Palestine Journal of Mathematics

Vol 15(2)(2026) , 662–671

A KIND OF NONCOMMUTING GRAPH OF FINITE-DIMENSIONAL LIE ALGEBRAS

A. Shamsaki, G. Rachmaputri, I. Muchtadi Alamsyah and A. Erfanian

Communicated by: T. Asir

MSC 2020 Classifications: Primary 17B60; Secondary 05C60, 05C10.

Keywords and phrases: Lie algebra, g-noncommuting graph, non-commuting graph.

This research is supported by grant funds for scientific activities of the Emerging Regional Center of Excellence (ERCE) ITB 2023 from The European Mathematical Society (EMS).

Corresponding Author: A. Erfanian

Abstract. Let L be a finite-dimensional Lie algebra over the field $\mathbb{F}_q$ and g be a fixed element of L. In this paper, we define the g-noncommuting graph G_L^g and obtain the structures of G_L^g when they contain a vertex x with $deg(x) \leq 6$. Moreover, we characterize all g-noncommuting graph G_L^g when this graph is K_n-free such that $n \leq 5$.

1 Introduction

Lie algebra is one of the most interesting algebraic structures. Recently, their importance appears in other sciences including physics, especially quantum physics, which deals with physical phenomena on a microscopic scale. Besides the relation with the group theory, the probability theory and the graph theory have been studied in several papers. For instance, in [5], the commutativity degree of a finite-dimensional Lie algebra L, denoted by $d(L)$, is defined as the probability of commuting chosen two random elements of L and the structure of Lie algebra L such that $d(L) \in [\frac{1}{q}, \frac{q^2+q-1}{q^3}]$ for some $q \geq 2$ is obtained from [6]. Also, there exists a relation between Lie algebras and the graph theory. The notion of the non-commuting graph G_L associated to a finite-dimensional Lie algebra has been firstly defined in [3] as a graph whose vertices are in $L \setminus Z(L)$, two vertices x and y join if and only if $[x, y] \neq 0$.
In the following, we just recall some necessary concepts of graph theory from [1, 2] and assume that the reader is familiar with some basic definitions of Lie algebras for example subalgebra, ideal, and so on. Otherwise, we refer to [4]. A graph consists of $V(G)$ as vertex set and a set $E(G)$ of edges. Two vertices of G are adjacent if and only if there is an edge between them. The number of edges of G incident with the vertex v is called the degree v and denoted by $\deg(v)$. A graph G is r-regular if $deg(v) = r$ for all $v \in V(G)$. Let $|V(G)| = n$. Then if the degree of every vertex is $n-1$, then G is a complete graph denoted by K_n. A graph is K_n-free when it does not contain K_n as an induced subgraph. A path is a set of alternating vertices $v_0, \dots, v_n$ and edges $e_1, \dots, e_n$, where e_i is an edge between v_{i-1} to v_i for $1 \leq i \leq n$ such that all vertices and edges are distinct. A closed path is called a cycle. A graph is said to be connected if every pair of vertices in the graph is connected. A tree is a connected graph containing no cycles. A vertex subset D of $V(G)$ is a dominating set if every vertex of G either belongs to D or adjacent to a vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality D among all dominating sets. A clique of G is an induced subgraph H of G that is a complete graph. The clique number $\omega(G)$ of G is the largest order of a clique in G. If a graph can be drawn in such a way that any edge does not cross each other, then it is called planar.
In this paper, We are going to present a generalization of the non-commuting graph of a finite-dimensional Lie algebra. Let g be a fixed element of L. Then g-noncommuting of L with respect an element g is defined as the following:
We denote the g-noncommuting graph of L by G_L^g as a undirected simple graph with the vertex

set L and two distinct vertices x and y join by an edge if and only if $[x, y] \neq g$ and $-g$.
We organize this paper into three sections. In the second section, we determine several propositions of G_L^g such as domination number, regularity and completeness. In addition, the structures of G_L^g are obtained when they have a vertex such as x with $\deg(x) \leq 6$. In the final section, we discuss the existence of certain forbidden subgraphs in G_L^g. Specifically, we determine the structure of G_L^g provided that G_L^g is K_n-free for $n \leq 5$.

2 Some properties of g-noncommuting graphs

In this section, we begin by stating an important lemma that plays a crucial role. Subsequently, we derive a formula for the degree of any vertex in G_L^g.

Lemma 2.1. *Let G_L^g be the g-noncommuting graph of a Lie algebra L over the field $\mathbb{F}_q$ and $x \in L$.*

(i) If $g \in \operatorname{Im} ad_x$ and $g \neq 0$, then $deg(x) = \begin{cases} |L| - 2|C_L(x)| - 1 \textit{if} & \mathrm{Char}\mathbb{F}_q \neq 2, \\ |L| - |C_L(x)| - 1 \ \textit{if} & \mathrm{Char}\mathbb{F}_q = 2. \end{cases}$

(ii) If $g = 0$, then $deg(x) = |L| - |C_L(x)|$.

(iii) If $g \notin \operatorname{Im} ad_x$, then $deg(x) = |L| - 1$.

