boldsymbol{x} - \boldsymbol{x}_0)_{\alpha}}{\Gamma(\alpha + 1)} + O(|\boldsymbol{x} - \boldsymbol{x}_0|_{2\alpha}). \\ \boldsymbol{b}(\boldsymbol{x}, t) &= \boldsymbol{b}(\boldsymbol{x}_0, t) + \frac{\nabla^{\bar{\alpha}} \boldsymbol{b}|_{(\boldsymbol{x}_0, t)} (\boldsymbol{x} - \boldsymbol{x}_0)_{\alpha}}{\Gamma(\alpha + 1)} + O(|\boldsymbol{x} - \boldsymbol{x}_0|_{2\alpha}). \end{aligned} \tag{3.1}$$

We are developing the solutions using the Taylor series approximation (3.1) represented as

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \left(\sum_{j=1}^{3} m_{1j}(t)\, x_j^{\alpha}, \sum_{j=1}^{3} m_{2j}(t)\, x_j^{\alpha}, \sum_{j=1}^{3} m_{3j}(t)\, x_j^{\alpha} \right) \\ \boldsymbol{b}(\boldsymbol{x}, t) &= \left(\sum_{j=1}^{3} n_{1j}(t)\, x_j^{\alpha}, \sum_{j=1}^{3} n_{2j}(t)\, x_j^{\alpha}, \sum_{j=1}^{3} n_{3j}(t)\, x_j^{\alpha} \right) \end{aligned} \tag{3.2}$$

where $m_{ij}(t)$ and $n_{ij}(t)$ $(i, j = 1, 2, 3)$ are functions continuous in time $t \in \mathbb{R}$. Contributions arising from higher-order powers of x_1, x_2, x_3 are regarded as negligible and thus omitted.

Both $\nabla^{\bar{\alpha}}\boldsymbol{u}$ and $\nabla^{\bar{\alpha}}\boldsymbol{b}$ can be uniquely expressed as the sum of a symmetric part and a skew-symmetric part, namely

$$\begin{aligned} \nabla^{\bar{\alpha}}\boldsymbol{u} &= \left(\frac{\nabla^{\bar{\alpha}}\boldsymbol{u} + (\nabla^{\bar{\alpha}}\boldsymbol{u})^T}{2}\right) + \left(\frac{\nabla^{\bar{\alpha}}\boldsymbol{u} - (\nabla^{\bar{\alpha}}\boldsymbol{u})^T}{2}\right), \\ &= \mathcal{D}_\alpha(\boldsymbol{x}_0, t) + \boldsymbol{\Omega}_\alpha(\boldsymbol{x}_0, t). \\ \nabla^{\bar{\alpha}}\boldsymbol{b} &= \left(\frac{\nabla^{\bar{\alpha}}\boldsymbol{b} + (\nabla^{\bar{\alpha}}\boldsymbol{b})^T}{2}\right) + \left(\frac{\nabla^{\bar{\alpha}}\boldsymbol{b} - (\nabla^{\bar{\alpha}}\boldsymbol{b})^T}{2}\right), \\ &= \mathcal{K}_\alpha(\boldsymbol{x}_0, t) + \mathcal{L}_\alpha(\boldsymbol{x}_0, t). \end{aligned} \tag{3.3}$$

where $\mathcal{D}_\alpha(\boldsymbol{x}_0, t)$, $\mathcal{K}_\alpha(\boldsymbol{x}_0, t)$ are the symmetric parts and $\boldsymbol{\Omega}_\alpha(\boldsymbol{x}_0, t)$, $\mathcal{L}_\alpha(\boldsymbol{x}_0, t)$ are the skew-symmetric parts of $\nabla^{\bar{\alpha}}\boldsymbol{u}$ and $\nabla^{\bar{\alpha}}\boldsymbol{b}$ respectively. Further, we see that the skew-symmetric parts of equation (3.3) have the following property
For any $\boldsymbol{h} = (h_1, h_2, h_3) \in \mathbb{R}^3$ we have

$$\boldsymbol{\Omega}_\alpha \boldsymbol{h} = \frac{1}{2}\boldsymbol{\omega_\alpha} \times \boldsymbol{h}, \qquad \mathcal{L}_\alpha \boldsymbol{h} = \frac{1}{2}\boldsymbol{j_\alpha} \times \boldsymbol{h}. \tag{3.4}$$

where the fractional current density vector $\boldsymbol{j_\alpha} = \nabla^{\bar{\alpha}} \times \boldsymbol{b} = (j_1, j_2, j_3)^t$ and fractional vorticity vector $\boldsymbol{\omega_\alpha} = \nabla^{\bar{\alpha}} \times \boldsymbol{u} = (w_1, w_2, w_3)^t$
Let us assume that the matrix

$$\mathcal{D}_\alpha = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}, \qquad \mathcal{K}_\alpha = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{3.5}$$

By the definition of $\mathcal{D}_\alpha$, we have $d_{11} = 2D^\alpha_{x_1}u_1$, $d_{22} = 2D^\alpha_{x_2}u_2$ and $d_{33} = 2D^\alpha_{x_3}u_3$. Since $\nabla^{\bar{\alpha}} \cdot \boldsymbol{u} = 0$ and $\nabla^{\bar{\alpha}} \cdot \boldsymbol{b} = 0$, we obtain $d_{11} + d_{22} + d_{33} = 0$ and $k_{11} + k_{22} + k_{33} = 0$.

The matrices $\boldsymbol{\Omega}_\alpha$ and $\mathcal{L}_\alpha$ can be expressed in terms of $\boldsymbol{\omega_\alpha}$ and $\boldsymbol{j_\alpha}$ as follows.

$$\boldsymbol{\Omega}_\alpha = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}, \qquad \mathcal{L}_\alpha = \begin{bmatrix} 0 & -j_3 & j_2 \\ j_3 & 0 & -j_1 \\ -j_2 & j_1 & 0 \end{bmatrix}. \tag{3.6}$$

Now we can use fractional vorticity vector $\boldsymbol{\omega_\alpha}$ and fractional current density vector $\boldsymbol{j_\alpha}$ in (3.1) we get

$$\begin{aligned} \nabla^{\bar{\alpha}}\boldsymbol{u}|_{(\boldsymbol{x}_0,t)}\boldsymbol{h} &= \mathcal{D}_\alpha(\boldsymbol{x}_0, t)\boldsymbol{h} + \frac{1}{2}\boldsymbol{\omega_\alpha}(\boldsymbol{x}_0, t) \times \boldsymbol{h}, \\ \nabla^{\bar{\alpha}}\boldsymbol{b}|_{(x_0,t)}\boldsymbol{h} &= \mathcal{K}_\alpha(\boldsymbol{x}_0, t)\boldsymbol{h} + \frac{1}{2}\boldsymbol{j_\alpha}(\boldsymbol{x}_0, t) \times \boldsymbol{h}. \end{aligned} \tag{3.7}$$

