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On the Diophantine equation $43^x + 87^y = z^2$ when x, y are consecutive positive integers

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Abstract. In this paper, we solve that the Diophantine equation $43^x + 87^y = z^2$ in the positive integers (x, y, z) with consecutives x and y. Concretely, we investigate all the possibilities where x is even, odd and if $x > y$, $x < y$. It is proved that the equation has a unique solution which is showcased. As a corollary, there exist an equation $p^x + (2p + 1)^y = z^2$ where consecutives x, y and $z = p + 1$ with p is a prime.

1 Introduction

In recent years, Diophantine equation of the form $a^x + b^y = z^2$ has been studying of many mathematicians when a, b are fixed. In 2013, Chotchaisthit [3] found that the Diophantine equation $2^x + 11^y = z^2$ has the only solution $(x, y, z) = (3, 0, 3)$ in nonnegative integers x, y and z. Following that, Sroysang [9] obtained that the Diophantine equation $5^x + 63^y = z^2$ has a unique solution $(x, y, z) = (0, 1, 8)$ in nonnegative integers x, y and z. In the same years, Suvarnamani [10] proved that the Diophantine equation $p^x + (p + 1)^y = z^2$ admits a unique solution $(p, x, y, z) = (3, 1, 0, 2)$ in nonnegative integers x, y, z and p is an odd prime number. In 2017, Rabago [8] showed that the Diophantine equation $2^x + 17^y = z^2$ has the only solutions $(x, y, z) \in \{(3, 1, 5), (5, 1, 7), (6, 1, 9), (7, 3, 71), (9, 1, 23)\}$ in positive integers x, y and z. In the same years, Nathanson [7] constructed infinite family of positive integral solutions of the Diophantine equation $x^n - y^n = z^{n+1}$ for every positive integer n. Two years later, Leung [5] investigated that the Diophantine equation $x^n - y^n = z^{n+k}$ has positive integral solutions for every positive integers n and k. In recent times, Mina and Bacani [6] solved that the Diophantine equation $p^x + (p+4k)^y = z^2$ admits a unique solution $(p, p+4k, x, y, z) = (3, 11, 5, 2, 122)$ for prime pairs $(p, p + 4k)$ with $k \geq 2$ and whenever $x > 1$, $y > 1$. Moreover, Borah and Dutta [1] showed that the Diophantine equation $7^x + 32^y = z^2$ has a unique solution $(x, y, z) = (2, 1, 9)$ in nonnegative integers x, y and z. Additionally, Buosi et al. [2] showed all the nontrivial nonnegative integer solutions to the Diophantine equation $p^x - 2^y = z^2$ where $p = k^2 + 4$ is a prime number and $k \geq 1$, are given by $(1, 2, k)$ if $p \geq 13$ and $\{(1, 0, 2), (1, 2, 1), (3, 2, 11)\}$ if $p = 5$. Recently, Gaha and Mezroui [4] proved that the four Diophantine equations $13^x + 17^y = z^2$, $13^x + 37^y = z^2$, $17^x + 37^y = z^2$ and $17^x + 91^y = z^2$ have no solutions in the nonnegative integers (x, y, z).

In the present paper, we study the Diophantine equation

$$43^x + 87^y = z^2 \tag{1.1}$$

in the positive integers (x, y, z) with consecutives x and y. Following that, we examine that the equation has a unique solution which is demonstrated. The main theorem of this paper is the following.

Theorem 1.1. *Let x, y and z are positive integers. Assume that the exponential Diophantine equation (1.1) when x, y are consecutive integers. For an integer $n \geq 0$, then we have the following statements hold.*

(i) If $x = 2n + 1$, $y = 2n + 2$, $n \geq 0$, then the equation (1.1) has no solution.

(ii) If $x = 2n + 2$, $y = 2n + 3$, $n \geq 0$, then the equation (1.1) has no solution.

(iii) If $x = 2n + 2$, $y = 2n + 1$, $n \geq 0$, then the equation (1.1) has a unique solution.