Proof. **(i)** Assume that $g \neq 0$ and $x \in L$ such that $g \in \operatorname{Im} ad_x$ and the field $\mathbb{F}_q$ does not have characteristic 2. Put $A = \{y \in L \mid [x, y] = g\}$ and $B = \{y \in L \mid [x, y] = -g\}$. Then

$$\begin{aligned} deg(x) &= |\{y \in V(G_L^g) \mid [x, y] \neq g \text{ and } -g\}| = |V(G_L^g)| - |A \cup B| - 1 \\ &= |L| - |A| - |B| + |A \cap B| - 1. \end{aligned} \tag{2.1}$$

Clearly, $A \cap B = \emptyset$. Otherwise, if there exists $y \in A \cap B$ such that $[x, y]$ is equal to both g and $-g$. Then $2g = 0$, which is a contradiction with $\mathbb{F}_q$ does not have characteristic 2. Therefore, $deg(x) = |L| - |A| - |B| - 1$. Suppose that $A = \{y_1, \ldots, y_n\}$ has n distinct elements. Since $g \in \operatorname{Im} ad_x$, there exists $y_1 \in V(G_L^g)$ such that $[x, y_1] = g$. If $y \in C_L(x)$, then $y + y_1 \in A$. Also, if $y_i \in A$ for all $1 \leq i \leq n$, then $y_i - y_n \in C_L(x)$. So, $|A| = |C_L(x)|$. By a similar method, one can see that $|B| = |C_L(x)|$. Hence $deg(x) = |L| - 2|C_L(x)| - 1$, by (2.1). If $\mathrm{Char}\mathbb{F}_q = 2$, then $g = -g$ and so $A = B$. On the other hand, $|A| = |C_L(x)|$. Thus $deg(x) = |L| - |C_L(x)| - 1$, by (2.1).
(ii) Suppose that $g = 0$. Then

$$deg(x) = |\{y \in V(G_L^g) \mid [x, y] \neq 0\}| = |V(G_L^g)| - |C_L(x)| = |L| - |C_L(x)|.$$

(iii) Since $g \notin \operatorname{Im} ad_x$, then $-g \notin \operatorname{Im} ad_x$ and there is no element y of L such that $[x, y] = g$ and $-g$. Hence x is the vertex of G_L^g that is adjacent to all vertices of G_L^g. Therefore $deg(x) = |L| - 1$. □

Note that if $g = 0$, then the non-commuting graph is a subgraph of G_L^g. In addition, G_L^g is a null graph when $g = 0$ and L is abelian.
In the next lemma, we show that G_L^g is a complete graph whenever $g \neq 0$ and L is abelian.

Lemma 2.2. *Let L be an abelian Lie algebra and g be a non-zero element. Then G_L^g is isomorphic $K_{|L|}$ and $\gamma(G_L^g) = 1$.*

Proof. Since L is abelian, we have $[x, y] = 0$ for every $x, y \in V(G_L^g)$. Also, g is non-zero by our assumption thus $[x, y] \neq g$ and $-g$ for all $x, y \in V(G_L^g)$. Hence all vertices are adjacent and so G_L^g is a complete graph. Therefore $G_L^g \cong K_{|L|}$. It is obvious that the domination number of the complete graph $K_{|L|}$ is equal to 1. □

From now on, we assume that all Lie algebras are non-abelian and g is a non-zero element.

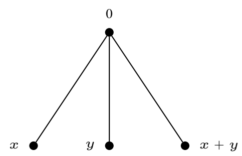
Proposition 2.3. *Let L be a Lie algebra. Then $\gamma(G_L^g) = 1$.*

Proof. We know that $0 \in Z(L)$ thus $[x, 0] = 0$ for every $x \in V(G^g_L)$. Since g is a non-zero element, therefore $[x, 0] \neq g$ and $-g$ for all $x \in V(G^g_L)$. So, $\{0\}$ is the dominating set of G^g_L and consequently $\gamma(G^g_L) = 1$. □

In the following proposition, we determine the graph G^g_L when L is a Lie algebra of dimension two or three.

Proposition 2.4. *Let L be a 2-dimensional Lie algebra over the field $\mathbb{F}_q$.*

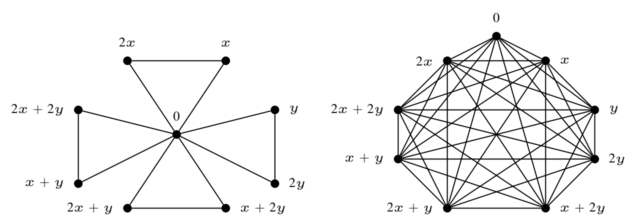
(i) If $q = 2$, then G^g_L is isomorphic to one of graphs in Figure 1.



(A) G^x_L (B) $G^g_L(g \notin L^2)$

Figure 1

(ii) If $q = 3$, then G^g_L is isomorphic to one of graphs in Figure 2.



(A) G^x_L (B) $G^g_L(g \notin L^2)$

Figure 2

(iii) If $q = 5$, then G^g_L has a subgraph that is isomorphic to K_6 or K_{25}.