Without loss of generality, we assume that $\boldsymbol{u}(\boldsymbol{x}_0, t) = \boldsymbol{b}(\boldsymbol{x}_0, t) = 0$. Now, we take the fractional gradient of the equation of motion of magnetic flow of fluid (1.2), and we get

$$\begin{aligned} \nabla^{\bar{\alpha}}(\partial_t\boldsymbol{u} + \boldsymbol{u} \cdot \nabla^{\bar{\alpha}}\boldsymbol{u}) &= \nabla^{\bar{\alpha}}(-\nabla^{\bar{\alpha}}P + \boldsymbol{b} \cdot \nabla^{\bar{\alpha}}\boldsymbol{b}), \\ \nabla^{\bar{\alpha}}(\partial_t\boldsymbol{b} + \boldsymbol{u} \cdot \nabla^{\bar{\alpha}}\boldsymbol{b}) &= \nabla^{\bar{\alpha}}(\boldsymbol{b} \cdot \nabla^{\bar{\alpha}}\boldsymbol{u}). \end{aligned} \tag{3.8}$$

We denote $\mathbf{U}_\alpha = \nabla^{\bar{\alpha}}\boldsymbol{u}$ and $\mathbf{B}_\alpha = \nabla^{\bar{\alpha}}\boldsymbol{b}$. Using the commutativity property of GL-FD, we get

$$\nabla^{\bar{\alpha}}(\partial_t\boldsymbol{u}) = \partial_t(\nabla^{\bar{\alpha}}\boldsymbol{u}) = \partial_t\mathbf{U}_\alpha, \qquad \nabla^{\bar{\alpha}}(\partial_t\boldsymbol{b}) = \partial_t(\nabla^{\bar{\alpha}}\boldsymbol{b}) = \partial_t\mathbf{B}_\alpha$$

Consider the term $\nabla^{\bar{\alpha}}(\boldsymbol{u} \cdot \nabla^{\bar{\alpha}}\boldsymbol{u})$

$$\begin{aligned} \nabla^{\bar{\alpha}}(\boldsymbol{u} \cdot \nabla^{\bar{\alpha}}\boldsymbol{u}) &= \nabla^{\bar{\alpha}} \begin{bmatrix} u_1 D^\alpha_{x_1}u_1 + u_2 D^\alpha_{x_2}u_1 + u_3 D^\alpha_{x_3}u_1 \\ u_1 D^\alpha_{x_1}u_2 + u_2 D^\alpha_{x_2}u_2 + u_3 D^\alpha_{x_3}u_2 \\ u_1 D^\alpha_{x_1}u_3 + u_2 D^\alpha_{x_2}u_3 + u_3 D^\alpha_{x_3}u_3 \end{bmatrix}^t \\ &= \begin{bmatrix} a_{ij} \end{bmatrix}_{3\times 3}, \end{aligned} \tag{3.9}$$

where using fractional Leibnitz rule (2.4), we obtain

$$\begin{aligned}
a_{ij} &= D^{\alpha}_{x_j} \sum_{k=1}^{3} (u_k D^{\alpha}_{x_k} u_i), \\
&= \sum_{k=1}^{3} D^{\alpha}_{x_j} (u_k D^{\alpha}_{x_k} u_i), \\
&= \sum_{k=1}^{3} \sum_{r=0}^{\infty} \binom{\alpha}{r} D^{r}_{x_j} (D^{\alpha}_{x_k} u_i) D^{\alpha-r}_{x_j} u_k.
\end{aligned} \tag{3.10}$$

Rearranging the terms, we obtain

$$\nabla^{\bar{\alpha}}(\boldsymbol{u} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u}) = \sum_{k=0}^{\infty} \binom{\alpha}{k} A_k, \tag{3.11}$$

where $A_k = [a^{(k)}_{ij}]_{3\times 3}$ and

$$\begin{aligned}
a^{(0)}_{ij} &= \sum_{l=1}^{3} D^{\alpha}_{l} u_i D^{\alpha}_{j} u_l, \\
a^{(k)}_{ij} &= \sum_{l=1}^{3} D^{k}_{x_j} D^{\alpha}_{l} u_i D^{\alpha-k}_{j} u_l.
\end{aligned} \tag{3.12}$$

Hence, we observe that $A_0 = \mathbf{U}^2_{\alpha}$.
As $\boldsymbol{u}$ and $\boldsymbol{b}$ are of the form (3.2) and

$$D^{\alpha}_{x} x^{\gamma} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)} x^{\alpha-\gamma},$$

it follows that A_k vanish for $k \geq 1$.

Hence,

$$\nabla^{\bar{\alpha}}(\boldsymbol{u} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u}) = \binom{\alpha}{0} \mathbf{U}^2_{\alpha}. \tag{3.13}$$

Consequently, the remaining fractional derivative terms can be expressed as

$$\nabla^{\bar{\alpha}}(\boldsymbol{u} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u}) = \mathbf{U}^2_{\alpha}, \quad \nabla^{\bar{\alpha}}(\boldsymbol{u} \cdot \nabla^{\bar{\alpha}} \boldsymbol{b}) = \mathbf{B}_{\alpha}\mathbf{U}_{\alpha}, \quad \nabla^{\bar{\alpha}}(\boldsymbol{b} \cdot \nabla^{\bar{\alpha}} \boldsymbol{U}) = \mathbf{U}_{\alpha}\mathbf{B}_{\alpha}, \quad \nabla^{\bar{\alpha}}(\boldsymbol{b} \cdot \nabla^{\bar{\alpha}} \boldsymbol{b}) = \mathbf{B}^2_{\alpha}, \tag{3.14}$$

On substituting (3.14), we obtain

$$\begin{aligned}
\partial_t \mathbf{U}_{\alpha} + \mathbf{U}^2_{\alpha} &= -\hat{P} + \mathbf{B}^2_{\alpha}, \\
\partial_t \mathbf{B}_{\alpha} + \mathbf{B}_{\alpha}\mathbf{U}_{\alpha} &= \mathbf{U}_{\alpha}\mathbf{B}_{\alpha},
\end{aligned} \tag{3.15}$$

where $\hat{P} = \nabla^{\bar{\alpha}}(\nabla^{\bar{\alpha}} P)$ is the fractional hessian matrix for P. Here we notice the following observations, which shall be used to prove the following Proposition 3.3.

Matrix	Symmmetric part of the matrix	Skew symmetric part of the matrix
$\partial_t \mathbf{U}_{\alpha}$	$\partial_t \mathcal{D}_{\alpha}$	$\partial_t \boldsymbol{\Omega}_{\alpha}$
$\mathbf{U}_{\alpha}.\mathbf{U}_{\alpha}$	$\mathcal{D}^2_{\alpha} + \boldsymbol{\Omega}^2_{\alpha}$	$\mathcal{D}_{\alpha}\boldsymbol{\Omega}_{\alpha} + \boldsymbol{\Omega}_{\alpha}\mathcal{D}_{\alpha}$
$\mathbf{B}_{\alpha}.\mathbf{B}_{\alpha}$	$\mathcal{K}^2_{\alpha} + \mathcal{L}^2_{\alpha}$	$\mathcal{K}_{\alpha}\mathcal{L}_{\alpha} + \mathcal{L}_{\alpha}\mathcal{K}_{\alpha}$
$\mathbf{U}_{\alpha}.\mathbf{B}_{\alpha} - \mathbf{B}_{\alpha}\mathbf{U}_{\alpha}$	$\mathcal{D}_{\alpha}\mathcal{L}_{\alpha} - \mathcal{L}_{\alpha}\mathcal{D}_{\alpha} + \boldsymbol{\Omega}_{\alpha}\mathcal{K}_{\alpha} - \mathcal{K}_{\alpha}\boldsymbol{\Omega}_{\alpha}$	$\mathcal{D}_{\alpha}\mathcal{K}_{\alpha} - \mathcal{K}_{\alpha}\mathcal{D}_{\alpha} + \boldsymbol{\Omega}_{\alpha}\mathcal{L}_{\alpha} - \mathcal{L}_{\alpha}\boldsymbol{\Omega}_{\alpha}$
$\partial_t \mathbf{B}_{\alpha}$	$\partial_t \mathcal{K}_{\alpha}$	$\partial_t \mathcal{L}_{\alpha}$

Lemma 3.1. *The fractional vorticity vector* $\boldsymbol{\omega_\alpha}$ *satisfies the relation* $\boldsymbol{\omega_\alpha} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u} = \boldsymbol{\omega}_\alpha \cdot (\nabla^{\bar{\alpha}} \boldsymbol{u})^t$ *where* $0 < \alpha \leq 1$.