(iv) If $x = 2n + 1$, $y = 2n$, $n \geq 1$, then the equation (1.1) has no solution.

As with the results, then we have the following corollary.

Corollary 1.2. *The Diophantine equation $a^x + b^y = z^2$ has a unique solution. Including, $(x, y, z) = (2, 1, 44)$ wherein the couple $(a, b) = (43, 87) = (p, 2p + 1)$ and consecutives $x = 2$, $y = 1$. For each prime p, there exist an equation $p^x + (2p + 1)^y = z^2$ has exactly one solution when consecutives $x = 2$, $y = 1$ and $z = p+1$. As a result, the equality $p^2+(2p+1)^1 = (p+1)^2$ is an identity valid for each p.*

Specifically, we prove the following main results.

2 The proof of main theorem

Proof. Although similarities exist all four possibilities are listed in Theorem 1.1, we examine separately so that each possibility is self-contained in them.

(i) Suppose now that

$$43^{2n+1} + 87^{2n+2} = z^2, \quad n \geq 0. \tag{2.1}$$

If $n = 0$, we get by the equation (2.1) that $43^1 + 87^2 = 7612 \neq z^2$ (because the number 7612 is not a square), then this equation has no solution. If $n \geq 1$, by the equation (2.1) shows that

$$43^{2n+1} = z^2 - 87^{2n+2} = z^2 - 87^{2(n+1)} = (z - 87^{n+1})(z + 87^{n+1}).$$

Notice that $z - 87^{n+1} = 43^\lambda$ and $z + 87^{n+1} = 43^\mu$, where $0 \leq \lambda < \mu$ and $\lambda + \mu = 2n + 1$. Thus, $43^\mu - 43^\lambda$ implies that

$$43^\lambda(43^{\mu-\lambda} - 1) = 2 \cdot 87^{n+1}. \tag{2.2}$$

If $\lambda > 0$, then the power 43^λ does not divide the right side of the equation (2.2). Therefore, $\lambda = 0$ and $\mu = 2n + 1$. So, from the equation (2.2), we acquire

$$43^{2n+1} - 1 = 2 \cdot 87^{n+1}. \tag{2.3}$$

For all values $n \geq 1$, then the power 43^{2n+1} has a last digit equal to 3 and 7. Hence, $43^{2n+1} - 1$ ends in the digit 2 and 6. Therefore, $43^{2n+1} - 1$ is a multiple of 7. Since the right side of the equation (2.3) is not a multiple of 7, it follows that $43^{2n+1} - 1 \neq 2 \cdot 87^{n+1}$. So, the equation $43^{2n+1} + 87^{2n+2} = z^2$ has no solution.

(ii) If $x = 2n + 1$ and $y = 2n$ with $n \geq 1$, we have $43^{2n+1} + 87^{2n} = z^2$, then

$$43^{2n+1} = z^2 - 87^{2n} = (z - 87^n)(z + 87^n).$$

Notice that $z - 87^n = 43^\alpha$ and $z + 87^n = 43^\beta$, where $0 \leq \alpha < \beta$ and $\alpha + \beta = 2n + 1$. Thus, $43^\alpha(43^{\beta-\alpha} - 1) = 2 \cdot 87^n$. This implies that $43^\alpha = 1$ and $43^{\beta-\alpha} - 1 = 2 \cdot 87^n$, it follows that $\alpha = 0$ and $43^{2n+1} - 1 = 2 \cdot 87^n$. For $n = 1$, then $79506 = 174$, which is not acceptable. For $n \geq 2$, we have

$$2 \cdot 87^n = 43^{2n+1} - 1 = (43 - 1)(43^{2n} + 43^{2n-1} + \cdots + 43 + 1)$$

and so

$$87^n = 21(43^{2n} + 43^{2n-1} + \cdots + 43 + 1),$$

which is not possible. Hence, the equation $43^x + 87^y = z^2$ has no solution in the positive integers (x, y, z).