Proof. **(i)** We know that every 2-dimensional non-abelian Lie algebra L is isomorphic to $\langle x, y \mid [x, y] = x\rangle$, by [4, Theorem 3.1]. So, $Z(L) = \{0\}$. Since $q = 2$, we have $V(G^g_L) = \{0, x, y, x + y\}$. On the other hand, $L^2 = \langle x\rangle$ and $g \notin Z(L)$ thus $g = x$ or $g \notin L^2$. If $g = x$, then the vertices x, y and $x + y$ are only adjacent to 0. So, G^g_L is isomorphic to the part A of Figure 1. If $g \notin L^2$, then $g \notin \operatorname{Im} ad_x$ for all $x \in V(G^g_L)$ and so all vertices are adjacent by the part (iii) of Lemma 2.1. Hence G^g_L isomorphic to the part B of Figure 1.
(ii) By a similar method of the part (i), L is isomorphic to $\langle x, y \mid [x, y] = x\rangle$, $Z(L) = \{0\}$, $V(G^g_L) = \{0, x, 2x, y, 2y, x + 2y, 2x + y, x + y, 2x + 2y\}$, and $L^2 = \{0, x, 2x\}$. Since g is non-central and $q = 3$, then $g = x$ and $-g = 2x$. Therefore $g = x$ or $g \notin L^2$. If $g = x$ and $-g = 2x$, then G^g_L is isomorphic to the part A of Figure 2. If $g \notin L^2$, then $g \notin \operatorname{Im} ad_x$ for all $x \in V(G^g_L)$. Therefore the degree of all vertices are equal to 8 by the part (iii) of Lemma 2.1, so $G^g_L \cong K_9$. Hence G^g_L is isomorphic to the part B of Figure 2.
(iii) As the same method of the part (i), $Z(L) = \{0\}$, $L^2 = \langle x\rangle$. So, $L^2 = \{0, x, 2x, 3x, 4x\}$. Assume that $g \in L^2$. If g is x or $2x$, then $-g$ is $4x$ or $3x$, respectively and the vertices $\{0, x, 4x, 3y, 2y, x+2y\}$ or $\{0, x, 4x, y, 4y, x+y\}$ make a subgraph of G^g_L which is isomorphic to K_6. Also, the degree of all vertices is equal to 24 by the part (iii) of Lemma 2.1 and $G^g_L \cong K_{25}$ when $g \notin L^2$. □

Proposition 2.5. *Let L be a 3-dimensional Lie algebra over the field $\mathbb{F}_2$ with $\dim L^2 = 2$. Then G^g_L is isomorphic to K_8 when $g \notin L^2$ or the following graph if $g \in L^2$.*

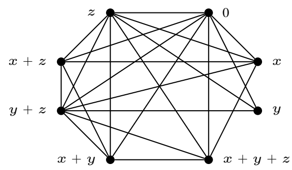


Figure 3

Proof. Assume that the set $\{y, z\}$ be a basis of L^2 and expand it to $\{x, y, z\}$ as a basis of L. Then $\{[x, y], [x, z], [y, z]\}$ generates L^2. Since L^2 is abelian by [4, Lemma 3.3], we have $[y, z] = 0$. On the other hand, dim $L^2 = 2$ thus $\{[x, y], [x, z]\}$ is another basis of L^2 and so the set $\{[x, y], [x, z]\}$ is linearly independent. Also, $V(G_L) = \{0, x, y, z, x+y, x+z, y+z, x+y+z\}$ thus we have the following brackets

$$\begin{aligned}
&[x, y], \quad [x, z], \quad [x, x+y] = [x, y], \quad [x, x+z] = [x, z], \quad [x, y+z] = [x, y] + [x, z],\\
&[x, x+y+z] = [x, y] + [x, z], \quad [y, z] = 0, \quad [y, x+y] = [y, x], [y, x+z] = [y, x],\\
&[y, y+z] = 0, \quad [y, x+y+z] = [y, x], \quad [z, x+y] = [z, x],\\
&[z, x+z] = [z, x], \quad [z, y+z] = 0, \quad [z, x+y+z] = [z, x],\\
&[x+y, x+z] = [x, z] + [y, x], \quad [x+y, y+z] = [x, y] + [x, z],\\
&[x+y, x+y+z] = [x, z], \quad [x+z, y+z] = [x, y] + [x, z],\\
&[x+z, x+y+z] = [x, y], \quad [y+z, x+y+z] = [y, x] + [z, x],
\end{aligned} \tag{2.2}$$

and $[0, L] = 0$. If $g \in L^2 = \langle [x, y], [x, z] \rangle \setminus \{0\}$, then $g \in \{[x, y], [x, z], [x, y] + [x, z]\}$. Without loss generality, let $g = [x, y]$. Then G_L^g is isomorphic to Figure 3 by the brackets (2.2). If $g \notin L^2$, then G_L^g is isomorphic to K_8 by the part (iii) of Lemma 2.1. □

Proposition 2.6. *Let L be a 3-dimensional Lie algebra over the field $\mathbb{F}_2$ with* dim $L^2 = 3$. *Then G_L^g is isomorphic to K_8 when $g \notin L^2$, or the following graph when $g \in L^2$.*

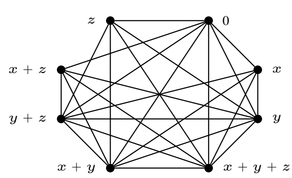


Figure 4

Proof. The proof is obtained similar to the proof [3, Proposition 3.3] and Proposition 2.5. □