Proof. For any $\boldsymbol{h} \in \mathbb{R}^3$ we have an identification in (3.4) as

$$\begin{aligned} \frac{1}{2}\boldsymbol{\omega_\alpha} \cdot (\boldsymbol{\omega_\alpha} \times \boldsymbol{h}) &= \frac{1}{2}\boldsymbol{\omega_\alpha} \cdot (\nabla^{\bar{\alpha}} \boldsymbol{u} - (\nabla^{\bar{\alpha}} \boldsymbol{u})^t)\boldsymbol{h}, \\ &= 0. \end{aligned} \tag{3.16}$$

since $\boldsymbol{h}$ is an arbitrary vector, the result follows. □

Lemma 3.2. *A fractional current density vector* $\boldsymbol{j}_\alpha$ *satisfies the relation* $\boldsymbol{j_\alpha} \cdot \nabla^{\bar{\alpha}} \boldsymbol{b} = \boldsymbol{j}_\alpha \cdot (\nabla^{\bar{\alpha}} \boldsymbol{b})^t$ *where* $0 < \alpha \leq 1$.

Proof. The proof of this lemma follows from (3.4) □

Proposition 3.3. *The evolution of the vorticity vector* $\boldsymbol{\omega_\alpha}$ *and the current density* $\boldsymbol{j_\alpha}$ *are determined by the following system of equations:*

$$\frac{\partial \boldsymbol{\omega_\alpha}}{\partial t} - \boldsymbol{\omega_\alpha} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u} = -\boldsymbol{j}_\alpha \cdot \nabla^{\bar{\alpha}} \boldsymbol{b}, \quad \frac{\partial \boldsymbol{j}_\alpha}{\partial t} = 2T + \frac{1}{2}\boldsymbol{\omega_\alpha} \times \boldsymbol{j_\alpha}, \tag{3.17}$$

where the vector $T = (T_{23}, T_{13}, T_{12})^t \in \mathbb{R}^3$, *has the property that for any* $\boldsymbol{h} \in \mathbb{R}^3$,

$$T \times \boldsymbol{h} = (\mathcal{D}_\alpha \mathcal{K}_\alpha - \mathcal{K}_\alpha \mathcal{D}_\alpha)\boldsymbol{h}. \tag{3.18}$$

Proof. Equating the skew-symmetric components of equation (3.15) we obtain,

$$\begin{aligned} \frac{\partial \Omega_\alpha}{\partial t} + \mathcal{D}_\alpha \Omega_\alpha + \Omega_\alpha \mathcal{D}_\alpha &= \mathcal{K}_\alpha \mathcal{L}_\alpha + \mathcal{L}_\alpha \mathcal{K}_\alpha, \\ \frac{\partial \mathcal{L}_\alpha}{\partial t} &= (\mathcal{D}_\alpha \mathcal{K}_\alpha - \mathcal{K}_\alpha \mathcal{D}_\alpha) + (\Omega_\alpha \mathcal{L}_\alpha - \mathcal{L}_\alpha \Omega_\alpha). \end{aligned} \tag{3.19}$$

Hence, the action of any $\boldsymbol{h} \in \mathbb{R}^3$, by post multiplication to (3.19) We have,

$$\begin{aligned} \frac{1}{2}\frac{\partial \boldsymbol{\omega}_\alpha}{\partial t} \times \boldsymbol{h} + \big(\mathcal{D}_\alpha \Omega_\alpha + \Omega_\alpha \mathcal{D}_\alpha\big)\boldsymbol{h} &= \big(\mathcal{K}_\alpha \mathcal{L}_\alpha + \mathcal{L}_\alpha \mathcal{K}_\alpha\big)\boldsymbol{h}, \\ \frac{\partial \boldsymbol{j}_\alpha}{\partial t} \times \boldsymbol{h} &= (\mathcal{D}_\alpha \mathcal{K}_\alpha - \mathcal{K}_\alpha \mathcal{D}_\alpha)\boldsymbol{h} + (\Omega_\alpha \mathcal{L}_\alpha - \mathcal{L}_\alpha \Omega_\alpha)\boldsymbol{h}. \end{aligned} \tag{3.20}$$

We shall use the fact that the trace of the matrix $\mathcal{D}_\alpha$ and Ω_α is zero to compute the matrix $\mathcal{D}_\alpha \Omega_\alpha + \Omega_\alpha \mathcal{D}_\alpha$ as follows

$$\begin{aligned} &\mathcal{D}_\alpha \Omega_\alpha + \Omega_\alpha \mathcal{D}_\alpha = \\ &\frac{1}{2}\begin{bmatrix} 0 & d_{13}w_1 + d_{23}w_2 + d_{33}w_3 & -d_{12}w_1 - d_{22}w_2 - d_{23}w_3 \\ -d_{13}w_1 - d_{23}w_2 - d_{33}w_3 & 0 & d_{11}w_1 + d_{12}w_2 + d_{13}w_3 \\ d_{12}w_1 + d_{22}w_2 + d_{23}w_3 & -d_{11}w_1 - d_{12}w_2 - d_{13}w_3 & 0 \end{bmatrix} \end{aligned} \tag{3.21}$$

Now we define a vector $\boldsymbol{c}$ as

$$\boldsymbol{c} = \begin{bmatrix} c_{23} \\ c_{13} \\ c_{12} \end{bmatrix} = \begin{bmatrix} -d_{11}w_1 - d_{12}w_2 - d_{13}w_3 \\ -d_{12}w_1 - d_{22}w_2 - d_{23}w_3 \\ -d_{13}w_1 - d_{23}w_2 - d_{33}w_3 \end{bmatrix}, \tag{3.22}$$

$$\mathcal{D}_\alpha \Omega_\alpha + \Omega_\alpha \mathcal{D}_\alpha = \frac{1}{2}\begin{bmatrix} 0 & -c_{12} & c_{13} \\ c_{12} & 0 & -c_{23} \\ -c_{13} & c_{23} & 0 \end{bmatrix}. \tag{3.23}$$

Hence for any $\boldsymbol{h} \in \mathbb{R}^3$, we have

$$(\mathcal{D}_\alpha \boldsymbol{\Omega}_\alpha + \boldsymbol{\Omega}_\alpha \mathcal{D}_\alpha)\boldsymbol{h} = \frac{1}{2}\boldsymbol{c} \times \boldsymbol{h}. \tag{3.24}$$