(iii) Suppose now that

$$43^{2n+2} + 87^{2n+3} = z^2, \quad n \geq 0. \tag{2.4}$$

If $n = 0$, we have by the equation (2.4) that $43^2 + 87^3 = 660352 \neq z^2$ (because the number 660352 is not a square), then this equation has no solution. If $n \geq 1$, by the equation (2.4) gives that

$$87^{2n+3} = z^2 - 43^{2n+2} = z^2 - 43^{2(n+1)} = (z - 43^{n+1})(z + 43^{n+1}).$$

Notice that $z - 43^{n+1} = 87^\gamma$ and $z + 43^{m+1} = 87^\delta$, where $0 \leq \gamma < \delta$ and $\gamma + \delta = 2n + 3$. Thus, $87^\delta - 87^\gamma$ indicates that

$$87^\gamma(87^{\delta-\gamma} - 1) = 2 \cdot 43^{n+1}. \tag{2.5}$$

If $\gamma > 0$, then the power 87^γ does not divide the right side of the equation (2.5). Therefore, $\gamma = 0$ and $\delta = 2n + 3$. So, from the equation (2.5), we get

$$87^{2n+3} - 1 = 2 \cdot 43^{n+1}. \tag{2.6}$$

For all values $n \geq 1$. By the equation (2.6), we acquire

$$2 \cdot 43^{n+1} + 1 < 43 \cdot 43^{n+1} + 1 = 43^{n+2} + 1 < 87^{n+2} \cdot 87^{n+1} = 87^{2n+3},$$

and therefore $87^{2n+3} - 1 \neq 2 \cdot 43^{n+1}$. So, the equation $43^{2n+2} + 87^{2n+3} = z^2$ has no solution.

(iv) Suppose now that

$$43^{2n+2} + 87^{2n+1} = z^2, \quad n \geq 0. \tag{2.7}$$

If $n = 0$, we get by the equation (2.7) that $43^2 + 87^1 = 1936 = 44^2 = z^2$, then we have that $(x, y, z) = (2, 1, 44)$ is solution of the Diophantine equation $43^x + 87^y = z^2$ where $x = 2n + 2$ and $y = 2n + 1$. If $m \geq 1$, by the equation (2.7) becomes

$$87^{2n+1} = z^2 - 43^{2n+2} = z^2 - 43^{2(n+1)} = (z - 43^{n+1})(z + 43^{n+1}).$$

Notice that $z - 43^{n+1} = 87^\nu$ and $z + 43^{n+1} = 87^\xi$, where $0 \leq \nu < \xi$ and $\nu + \xi = 2n + 1$. Thus, $87^\xi - 87^\nu$ appears that

$$87^\nu(87^{\xi-\nu} - 1) = 2 \cdot 43^{n+1}. \tag{2.8}$$

If $\nu > 0$, then the power 87^ν does not divide the right side of the equation (2.8). Therefore, $\nu = 0$ and $\xi = 2n + 1$. So, from the equation (2.8) reveals

$$87^{2n+1} - 1 = 2 \cdot 43^{n+1}. \tag{2.9}$$

For all values $n \geq 1$. From the equation (2.9), it is seen that

$$2 \cdot 43^{n+1} + 1 < 43 \cdot 43^{n+1} + 1 = 43^{n+2} + 1 < 87^{n+2} \cdot 87^{n-1} = 87^{2n+1},$$

it follows that $87^{2n+1} - 1 \neq 2 \cdot 43^{n+1}$. So, the equation $43^{2n+2} + 87^{2n+1} = z^2$ has no solution, completing the proof.

$\square$

3 Conclusion

This paper aims is to get shown that the Diophantine equation $43^x + 87^y = z^2$ admits a unique solution when x, y are consecutives, namely $(x, y, z) = (2, 1, 44)$. Subsequently, we determined there exists an equation $p^x + (2p + 1)^y = z^2$ has a unique solution where consecutives $x = 2$, $y = 1$ and $z = p + 1$ for all prime p.

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