Proposition 2.7. *Let L be a 3-dimensional Lie algebra over the field $\mathbb{F}_2$ with* dim $L^2 = 1$. *Then G_L^g is isomorphic to K_8, or the following graph when $g \notin L^2$, or $g \in L^2$, respectively.*

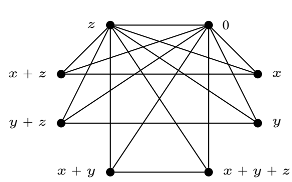


Figure 5

Proof. The proof is proved by a similar method in the proof of [3, Proposition 3.4] and Proposition 2.5. □

Theorem 2.8. *Let L be a 3-dimensional Lie algebra over the field $\mathbb{F}_2$. Then G^g_L is isomorphic to K_8 or one of the Figures 3, 4 or 5.*

Proof. Since L is non-abelian, we have $\dim L^2 \neq 0$ and $\dim L/Z(L) \geq 2$. Also, $\dim L = 3$ which implies that $1 \leq \dim L^2 \leq 3$ and $\dim Z(L) \leq 1$. Now, the proof follows by [3, Lemma 3.1], Propositions 2.5, 2.6 and 2.7. □

Proposition 2.9. *Let L be a Lie algebra over the field $\mathbb{F}_q$. Then*

(i) G^g_L has a pendant vertex if and only if G^g_L is isomorphic to the graph A of Figure 1.

(ii) G^g_L has a vertex of degree of 2 if and only if G^g_L is isomorphic to the graph A of Figure 2.

(iii) G^g_L has a vertex of degree of 3 if and only if G^g_L is isomorphic to the graphs B of Figure 1 and Figures 5 and 3.

(iv) There is no G^g_L with a vertex of degree of 4.

(v) G^g_L has a vertex of degree of 5 if and only if G^g_L is isomorphic to the graphs of Figures 3 and 4.

(vi) There is no G^g_L with a vertex of degree of 6.

Proof. We just prove the part (i). Other parts can be proved by a similar method.
(i) Since there is a vertex x such that $deg(x) = 1$ and $g \neq 0$, then $deg(x)$ is equal to $|L| - 2|C_L(x)| - 1$, $|L| - |C_L(x)| - 1$ or $|L| - 1$, by Lemma 2.1. Suppose that $|L| = q^n$ and $|C_L(x)| = q^m$ such that $n > m$. If $|L| - 2|C_L(x)| - 1 = 1$, then $q^m(q^{n-m} - 2) = 2$. In this case, we know that $q \neq 2$ and so $m = 0$, which is a contradiction. Let $|L| - |C_L(x)| - 1 = 1$. Then $q^m(q^{n-m} - 1) = 2$. On the other hand, this case occurs when $q = 2$. Thus $m = 1$ and $n = 2$. Hence L is a 2-dimensional Lie algebra over the field $\mathbb{F}_2$ and so G^g_L is isomorphic to the graph A in the Figure 1, by Proposition 2.4. Also, if $|L| - 1 = 1$, then $|L| = 2$ and L is abelian. It is a contradiction, by our assumptions. □

We state some facts about being regular of G^g_L.

Proposition 2.10. *Let L be a Lie algebra. Then G^g_L is regular if and only if $g \notin L^2 \setminus \{0\}$.*

Proof. By contrary, let $g \in L^2 \setminus \{0\}$. Then there are two vertices $x_1, x_2 \in L$ such that $[x_1, x_2] = g$. On the other hand, the zero element is adjacent to all elements of $L \setminus \{0\}$. Hence G^g_L is not regular, which is a contradiction. Conversely, assume that $g \notin L^2 \setminus \{0\}$. Then $g \notin \operatorname{Im} ad_x$ for all $x \in V(G^g_L)$. Hence $deg(x) = |L| - 1$, by the part (iii) of Lemma 2.1 for every $x \in V(G^g_L)$ and so G^g_L is regular. □

Lemma 2.11. *Let L be a Lie algebra. Then concepts complete and regular are equivalent for the graph G^g_L.*

Proof. It is clear that every complete graph is regular. Assume that G^g_L is regular. Since $g \neq 0$ and $[0, x] = 0$ for every $x \in L$, we have $deg(0) = |L| - 1$. On the other hand, G^g_L is regular thus every vertex is equal to $|L| - 1$. Hence G^g_L is complete. □

Corollary 2.12. *Let L be an n-dimensional Lie algebra over the field $\mathbb{F}_q$ and G^g_L be r-regular. Then $G^g_L \cong K_{q^{r+1}}$.*

Proof. By the proof of previous lemma, $deg(x) = |L| - 1$ for all $x \in L$. Also, $|L| = q^n$ and G^g_L is r-regular thus $r = q^n - 1$. Hence $G^g_L \cong K_{q^{r+1}}$, by Lemma 2.11. □

3 The characterization of K_n-free g-noncommuting graph

In this section, we are going to obtain the structures of G^g_L when this graph is K_n-free for all $n \leq 5$ and g is a non-zero element.
In the next proposition, we obtain a lower bound for the clique number of G^g_L provided that $g \notin Z(L)$.