Further, we notice that.

$$\begin{aligned} \boldsymbol{c} &= \begin{bmatrix} -d_{11}w_1 - d_{12}w_2 - d_{13}w_3 \\ -d_{12}w_1 - d_{22}w_2 - d_{23}w_3 \\ -d_{13}w_1 - d_{23}w_2 - d_{33}w_3 \end{bmatrix} = - \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ &= \mathcal{D}_\alpha \boldsymbol{\omega}_\alpha. \end{aligned} \tag{3.25}$$

Since $\nabla^{\bar{\alpha}}\boldsymbol{u} = \mathcal{D}_\alpha + \Omega_\alpha$ and $\Omega_\alpha \boldsymbol{\omega}_\alpha = \mathbf{0}$, we have

$$(\nabla^{\bar{\alpha}}\boldsymbol{u})\boldsymbol{\omega}_\alpha = \mathcal{D}_\alpha \boldsymbol{\omega}_\alpha.$$

Hence,

$$\boldsymbol{c} = -\mathcal{D}_\alpha \boldsymbol{\omega}_\alpha = -(\nabla^{\bar{\alpha}}\boldsymbol{u})\,\boldsymbol{\omega}_\alpha = -\,\boldsymbol{\omega}_\alpha \cdot \nabla^{\bar{\alpha}}\boldsymbol{u}.$$

Thus we have,

$$(\mathcal{D}_\alpha \boldsymbol{\Omega}_\alpha + \boldsymbol{\Omega}_\alpha \mathcal{D}_\alpha)\boldsymbol{h} = -\frac{1}{2}(\boldsymbol{\omega}_\alpha \cdot \nabla^{\bar{\alpha}}\boldsymbol{u}) \times \boldsymbol{h}. \tag{3.26}$$

Similiarly we can compute the matrices $(\boldsymbol{\Omega}_\alpha \mathcal{L}_\alpha - \mathcal{L}_\alpha \boldsymbol{\Omega}_\alpha)$ and $(\mathcal{K}_\alpha \mathcal{L}_\alpha + \mathcal{L}_\alpha \mathcal{K}_\alpha)$. And action of any $\boldsymbol{h} \in \mathbb{R}^3$ by post multiplication on these matrices is as follows:

$$\begin{aligned} (\mathcal{K}_\alpha \mathcal{L}_\alpha + \mathcal{L}_\alpha \mathcal{K}_\alpha)\boldsymbol{h} &= -\frac{1}{2}(\boldsymbol{j}_\alpha \cdot \nabla^{\bar{\alpha}}\boldsymbol{b}) \times \boldsymbol{h}, \\ (\boldsymbol{\Omega}_\alpha \mathcal{L}_\alpha - \mathcal{L}_\alpha \boldsymbol{\Omega}_\alpha)\boldsymbol{h} &= \frac{1}{4}(\boldsymbol{\omega}_\alpha \times \boldsymbol{j}_\alpha) \times \boldsymbol{h}. \end{aligned} \tag{3.27}$$

Further, we also compute,

$$(\mathcal{D}_\alpha \mathcal{K}_\alpha - \mathcal{K}_\alpha \mathcal{D}_\alpha) = \begin{bmatrix} 0 & -T_{12} & T_{13} \\ T_{12} & 0 & -T_{23} \\ -T_{13} & T_{23} & 0 \end{bmatrix}. \tag{3.28}$$

Hence for $\boldsymbol{h} \in \mathbb{R}^3$ the matrix $(\mathcal{D}_\alpha \mathcal{K}_\alpha - \mathcal{K}_\alpha \mathcal{D}_\alpha)$ satisfies the equation (3.18) i.e.,

$$T \times \boldsymbol{h} = (\mathcal{D}_\alpha \mathcal{K}_\alpha - \mathcal{K}_\alpha \mathcal{D}_\alpha)\boldsymbol{h}, \tag{3.29}$$

By (3.18), (3.27), (3.26) to (3.20) we get

$$\begin{aligned} \frac{1}{2}\frac{\partial \boldsymbol{\omega}_\alpha}{\partial t} \times \boldsymbol{h} - \frac{1}{2}(\boldsymbol{\omega}_\alpha \cdot \nabla^{\bar{\alpha}}\boldsymbol{u}) \times \boldsymbol{h} &= -\frac{1}{2}(\boldsymbol{j}_\alpha \cdot \nabla^{\bar{\alpha}}\boldsymbol{b}) \times \boldsymbol{h}, \\ \frac{1}{2}\frac{\partial \boldsymbol{j}_\alpha}{\partial t} \times \boldsymbol{h} &= T \times \boldsymbol{h} + \frac{1}{4}(\boldsymbol{\omega}_\alpha \times \boldsymbol{j}_\alpha) \times \boldsymbol{h}. \end{aligned} \tag{3.30}$$

As $\boldsymbol{h}$ is arbitrary, we conclude the result that the evolution of fractional vorticity $\boldsymbol{\omega}_\alpha$ and current density $\boldsymbol{j}_\alpha$ are determined by equations in Proposition (3.17). □

Theorem 3.4. *If the fractional gradients of the magnetic field and velocity commute, then the equations (1.2) admit a special solution of the form*

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t}) &= \frac{1}{\Gamma(\alpha+1)}\mathcal{D}_\alpha(t)\boldsymbol{x}_{\boldsymbol{\alpha}} + \frac{1}{2\Gamma(\alpha+1)}\boldsymbol{\omega}_{\boldsymbol{\alpha}}(t) \times \boldsymbol{x}_{\boldsymbol{\alpha}}, \\ \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{t}) &= \frac{1}{\Gamma(\alpha+1)}\mathcal{K}_\alpha(t)\boldsymbol{x}_{\boldsymbol{\alpha}} + \frac{1}{2\Gamma(\alpha+1)}\boldsymbol{j}_{\boldsymbol{\alpha}}(t) \times \boldsymbol{x}_{\boldsymbol{\alpha}}, \\ P &= \frac{1}{2\Gamma(\alpha+1)}(\hat{P}_\alpha)\boldsymbol{x}_{\boldsymbol{\alpha}} \cdot \boldsymbol{x}_{\boldsymbol{\alpha}}, \end{aligned} \tag{3.31}$$

where $\boldsymbol{x}_{\boldsymbol{\alpha}} = (x_1^\alpha, x_2^\alpha, x_3^\alpha)$ *and* $\mathcal{D}_\alpha(t)$, $\mathcal{K}_\alpha(t)$ *are symmetric matrices with zero trace and are defined by (3.3); furthermore the vorticity vector* $\boldsymbol{\omega}_{\boldsymbol{\alpha}}(t)$ *and the current density* $\boldsymbol{j}_\alpha(t)$ *satisfy the system of fractional ODEs.*

$$\begin{aligned} \frac{d\boldsymbol{\omega}_{\boldsymbol{\alpha}}}{dt} &= \mathcal{D}_\alpha(t)\boldsymbol{\omega}_{\boldsymbol{\alpha}}(\boldsymbol{t}) - \mathcal{K}_\alpha(t)\boldsymbol{j}_{\boldsymbol{\alpha}}(\boldsymbol{t}),\\ \frac{d\boldsymbol{j}_{\boldsymbol{\alpha}}}{dt} &= 0, \end{aligned} \tag{3.32}$$

and the matrix $\hat{P}_\alpha(t)$ *is obtained by*

$$\hat{P}_\alpha(t) = \frac{d\mathcal{D}_\alpha}{dt} + {\mathcal{D}_\alpha}^2 + {\boldsymbol{\Omega}_\alpha}^2 - {\mathcal{K}_\alpha}^2 - {\mathcal{L}_\alpha}^2, \tag{3.33}$$

where the matrices $\boldsymbol{\Omega}_\alpha$ *and* $\mathcal{L}_\alpha$ *are defined in (3.3) through the linear maps given in (3.4)*

Proof. We proceed to show that the velocity, magnetic field, and pressure functions defined by the equation (3.31) satisfy the system of equations (1.2).

To verify the momentum of equations of fluid velocity and magnetic field, we note that $\boldsymbol{u} = \frac{1}{\Gamma(\alpha+1)}\mathbf{U}_\alpha \boldsymbol{x}_\alpha$ and $\boldsymbol{b} = \frac{1}{\Gamma(\alpha+1)}\mathbf{B}_\alpha \boldsymbol{x}_\alpha$, where $\mathbf{U}_\alpha = \nabla^{\bar{\alpha}}\boldsymbol{u} = \mathcal{D}_\alpha + \boldsymbol{\Omega}_\alpha$ and $\mathbf{B}_\alpha = \nabla^{\bar{\alpha}}\boldsymbol{b} = \mathcal{K}_\alpha + \mathcal{L}_\alpha$ are functions of time alone. Hence, the following advection terms become

$$\begin{aligned} (\boldsymbol{u}\cdot\nabla^{\bar{\alpha}})\boldsymbol{u} &= \left(\frac{1}{\Gamma(\alpha+1)}\mathbf{U}_\alpha \boldsymbol{x}_{\boldsymbol{\alpha}}\cdot\nabla^{\bar{\alpha}}\right)\frac{1}{\Gamma(\alpha+1)}\mathbf{U}_\alpha \boldsymbol{x}_{\boldsymbol{\alpha}} = \frac{1}{\Gamma(\alpha+1)}\mathbf{U}_\alpha^2 \boldsymbol{x}_{\boldsymbol{\alpha}},\\ (\boldsymbol{u}\cdot\nabla^{\bar{\alpha}})\boldsymbol{b} &= \left(\frac{1}{\Gamma(\alpha+1)}\mathbf{U}_\alpha \boldsymbol{x}_{\boldsymbol{\alpha}}\cdot\nabla^{\bar{\alpha}}\right)\frac{1}{\Gamma(\alpha+1)}\mathbf{B}_\alpha \boldsymbol{x}_{\boldsymbol{\alpha}} = \frac{1}{\Gamma(\alpha+1)}\mathbf{U}_\alpha\mathbf{B}_\alpha \boldsymbol{x}_{\boldsymbol{\alpha}}. \end{aligned} \tag{3.34}$$

So the following terms result in

$$\begin{aligned} \partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla^{\bar{\alpha}})\boldsymbol{u} &= \frac{d\mathcal{D}_\alpha}{dt}\boldsymbol{x}_{\boldsymbol{\alpha}} + \frac{d\boldsymbol{\Omega}_\alpha}{dt}\boldsymbol{x}_{\boldsymbol{\alpha}} + \frac{1}{\Gamma(\alpha+1)}\mathbf{U}_\alpha^2\boldsymbol{x}_{\boldsymbol{\alpha}},\\ \partial_t\boldsymbol{b} + (\boldsymbol{u}\cdot\nabla^{\bar{\alpha}})\boldsymbol{b} &= \frac{d\mathcal{K}_\alpha}{dt}\boldsymbol{x}_{\boldsymbol{\alpha}} + \frac{d\mathcal{L}_\alpha}{dt}\boldsymbol{x}_{\boldsymbol{\alpha}} + \frac{1}{\Gamma(\alpha+1)}\mathbf{U}_\alpha\mathbf{B}_\alpha\boldsymbol{x}_{\boldsymbol{\alpha}}. \end{aligned} \tag{3.35}$$

The symmetric parts of (3.15) can be recast as follows

$$\begin{aligned} \frac{d\mathcal{D}_\alpha}{dt} + \mathcal{D}_\alpha + \boldsymbol{\Omega}_\alpha^2 &= -\hat{P}_\alpha + {\mathcal{K}_\alpha}^2 + {\mathcal{L}_\alpha}^2,\\ \frac{d\mathcal{K}_\alpha}{dt} &= \mathcal{D}_\alpha\mathcal{L}_\alpha - \mathcal{L}_\alpha\mathcal{D}_\alpha + \boldsymbol{\Omega}_\alpha\mathcal{K}_\alpha - \mathcal{K}_\alpha\boldsymbol{\Omega}_\alpha. \end{aligned} \tag{3.36}$$

As the gradient of velocity and magnetic field commute, so $\mathbf{U}_\alpha\mathbf{B}_\alpha - \mathbf{B}_\alpha\mathbf{U}_\alpha = 0$ and hence we get $\mathcal{D}_\alpha\mathcal{L}_\alpha - \mathcal{L}_\alpha\mathcal{D}_\alpha + \boldsymbol{\Omega}_\alpha\mathcal{K}_\alpha - \mathcal{K}_\alpha\boldsymbol{\Omega}_\alpha = 0$. Hence, we rewrite the above equation (3.36) as

$$\begin{aligned} \frac{d\mathcal{D}_\alpha}{dt} + \mathcal{D}_\alpha + \boldsymbol{\Omega}_\alpha^2 &= -\hat{P}_\alpha + {\mathcal{K}_\alpha}^2 + {\mathcal{L}_\alpha}^2,\\ \frac{d\mathcal{K}_\alpha}{dt} &= 0. \end{aligned} \tag{3.37}$$

For equation (3.17), we conclude that the term $2T + \frac{1}{2}\boldsymbol{\omega}_\alpha \times \boldsymbol{j}_\alpha$ vanishes. Hence (3.17) can be rewritten as

$$\begin{aligned} \frac{\partial\boldsymbol{\omega}_\alpha}{\partial t} - \boldsymbol{\omega}_\alpha\cdot\nabla^{\bar{\alpha}}\boldsymbol{u} &= -\boldsymbol{j}_\alpha\cdot\nabla^{\bar{\alpha}}\boldsymbol{b},\\ \frac{\partial\boldsymbol{j}_\alpha}{\partial t} &= 0. \end{aligned} \tag{3.38}$$

As $\mathcal{D}_\alpha$, $\mathcal{K}_\alpha$, $\mathcal{L}_\alpha$, $\boldsymbol{\Omega}_\alpha$ are functions of time alone so that we have

$$\begin{aligned} \frac{d\boldsymbol{\omega}_\alpha}{dt} &= \boldsymbol{\omega}_\alpha\cdot\nabla^{\bar{\alpha}}\boldsymbol{u} - \boldsymbol{j}_\alpha\cdot\nabla^{\bar{\alpha}}\boldsymbol{b},\\ \frac{d\boldsymbol{j}_\alpha}{dt} &= 0. \end{aligned} \tag{3.39}$$