Proposition 3.1. *Let L be a Lie algebra over the field $\mathbb{F}_q$ and $g \notin Z(L)$. Then $\omega(G_L^g) \geq q|Z(L)|$.*

Proof. We claim that $\langle Z(L), g\rangle = \{z + \alpha g \mid \alpha \in \mathbb{F}_q, z \in Z(L)\}$ is a clique of G_L^g. Since $[z_1 + \alpha g, z_2 + \beta g] = 0$ for every $z_1 + \alpha g, z_2 + \beta g \in \langle Z(L), g\rangle$, so all vertices of $\langle Z(L), g\rangle$ are adjacent. Hence $\omega(G_L^g) \geq q|Z(L)|$. □

The following propositions and theorems play an important role in classification of Lie algebras when G_L^g is K_n-free such that $n \leq 5$.

Theorem 3.2. *Let L be an n-dimensional Lie algebra over the field $\mathbb{F}_2$ for some $n \geq 3$ and $|Z(L)| = 1$. Then G_L^g has a subgraph isomorphic to K_5 or K_8 for every $g \notin Z(L)$.*

Proof. If $g \notin L^2$, then $deg(v) = |L| - 1$ for every $v \in L$, by the part (iii) of Lemma 2.1. So, G_L^g is complete and isomorphic to $K_{|L|}$ such that $|L| \geq 8$.
Since L is non-abelian, there are elements $x, y \in L$ such that $[x, y] \neq 0$. Also, there exist $z \in L$ such that $A = \{x, y, z\}$ is linearly independent. Thus $C = \{[x, y], [x, z], [y, z]\}$ and so $1 \leq \dim C \leq 3$. Now, we may consider the following cases:
Case (i) $\dim C = 1$. Then $[x, y], [x, z] = \alpha[x, y], [y, z] = \beta[x, y]$ for some $\alpha, \beta \in \mathbb{F}_2$. Put $z' = \beta x + \alpha y - z$. So, $\{x, y, z'\}$ is linearly independent and $[x, z'] = [y, z'] = 0$. It implies that C is $\{[x, y] = z, [x, z'] = [y, z'] = 0\}$, or $\{[x, y] = x, [x, z'] = [y, z'] = 0\}$. We know that there is no 3-dimensional Lie algebra over the field $\mathbb{F}_2$ with $\dim L^2 = 1$ and $|Z(L)| = 1$. So, $\dim L \geq 4$. Since $\dim L \geq 4$ and $|Z(L)| = 1$, there is an element $w \in L$ such that the set $\{x, y, z', w\}$ is linearly independent and $[z', w] \neq 0$. Assume that $\{[x, y] = z', [x, z'] = 0, [y, z'] = 0, [x, w], [y, w], [z', w]\}$. If $g \neq z'$, then $\{x, y, x + y, z', 0\}$ is a subgraph of G_L^g, which is isomorphic to K_5. Let $g = z'$. In this case, we claim that both of $[x, w]$ and $[y, w]$ are not belong to $\langle z'\rangle$. Otherwise, $[x, [y, w]] + [w, [x, y]] + [y, [w, x]] = 0$, by Jacobian identity. It implies that $[w, z'] = 0$, which is a contradiction. Hence $\{0, w, z', w + z', x\}$ or $\{0, w, z', w + z', y\}$ is a subgraph isomorphic to K_5 provided that $\{[x, y], [w, z']\}$ is linearly independent. Also, $\{0, x, w, x + w, x + z'\}$ or $\{0, y, w, y + w, y + z'\}$ is a subgraph isomorphic to K_5 when $\{[x, y], [w, z']\}$ is linearly dependent. Now, suppose that $C = \{[x, y] = x, [x, z'] = [y, z'] = 0\}$. If $g \neq x$, then it makes a subgraph of G_L^g isomorphic to K_5 by a similar method. If at least one of the brackets $[x, w]$ or $[w, y]$ is linearly independent with x, then one can see that there is subgraphs $\{0, w, z', w+z', x\}$ or $\{0, w, z', w+z', y\}$ that is isomorphic to K_5 if $\{[x, y], [w, z']\}$ is linearly independent. Furthermore, we have the subgraph $\{0, w, z', w + z', x + y\}$ isomorphic to K_5 when $[x, w] = [w, y] = x$ and $\{[x, y], [w, z']\}$ is linearly independent. Suppose that $\{[x, y], [w, z']\}$ is a dependent set. Also,

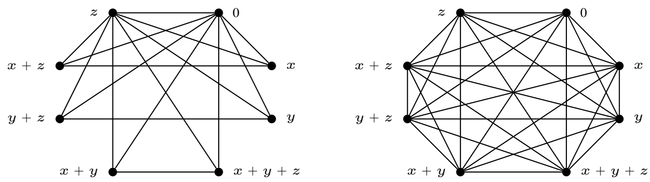
$$x = [x, y] = [y, [w, x]] = [z', [y, w]] + [w, [z', y]] = [z', [y, w]]$$

by Jacobian identity and if $[y, w] \in \langle x\rangle$, then $x = 0$. It is a contradiction. Therefore there exists subgraph $\{0, y, w, y + w, y + z'\}$ isomorphic to K_5 provided that $\{[x, y], [w, z']\}$ is linearly dependent.
Case (ii) $\dim C = 2$. Without loss of generality, let $\{[x, y], [x, z]\}$ be linearly independent. Let $[y, z] = 0$. Then we have the subgraphs $\{0, x, z, y + z, x + z\}$, $\{0, y, x, y + x, y + z\}$, or $\{0, x, y, z, x + y\}$ when g is equal to $[x, y]$, $[x, z]$, or $[x, y] + [x, z]$, respectively. Suppose that $[y, z] \neq 0$. So, there are the subgraphs $\{0, y, z, y + z, x + z\}$, $\{0, y, x, y + x, y + z\}$, or $\{0, x, y, z, x + z\}$ when g is equal to $[x, y]$, $[x, z]$, or $[x, y] + [x, z]$, respectively.
Case (iii) $\dim C = 3$. Then $\{[x, y], [x, z], [y, z]\}$ is linearly independent and by considering the brackets (2.2), one can see that there is at least a subgraph which is isomorphic to K_5. □

Theorem 3.3. *Let L be an n-dimensional Lie algebra over the field $\mathbb{F}_q$ for some $q \geq 3$, $n \geq 3$ and $|Z(L)| = 1$. Then G_L^g for every $g \notin Z(L)$ has a subgraph isomorphic to K_{q^2}.*

Proof. Since L is a Lie algebra over the field $\mathbb{F}_q$ for some $q \geq 3$, we have $g \neq -g$ for all $g \in L \setminus \{0\}$. On the other hand, $\dim L \geq 3$ and $|Z(L)| = 1$ thus $\dim L^2 \geq 2$, by [6, Proposition 2.1]. So, there exist elements g_1 and g_2 such that $[g_1, g_2]$ is not equal to $g, -g$ and 0. Hence $\{\alpha g_1 + \beta g_2 \mid \alpha, \beta \in \mathbb{F}_q\}$ makes a subgraph isomorphic to K_{q^2}. □

Theorem 3.4. *Let L be a Lie algebra such that $|Z(L)| = 2$ and $g \notin Z(L)$. Then G_L^g has a subgraph isomorphic to $K_{|L|}$, where $|L| \geq 8$ or is isomorphic to one of the following graphs:*



(A) G_L^x (B) $g \notin L^2$

Figure 6

Proof. Consider the following cases:
Case (i) $\dim L^2 = 1$. Since $|Z(L)| = 2$, therefore L is a Lie algebra over the field $\mathbb{F}_2$ and L is isomorphic to $H(m) = \langle x_i, y_i, z \mid [x_i, y_i] = z,\ 1 \le i \le m\rangle$ for some $m \ge 1$ or $\langle x, y, z \mid [x, y] = x\rangle$ by [6, Proposition 2.1]. If $L \cong H(m)$ for some $m \ge 1$, then $L^2 = Z(L) = \langle z\rangle$. As $g \notin Z(L)$, we have $g \notin L^2$ and all vertices are adjacent by the part (iii) of Lemma 2.1. Therefore G_L^g is isomorphic to $K_{|L|}$, where $|L| \ge 8$. If $\langle x, y, z \mid [x, y] = x\rangle$, then $L^2 = \langle x\rangle$ and $Z(L) = \langle z\rangle$. Since $g \notin Z(L)$, we have $g = x$ or $g \notin L^2$. Suppose that $g = x$ thus G_L^g is isomorphic to the part A of Figure 6. Also, G_L^g is isomorphic to K_8, by the part (iii) of Lemma 2.1 when $g \notin L^2 \setminus Z(L)$.
Case (ii) $\dim L^2 \ge 2$. In this case, there exist elements $x, y \in L$ such that $[x, y]$ is non-zero and $[x, y] \ne g$. Since $|Z(L)| = 2$, we can consider $Z(L) = \langle z\rangle$. Hence $\{\alpha x + \beta y + \gamma z \mid \alpha, \beta, \gamma \in \mathbb{F}_2\}$ forms a subgraph of G_L^g which is isomorphic to K_8. □

The following proposition gives a similar result to Theorem 3.4 where $g \in Z(L) \setminus \{0\}$. The proof is very similar and we omit here.

Proposition 3.5. *Let L be a Lie algebra such that $|Z(L)| = 2$ and $g \in Z(L) \setminus \{0\}$. Then G_L^g has a subgraph isomorphic to one of the graphs in Figure 6.*

Proposition 3.6. *Let L be a Lie algebra, $g \in Z(L) \setminus \{0\}$, and $|Z(L)| \ge 3$. Then G_L^g has a subgraph isomorphic to K_8.*

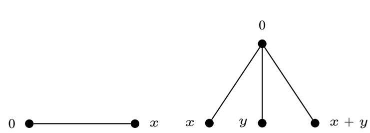
Proof. Since $|Z(L)| \ge 3$, we have $\{0, g_1, g_2\} \subseteq Z(L)$. Also, L is non-abelian thus there is an element $g \in L \setminus Z(L)$. Therefore $\{\alpha g + \beta g_1 + \gamma g_2 \mid \alpha, \beta, \gamma \in \mathbb{F}_2\}$ makes a subgraph of G_L^g like H such that $H \cong K_8$. □

In the next theorems, we determine the structure G_L^g when this graph is a K_n-free, where $n \le 5$.
Note that in the following propositions and theorems, we consider both abelian and non-abelian Lie algebras

Theorem 3.7. *Let L be a Lie algebra and g be a non-zero element. Then the following conditions are equivalent:*

(i) G_L^g is K_3-free,

(ii) G_L^g is isomorphic to one of the following graphs:



(A) (B)