By straightforward calculations, we obtain $\boldsymbol{\omega}_\alpha \cdot \nabla^{\bar{\alpha}} \boldsymbol{u} = \mathcal{D}_\alpha(t)\boldsymbol{\omega}_{\boldsymbol{\alpha}}$ and $\boldsymbol{j}_\alpha \cdot \nabla^{\bar{\alpha}} \boldsymbol{b} = \mathcal{K}_\alpha(t)\boldsymbol{j}_\alpha$. Therefore the vectors $\boldsymbol{\omega}_\alpha$ and $\boldsymbol{j}_\alpha$ satisfy the equation (4.3). Moreover, the skew-symmetric part of equation (3.19) is equivalent to the following equations.

$$\begin{aligned} &\frac{d\boldsymbol{\Omega}_\alpha}{dt} + \mathcal{D}_\alpha \boldsymbol{\Omega}_\alpha + \boldsymbol{\Omega}_\alpha \mathcal{D}_\alpha = \mathcal{K}_\alpha \mathcal{L}_\alpha + \mathcal{L}_\alpha \mathcal{K}_\alpha, \\ &\frac{d\mathcal{L}_\alpha}{dt} = 0. \end{aligned} \tag{3.40}$$

Using (3.37) and (3.40) to (3.35) we get

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u} = \frac{1}{\Gamma(\alpha+1)} [&-{\mathcal{D}_\alpha}^2 \boldsymbol{x}_\alpha - \boldsymbol{\Omega}_\alpha^2 \boldsymbol{x}_\alpha - \hat{P}_\alpha \boldsymbol{x}_\alpha + {\mathcal{K}_\alpha}^2 \boldsymbol{x}_\alpha + {\mathcal{L}_\alpha}^2 \boldsymbol{x}_\alpha \\ &- \mathcal{D}_\alpha \boldsymbol{\Omega}_\alpha \boldsymbol{x}_\alpha - \boldsymbol{\Omega}_\alpha \mathcal{D}_\alpha \boldsymbol{x}_\alpha + \mathcal{K}_\alpha \mathcal{L}_\alpha \boldsymbol{x}_\alpha + \mathcal{L}_\alpha \mathcal{K}_\alpha \boldsymbol{x}_\alpha + U_\alpha^2 \boldsymbol{x}_\alpha]. \end{aligned} \tag{3.41}$$

By simple calculations we have $\mathbf{U}_\alpha^2 \boldsymbol{x}_\alpha = [{\mathcal{D}_\alpha}^2 + \boldsymbol{\Omega}_\alpha^2 + \mathcal{D}_\alpha \boldsymbol{\Omega}_\alpha + \boldsymbol{\Omega}_\alpha \mathcal{D}_\alpha]\boldsymbol{x}_\alpha$ and $\mathbf{B}_\alpha^2 \boldsymbol{x}_\alpha = [{\mathcal{K}_\alpha}^2 + {\mathcal{L}_\alpha}^2 + \mathcal{K}_\alpha \mathcal{L}_\alpha + \mathcal{L}_\alpha \mathcal{K}_\alpha]\boldsymbol{x}_\alpha$ and substituting these terms in (3.41) results into the following equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla^{\bar{\alpha}} \boldsymbol{u} = -\hat{P}_\alpha \boldsymbol{x}_\alpha + \mathbf{B}_\alpha^2 \boldsymbol{x}_\alpha \tag{3.42}$$

So we determine the terms $(\boldsymbol{b} \cdot \nabla^{\bar{\alpha}})\boldsymbol{b}$ and the gradient of the pressure function P that results in the following

$$\begin{aligned} &(\boldsymbol{b}_\alpha \cdot \nabla^{\bar{\alpha}})\boldsymbol{b} = \mathbf{B}_\alpha^2 \boldsymbol{x}_\alpha, \\ &\nabla^{\bar{\alpha}} P = \nabla^{\bar{\alpha}} (\frac{1}{2} \hat{P}_\alpha(t) \boldsymbol{x}_\alpha) \boldsymbol{x}_\alpha = \hat{P}_\alpha \boldsymbol{x}_\alpha. \end{aligned} \tag{3.43}$$

Substituting values of (3.43) in (3.42) gives us the momentum equation of fluid velocity in (1.2). To verify the momentum equation of the magnetic field, we substitute (3.40) and (1.29) in the second equation of (3.35), and we get

$$\begin{aligned} \partial_t \boldsymbol{b} + (\boldsymbol{u} \cdot \nabla^{\bar{\alpha}})\boldsymbol{b} &= \frac{d\mathcal{K}_\alpha}{dt} \boldsymbol{x}_{\boldsymbol{\alpha}} + \frac{d\mathcal{L}_\alpha}{dt} \boldsymbol{x}_{\boldsymbol{\alpha}} + \frac{1}{\Gamma(\alpha+1)} \mathbf{U}_\alpha \mathbf{B}_\alpha \boldsymbol{x}_{\boldsymbol{\alpha}}, \\ &= \frac{1}{\Gamma(\alpha+1)} \mathbf{B}_\alpha \mathbf{U}_\alpha \boldsymbol{x}_{\boldsymbol{\alpha}} = (\boldsymbol{b} \cdot \nabla^{\bar{\alpha}})\boldsymbol{u}. \end{aligned} \tag{3.44}$$

Thus, we see that the fluid velocity, magnetic field, and pressure given by (3.31) satisfy the equations (1.2). Therefore $\boldsymbol{u}$, $\boldsymbol{b}$, and P given by (3.31) determined with the equations (1.2). This concludes the proof. □

4 Qualitative Analysis

In the previous section, we have determined the solutions of (1.2) in the form of (3.31) when the fractional gradient of the magnetic field of order α for $0 < \alpha \leq 1$ commutes with the fractional gradient of the fluid velocity of the same order α. Also the fractional vorticity vector $\boldsymbol{\omega}_{\boldsymbol{\alpha}}$ and fractional current density vector $\boldsymbol{j}_{\boldsymbol{\alpha}}$ must satisfy (4.3).

Now we suppose that the fractional gradients of velocity and magnetic fields commute and solutions of (1.2) exist in the form (3.31). If the deformation matrices $\mathcal{D}_\alpha$ and $\mathcal{K}_\alpha$ are continuous functions of time t, then the existence of such solutions is guaranteed.

4.1 Verification of the solutions obtained for the system if $\boldsymbol{\alpha} = 1$

For $\alpha = 1$, the system (1.2) results in the following

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla P + \boldsymbol{b} \cdot \nabla \boldsymbol{b}, \\ \partial_t \boldsymbol{b} + \boldsymbol{u} \cdot \nabla \boldsymbol{b} &= \boldsymbol{b} \cdot \nabla \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} &= 0, \\ \nabla \cdot \boldsymbol{b} &= 0, \end{aligned} \tag{4.1}$$

and the special solutions that we have obtained in Theorem (3.4) yield the following

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= (\mathcal{D}(t)\boldsymbol{x} + \frac{1}{2}\boldsymbol{\omega}(t) \times \boldsymbol{x})), \\ \boldsymbol{b}(\boldsymbol{x},t) &= (\mathcal{K}(t)\boldsymbol{x} + \frac{1}{2}\boldsymbol{j}(t) \times \boldsymbol{x}), \\ P &= \frac{1}{2}(P\hat{(t)}\boldsymbol{x}) \cdot \boldsymbol{x}, \end{aligned} \tag{4.2}$$

where $\mathcal{D}$ and $\mathcal{K}$ are symmetric matrices with zero trace and are defined by (3.3); furthermore, the vorticity vector $\boldsymbol{\omega}(t)$ and the current density $\boldsymbol{j}(t)$ satisfy the system of Fractional ODEs.

$$\begin{aligned} \frac{d\boldsymbol{\omega}}{dt} &= \mathcal{D}\boldsymbol{\omega}(t) - \mathcal{K}\boldsymbol{j}(t), \\ \frac{d\boldsymbol{j}(t)}{dt} &= 0, \end{aligned} \tag{4.3}$$

and the matrix $\hat{P}(t)$ is obtained by

$$-\hat{P}(t) = \frac{d\mathcal{D}}{dt} + \mathcal{D}^2 + \Omega^2 - \mathcal{K}^2 - \mathcal{L}^2, \tag{4.4}$$

where the matrices Ω and $\mathcal{L}$ are defined in (3.3) through the linear maps given in (3.4), provided the gradient of the magnetic fields and velocity commutes. This solution obtained above and the special solutions obtained in [6] are indistinguishable.

4.2 Integrability of fractional MHD equations

The fractional gradient of the fluid velocity and the magnetic field commute in the space-fractional MHD equations (1.2). Additionally, the deformation matrices $\mathcal{D}_\alpha$ and $\mathcal{K}_\alpha$ are continuous functions of time variables $t \in \mathbb{R}$. The next initial value problem is as follows:

$$\begin{aligned} \frac{d\boldsymbol{\omega}_\alpha}{dt} &= \mathcal{D}_\alpha\boldsymbol{\omega}_\alpha(t) - \mathcal{K}_\alpha\boldsymbol{j}_\alpha(t), \qquad \boldsymbol{\omega}_\alpha(0) = \boldsymbol{\omega}_0, \\ \frac{d\boldsymbol{j}_{\boldsymbol{\alpha}}(t)}{dt} &= 0, \qquad \boldsymbol{j}_\alpha(0) = \boldsymbol{j}_0. \end{aligned} \tag{4.5}$$

Assume that the pressure function P is given by $P = \frac{1}{2\Gamma(\alpha+1)}\hat{P_\alpha}(t)\boldsymbol{x}_\alpha \cdot \boldsymbol{x}_\alpha$ where $\hat{P_\alpha}(t)$ is determined by equation (4.4).

Consider R to be the domain of the system (4.5) i.e. $R = \{\boldsymbol{x} \in \mathbb{R}^3 : \nabla^{\bar{\alpha}}\boldsymbol{u}(\boldsymbol{x},t)\nabla^{\bar{\alpha}}\boldsymbol{b}(\boldsymbol{x},t) = \nabla^{\bar{\alpha}}\boldsymbol{b}(\boldsymbol{x},t)\nabla^{\bar{\alpha}}\boldsymbol{u}(\boldsymbol{x},t); \forall t \in \mathbb{R}\}$ in which $\mathcal{D}_\alpha$ and $\mathcal{K}_\alpha$ are continuous functions for all t. Now we examine the existence of a solution to the initial value problem (4.5) in the domain R. We see that (4.5) is a linear system of six coupled ODEs in variable t and we determine that $\boldsymbol{j}(t) = \boldsymbol{j}_0$. As a result, we obtain the first equation of (4.5) as follows:

$$\frac{d\boldsymbol{\omega}_\alpha(t)}{dt} = \mathcal{D}_\alpha\boldsymbol{\omega}_{\boldsymbol{\alpha}} - \mathcal{K}_\alpha\boldsymbol{j}_{\boldsymbol{0}}, \qquad \boldsymbol{\omega}_\alpha(0) = \boldsymbol{\omega}_{\boldsymbol{0}}. \tag{4.6}$$

This is a system of non-homogeneous linear differential equations, and as $\mathcal{D}_\alpha, \mathcal{K}_\alpha$ are continuous through the quadrature, we can obtain an analogous solution of the initial value problem stated in [6]. This implies that the system (4.5) is completely integrable in $R \subseteq \mathbb{R}^3$.

4.3 Critical Point Analysis

In this section, we find the critical points of the system (4.5), and we also examine their nature. The system we have considered is of the space fractional order gradient, but the components in the system of the equation (1.2) that are differentiated with respect to the time variable are of order one. So for the system (4.5), it is clear that $(\boldsymbol{\omega_\alpha}, \boldsymbol{j_\alpha}) = (0,0)$ is the critical point. At the critical point $(\boldsymbol{\omega_\alpha}, \boldsymbol{j_\alpha}) = (0,0)$, the system (4.5) can be written in the matrix form as follows:

$$\begin{bmatrix} \dfrac{d\boldsymbol{\omega_\alpha}}{dt} \\ \\ \dfrac{d\boldsymbol{j_\alpha}}{dt} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_\alpha(t) & -\mathcal{K}_\alpha(t) \\ O & O \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega_\alpha} \\ \boldsymbol{j_\alpha} \end{bmatrix} = \mathcal{A}(t) \begin{bmatrix} \boldsymbol{\omega_\alpha} \\ \boldsymbol{j_\alpha} \end{bmatrix}, \tag{4.7}$$

where O's are 3×3 null matrices and the matrix $\mathcal{A} = \mathcal{A}(t)$ is an upper triangular block matrix and its diagonal blocks are $\mathcal{D}_\alpha$, which is a deformation matrix, and a null matrix. We observe that the eigenvalues $\mathcal{A}$ are the sum of the eigenvalues of matrices $\mathcal{D}_\alpha$ and O. Further, we conclude that the eigenvalues of $\mathcal{D}_\alpha$ are real numbers for every t as $\mathcal{D}_\alpha$ is a real symmetric matrix of continuous real-valued functions. $\lambda = 0$ is at least three-fold eigenvalue of matrix $\mathcal{A}$ since O is a diagonal block matrix in $\mathcal{A}$. This shows that the critical point **(0, 0)** is degenerate.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the matrix $\mathcal{D}_\alpha$, then trace A= $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Thus, it follows that either all eigenvalues of A are zero, or there are at least two eigenvalues of the opposite sign. The stability of the critical point $(0,0)$ will be given by the following cases:

Case 1: If all the eigenvalues of the matrix $\mathcal{D}_\alpha$ are zeros then $(\boldsymbol{\omega_\alpha}, \boldsymbol{j_\alpha}) = (0,0)$ is a stable node.

Case 2: If atleast two eigenvalues of $\mathcal{D}_\alpha$ are nonzero and of opposite signs then $(\boldsymbol{\omega_\alpha}, \boldsymbol{j_\alpha}) = (0,0)$ is a saddle point.