Figure 7

Proof. (**i**) → (**ii**) Assume that G_L^g is K_3-free, L is non-abelian and $g \notin Z(L)$. Thus $\omega(G_L^g) \le 2$ and since $q|Z(L)| \le \omega(G_L^g)$ by Proposition 3.1, we have $|Z(L)| = 1$ and $q = 2$. We show that $\deg(g) = 1$. By contrary, let $\deg(g) \ge 2$. Then there is at least g_1 such that $[g, g_1] \ne g$

and $-g$. So, the set vertex $\{g, g_1, g+g_1\}$ makes a triangle, which is a contradiction. Hence $\deg(g) = 1$. On the other hand, $q = 2$ thus $\deg(g)$ is equal to $|L| - |C_L(g)| - 1$, or $|L| - 1$, by Lemma 2.1. If $\deg(g) = |L| - 1$, then $|L| = 2$ and L is abelian. It is a contradiction. Thus $\deg(g) = |L| - |C_L(g)| - 1$. Now, let $|L| = 2^n$ and $|C_L(g)| = 2^m$, where $n > m > 0$. Then $2^m(2^{n-m} - 1) = 2$. If $2^m = 1$, then $m = 0$, which is a contradiction. So, $2^m = 2$ and $2^{n-1} = 2$. Therefore $n = 2$ and G_L^g is isomorphic to the part B of Figure 7, by Proposition 2.4. If $g \in Z(L) \setminus \{0\}$ and L is non-abelian, then there exist $g_1 \in L \setminus Z(L)$. Hence the vertex set $\{g, g_1, g+g_1\}$ makes a subgraph that is isomorphic to K_3. It is a contradiction. Now, assume that L is an abelian Lie algebra. Thus $G_L^g \cong K_{|L|}$, by Lemma 2.2. Since G_L^g is K_3-free, G_L^g is isomorphic to the part A of Figure 7.
(**ii**) $\rightarrow$ (**i**) The proof is obvious. □

Theorem 3.8. *Let L be a Lie algebra and g be a non-zero element. Then the following conditions are equivalent:*

(i) G_L^g is K_4-free,

(ii) G_L^g is isomorphic to one of the following graphs:

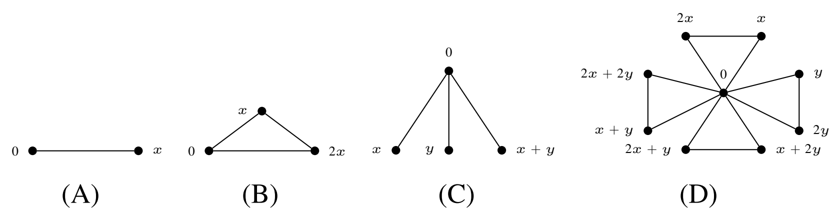


Figure 8

Proof. (**i**) $\rightarrow$ (**ii**) Suppose that G_L^g is K_4-free. If L is abelian, then $G_L^g \cong K_{|L|}$, by Lemma 2.2. As G_L^g is K_4-free, then $|L| \leq 3$ and so G_L^g is isomorphic to the parts A and B in Figure 8. Let L be non-abelian and $g \notin Z(L)$. Since G_L^g is K_4-free, we have $q|Z(L)| \leq \omega(G_L^g) \leq 3$, by Proposition 3.1. It implies that the followong cases:
Case (i) $q = 2$ and $|Z(L)| = 1$. Assume that $\dim L = 2$. Thus G_L^g is isomorphic to the part C of Figure 8, , by Proposition 2.4. If $\dim L \geq 3$, then this case does not happen, by Theorem 3.2.
Case (ii) $q = 3$ and $|Z(L)| = 1$. By Theorem 3.3, the graph G_L^g is not K_4-free when $\dim L \geq 3$, $q = 2$ and $|Z(L)| = 1$. Hence $\dim L = 2$ and so G_L^g is isomorphic to the part D of Figure 8, by Proposition 2.4.
Assume that L is non-abelian and $g \in Z(L) \setminus \{0\}$. Thus there is $g_1 \in L \setminus Z(L)$ and the vertex set $\{0, g, g_1, g+g_1\}$ makes a subgraph that is isomorphic to K_4. It is a contradiction.
(**i**) $\rightarrow$ (**ii**) The proof is obvious. □

Theorem 3.9. *Let L be a Lie algebra and g be a non-zero element. Then*

(i) G_L^g is K_5-free,

(ii) G_L^g is isomorphic to one of the following graphs:

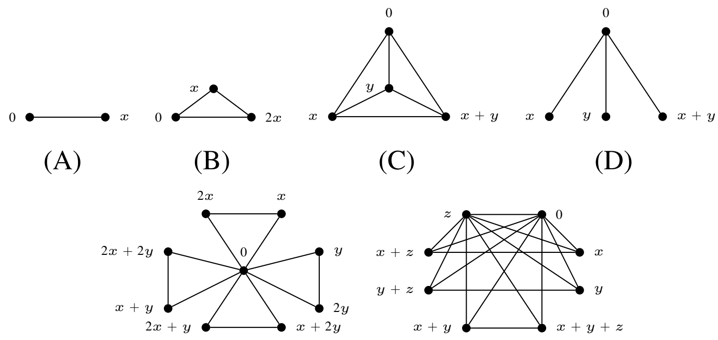


Figure 9

Proof. (**i**) → (**ii**) Suppose that L is non-abelian and $g \notin Z(L)$. As G_L^g is K_5-free, so $\omega(G_L^g) \leq 4$. Since $q|Z(L)| \leq \omega(G_L^g)$ by Proposition 3.1, we have the following cases:
Case (i)$|Z(L)| = 1$ and $q = 2$. If $\dim L = 2$, then G_L^g is isomorphic to the graphs C and D in Figure 9 by the part (i) of Proposition 2.4. If $\dim L \geq 3$, then there is no K_5-free graph G_L^g by Theorem 3.2.
Case (ii) $|Z(L)| = 1$ and $q = 3$. In this case, G_L^g is isomorphic to the graph E in Figure 9, by the part (ii) of Proposition 2.4 when $\dim L = 2$. Also, there does not exist G_L^g, by Theorem 3.3 when $\dim L \geq 3$.
Case (iii) $|Z(L)| = 2$ and $q = 2$. By Proposition 3.4, G_L^g is isomorphic to the part F of Figure 9 when $|Z(L)| = 2$ and $q = 2$.
Assume that L is non-abelian and $g \in Z(L) \setminus \{0\}$, therefore $|Z(L)| \geq 2$. If $|Z(L)| = 2$, then G_L^g is isomorphic to the part F in Figure 9, by Propositions 3.5. In addition, G_L^g has a subgraph isomorphic to K_8 when $|Z(L)| \geq 3$, by Proposition 3.6. Hence G_L^g is not K_5-free.
Let L be abelian and $g \neq 0$. Then $G_L^g \cong K_{|L|}$ by Lemma 2.2. Since G_L^g is K_5-free, so G_L^g is isomorphic to the parts A or B of Figure 9.
(**ii**) → (**i**) The proof of this part is clear. □

Proposition 3.10. *Let L be a Lie algebra and g be a non-zero element. Then G_L^g is tree if and only if G_L^g is isomorphic to one of the graphs Figure 7.*

Proof. If L is abelian, then $G_L^g \cong K_{|L|}$, by Lemma 2.2. So, G_L^g is tree when $|L| = 2$ and then G_L^g is isomorphic to the part A of Figure 7. Since G_L^g is tree, there is a vertex x of degree one. Suppose that L is a non-abelian Lie algebra over the field $\mathbb{F}_q$ with the characteristic 2. Then $deg(x) = |L| - |C_L(x)| - 1$, by the part (ii) of Lemma 2.1. Thus $\deg(x) = |L| - |C_L(x)| - 1$. Now, let $|L| = 2^n$ and $|C_L(g)| = 2^m$, where $n > m > 0$. Then $2^m(2^{n-m} - 1) = 2$. If $2^m = 1$, then $m = 0$, which is a contradiction. So, $2^m = 2$ and $2^{n-1} = 2$. It implies that $|L| = 4$ and so G_L^g is isomorphic to the part B of Figure 7, by Proposition 2.4. If $\mathbb{F}_q$ is a field of characteristic not equal to 2, then $deg(x) = |L| - 2|C_L(x)| - 1$, by the part (i) of Lemma 2.1. By considering $|L| = q^n$ and $|C_L(x)| = q^m$, we have $q^m(q^{n-m} - 2) = 2$. One can see that this case does not occur. Conversely, if G_L^g is isomorphic to one of the graphs in the Figure 7, then it is clear that G_L^g is tree. □

Corollary 3.11. *Let L be a Lie algebra and g be a non-zero element. Then the following conditions are equivalent.*

(i) G_L^g *is tree,*

(ii) G_L^g *is triangle free,*

(iii) G_L^g *is isomorphic to one of the graphs in Figure 7,*

Proof. The proof follows from Proposition 3.10 and Theorem 3.7. □

References

[1] D. F. Anderson, T. Asir, A. Badawi and T. T. Chelvam, *Graphs from rings*, Springer International Publishing, New York, NY, USA, (2021).

[2] R. B. Bapat, *Graphs and matrices*, London, Springer London, (2010).

[3] A. Charekhah Moghdddan, A. Erfanian and A. Shamsaki, *On the non-commuting graph associated to a finite-dimensional Lie algebra*, J. Indones. Math. Soc., **Accepted**, (2025).

[4] K. Erdmann and M. J. Wildon, *Introduction to Lie algebras*, Vol. 122, Springer, London, (2006).

[5] A. Shamsaki, A. Erfanian and M. Parvizi, *On the commutativity degree of a finite-dimensional Lie algebra*, **arXiv preprint arXiv:2406.10064**, (2024).

[6] A. Shamsaki, A. Erfanian and M. Parvizi, *On the values of commutativity degree of Lie algebras*, **arXiv preprint arXiv:2406.10077**, (2024).

Author information

A. Shamsaki, Department of Pure Mathematics Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran.
E-mail: Shamsaki.afsaneh@yahoo.com

G. Rachmaputri, Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia.
E-mail: gantina.r@itb.ac.id

I. Muchtadi Alamsyah, Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia.
E-mail: ntan@itb.ac.id

A. Erfanian, Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Iran.
E-mail: erfanian@um.ac.ir

Received: 2025-09-18
Accepted: 2026-02-02