The general condition for a critical point of system (4.5) is $\mathcal{D}_\alpha \boldsymbol{\omega}_\alpha = \mathcal{K}_\alpha \boldsymbol{j}_\alpha$. If we consider the set of critical points, satisfying

$$E = \{(\boldsymbol{\omega}_\alpha, \boldsymbol{j}_\alpha) | \boldsymbol{\omega}_\alpha = \boldsymbol{j}_\alpha \text{ and } \mathcal{D}_\alpha = \mathcal{K}_\alpha \text{ for all } t \in \mathbb{R}\}.$$

Then by considering the translation of the derivative field $(\boldsymbol{\omega}_\alpha, \boldsymbol{j}_\alpha) \to (\mathcal{D}_\alpha \boldsymbol{\omega}_\alpha - \mathcal{K}_\alpha \boldsymbol{j}_\alpha)$, we observe that each point of E is degenerate and has stability of the above cases. Thus, we conclude that the set E is a hyperplane of nodal points.

In the next section, we outline the examples of two and three-dimensional flow of fluid in the presence of the magnetic field for which we determine the solutions for the system (1.2) in the form of (3.31).

5 Examples

We consider the two-dimensional flow of fluid in the presence of a magnetic field, followed by an example of a three-dimensional flow.

Example 5.1. We shall consider the two-dimensional time-dependent flow for which the constant fractional vorticity and fractional current density vectors are unit vectors in the vertical direction, i.e., $\boldsymbol{\omega_\alpha} = (0,0,1)$ and $\boldsymbol{j_\alpha} = (0,0,1)$. For $\lambda \in \mathbb{R}$, let us assume the deformation matrix $\mathcal{D}_\alpha$ and $\mathcal{K}_\alpha$ be given by

$$\mathcal{D}_\alpha = \begin{bmatrix} \lambda+1 & 0 & 0 \\ 0 & -\lambda-1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{K}_\alpha = \begin{bmatrix} \lambda+1 & 0 & 0 \\ 0 & -\lambda-1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, we obtained the special solutions of two-dimensional space fractional MHD fluid as follows:

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \frac{1}{\Gamma(\alpha+1)} (\mathcal{D}_\alpha(t) \boldsymbol{x_\alpha} + \frac{1}{2} \boldsymbol{\omega_\alpha}(t) \times \boldsymbol{x_\alpha}) \\ &= \frac{1}{\Gamma(\alpha+1)} \left((\lambda+1) {x_1}^\alpha - \frac{1}{2} {x_2}^\alpha, (-\lambda-1) {x_2}^\alpha + \frac{1}{2} {x_1}^\alpha, 0 \right) \\ \boldsymbol{b}(\boldsymbol{x}, t) &= \frac{1}{\Gamma(\alpha+1)} (\mathcal{K}_\alpha(t) \boldsymbol{x_\alpha} + \frac{1}{2} \boldsymbol{j_\alpha}(t) \times \boldsymbol{x_\alpha}) \\ &= \frac{1}{\Gamma(\alpha+1)} \left((\lambda+1) {x_1}^\alpha - \frac{1}{2} {x_2}^\alpha, (-\lambda-1) {x_2}^\alpha + \frac{1}{2} {x_1}^\alpha, 0 \right) \\ P &= \frac{1}{2\Gamma(\alpha+1)} (\hat{P}_\alpha) \boldsymbol{x_\alpha} \cdot \boldsymbol{x_\alpha} = 0 \end{aligned} \tag{5.1}$$

This solution yields a mathematically consistent state of vanishing pressure ($P = 0$), a known feature in simplified models like pressureless cosmological fluids, high-temperature plasma flow. While non-physical, this result serves as a critical benchmark for the fractional MHD framework. The next example presents a three-dimensional flow with realistic deformation and a non-zero pressure field.

Example 5.2. For $\lambda \in \mathbb{R}$, consider a three-dimensional flow of fluid in the presence of a magnetic field in which deformation matrices of fluid velocity and magnetic field are given as follows:

$$\mathcal{D}_\alpha = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\lambda+1 \end{bmatrix}, \qquad \mathcal{K}_\alpha = \begin{bmatrix} \frac{1}{4}(-3-5\lambda) & 0 & 0 \\ 0 & \frac{1}{4}(5+3\lambda) & 0 \\ 0 & 0 & \frac{1}{2}(-1+\lambda) \end{bmatrix}.$$

The vorticity vector $\boldsymbol{\omega_\alpha} = (0,0,-2)$ and the current density vectors $\boldsymbol{j_\alpha} = (0,0,4)$. Using Theorem 3.4, the special solutions of the system are :

$$\begin{aligned}
\boldsymbol{u(x,t)} &= \frac{1}{\Gamma(\alpha+1)}(\mathcal{D}_\alpha(t)\boldsymbol{x_\alpha} + \frac{1}{2}\boldsymbol{\omega_\alpha}(t) \times \boldsymbol{x_\alpha}) \\
&= \frac{1}{\Gamma(\alpha+1)}\left(\lambda {x_1}^\alpha + {x_2}^\alpha, -x_1^\alpha - x_2^\alpha, (1-\lambda){x_3}^\alpha\right) \\
\boldsymbol{b(x,t)} &= \frac{1}{\Gamma(\alpha+1)}(\mathcal{K}_\alpha(t)\boldsymbol{x_\alpha} + \frac{1}{2}\boldsymbol{j_\alpha}(t) \times \boldsymbol{x_\alpha}) \\
&= \frac{1}{\Gamma(\alpha+1)}\left(- \frac{1}{4}(3+5\lambda){x_1}^\alpha - 2{x_2}^\alpha, 2x_1^\alpha + \frac{1}{4}(5+3\lambda){x_2}^\alpha, \frac{1}{2}(-1+\lambda){x_3}^\alpha\right) \\
P &= \frac{1}{2\Gamma(\alpha+1)}(\hat{P}_\alpha)\boldsymbol{x_\alpha} \cdot \boldsymbol{x_\alpha} \\
&= \frac{1}{2\Gamma(\alpha+1)}\frac{3}{16}(-1+\lambda)[(13+3\lambda)x_1^{2\alpha} + (13+3\lambda)x_2^{2\alpha} - 4(-1+\lambda)x_3^{2\alpha}]
\end{aligned} \tag{5.2}$$

6 Conclusion

We have considered the space-fractional MHD equations (1.2) and obtained special solutions in the form of (3.1) given in Theorem 3.4. Through Proposition 3.3, we showed that the vorticity $\boldsymbol{\omega_\alpha}$ and current density $\boldsymbol{j_\alpha}$ satisfy the corresponding fractional PDEs. Under the assumption of commutativity between the fractional gradient operators acting on velocity and magnetic fields, the system (4.3) admits a reduction to an integrable system of fractional ordinary differential equations. We identified the degenerate critical point $(\boldsymbol{\omega_\alpha}, \boldsymbol{j_\alpha}) = (0,0)$ and analyzed its stability. The derived solutions, which recover the traditional ideal MHD case for $\alpha = 1$, are consistent and relevant, as demonstrated by the examples.

By offering benchmark special solutions that broaden the theoretical understanding of fractional operator effects on incompressible fluids in the presence of a magnetic field, this article thus establishes a rigorous analytical framework for fractional MHD. Examining the fractional MHD equations for different types of fractional differential operators and performing numerical analysis of the reduced systems using recent computational methods for nonlinear FPDEs found in the literature are potential directions for future research. These studies might unfold more features of fractional MHD models, expanding their applications.

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