

Power series with epi-DCC/DCC_d on ideals

M. Khalifa

Communicated by: Ayman Badawi

MSC 2020 Classifications: Primary 13A15, 13E05, 13F05; Secondary 13C15.

Keywords and phrases: epi-DCC on ideals, DCC_d on ideals, Isoartinian ring, power series rings.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of the paper

Abstract. We prove that the power series ring $R[[X]]$ satisfies epi-DCC on ideals if and only if $R[[X]]$ satisfies DCC_d on ideals if and only if $R[[X]]$ is isoartinian if and only if R is a reduced Artinian ring. Let I be an ideal of R . We prove that the ring $R + XI[[X]]$ satisfies DCC_d on ideals if and only if R satisfies DCC_d on ideals, I is principal generated by an idempotent e and the ring $R/(1 - e)R$ is reduced and Artinian. We study also when each of the polynomial rings $R[X]$ and $R + XI[X]$ satisfies epi-DCC/DCC_d on ideals and when it is isoartinian.

1 Introduction

All rings considered are commutative with identity and modules are unital. Let R be a ring, I an ideal of R and M an R -module. R. Dastanpour and A. Ghorbani introduced two generalizations of Artinian rings as follows. A ring (not necessarily commutative) R satisfies divisibility on descending chain of right ideals (for short, DCC_d on right ideals) if, for every descending chain $I_1 \supseteq I_2 \supseteq \dots$ of right ideals of R , there exists a positive integer k such that for all $i \geq k$, $I_{i+1} = r_i I_i$ for some $r_i \in R$ [4, Definition 4.1]. An epimorphism of modules means a surjective homomorphism of modules. We say that M satisfies epi-DCC on submodules if, for every descending chain $M_1 \supseteq M_2 \supseteq \dots$ of submodules of M , there exists a positive integer k such that for each $i \geq k$, there exists an epimorphism $M_i \rightarrow M_{i+1}$ [5, Definition 4.1]. Artinian modules, semisimple modules and free modules over PIDs (principal ideal domains) are standard examples of modules with epi-DCC on submodules [5, Example 4.2]. We say that R satisfies epi-DCC on ideals if R , as a module over itself, satisfies epi-DCC on submodules [5]. A. Facchini and Z. Nazemian called M to be isoartinian if for every descending chain $M_1 \supseteq M_2 \supseteq \dots$ of submodules of M , there exists a positive integer n such that M_n is isomorphic to M_i for every $i \geq n$ [6]. The ring R is said to be isoartinian if R , as a module over itself, is isoartinian. The aim of this paper is to study when each of the following rings $R[[X]]$ (the power series ring), $R + XI[[X]]$, $R[X]$ (the polynomial ring) and $R + XI[X]$ satisfies epi-DCC on ideals/DCC_d on ideals/is isoartinian. Each isoartinian ring / ring with DCC_d on ideals satisfies epi-DCC on ideals but the converse is false (Example 2.2 and Example 2.3). In the second section, we prove that $R[[X]]$ satisfies epi-DCC on ideals if and only if $R[[X]]$ satisfies DCC_d on ideals if and only if $R[[X]]$ is isoartinian if and only if R is a reduced Artinian ring (Theorem 2.6). Let $\mathcal{A} = (A_i)_{i \geq 0}$ be an ascending chain of rings, $A = \bigcup_{i \geq 0} A_i$ and X an indeterminate over A . Let

$$\mathcal{A}[[X]] = \left\{ \sum_{i \geq 0} a_i X^i \in A[[X]] \mid \text{each } a_i \in A_i \right\}$$

be the ring of power series with coefficient of degree i in A_i and

$$\mathcal{A}[X] = \{a_0 + a_1 X + \dots + a_n X^n \in A[X] \mid n \in \mathbb{N}, \text{ each } a_i \in A_i\}$$

be the ring of polynomial with coefficient of degree i in A_i . Clearly, $\mathcal{A}[[X]]$ is a subring of $A[[X]]$ and containing $\mathcal{A}[X]$. We deduce that the ring $\mathcal{A}[[X]]$ satisfies epi-DCC on ideals if and

only if $\mathcal{A}[[X]]$ satisfies DCC_d on ideals if and only if $\mathcal{A}[[X]]$ is isoartinian if and only if for all i , $A_i = A_0$ is a reduced Artinian ring (Corollary 2.8). We prove that $R[X]$ satisfies epi-DCC on ideals if and only if $R[X]$ satisfies DCC_d on ideals if and only if $R[X]$ is isoartinian if and only if R is a reduced Artinian ring (Theorem 2.9). We deduce that the ring $\mathcal{A}[X]$ satisfies epi-DCC on ideals if and only if $\mathcal{A}[X]$ satisfies DCC_d on ideals if and only if $\mathcal{A}[X]$ is isoartinian if and only if for all i , $A_i = A_0$ is a reduced Artinian ring (Corollary 2.10).

The third section is devoted to study when each of the rings $R + XI[[X]]$ and $R + XI[X]$ satisfies epi-DCC/ DCC_d on ideals (respectively is isoartinian). We prove that if R is an integral domain and I is a proper ideal of R , then $R + XI[[X]]$ (respectively $R + XI[X]$) satisfies epi-DCC on ideals if and only if $I = 0$ and R is a PID (Corollary 3.2). We prove that if R is an integral domain, I is a proper ideal of R and $\mathcal{R} = R + XI[[X]]$ or $R + XI[X]$, then \mathcal{R} satisfies DCC_d on ideals (respectively is isoartinian) if and only if $I = 0$ and R is a PID (Corollary 3.3). We prove that if R satisfies epi-DCC on ideals, then R/P is a PID for each prime ideal P of R and R has Krull dimension ≤ 1 (Theorem 3.7). An ideal J of R is said to be an SFT-ideal of R if there exist a positive integer k and a finitely generated ideal F of R such that $F \subseteq J$ and $a^k \in F$ for all $a \in J$ ([2]). A ring R is said to be SFT if each ideal of R is an SFT-ideal. We deduce that if $R + XI[[X]]$ satisfies epi-DCC on ideals, then I is an SFT ideal of R (Corollary 3.8). We prove that the ring $R + XI[[X]]$ satisfies DCC_d on ideals if and only if R satisfies DCC_d on ideals, I is principal generated by an idempotent e and I is a reduced and Artinian ring (Theorem 3.10). We deduce that if R is a quasi-local ring, then $R + XI[[X]]$ satisfies DCC_d on ideals if and only if either ($I = 0$ and R satisfies DCC_d on ideals) or $I = R$ is a reduced Artinian ring (Corollary 3.14). We prove that if I is finitely generated, then $R + XI[X]$ satisfies DCC_d on ideals if and only if R satisfies DCC_d on ideals, I is principal generated by an idempotent e , $(1 - e)R$ is a radical ideal of R and every prime ideal of R containing $1 - e$ is maximal and finitely generated (Proposition 3.15). Let e be an idempotent element of R . We prove that $R + X(eR)[[X]]$ satisfies epi-DCC on ideals if and only if $R + X(eR)[X]$ satisfies epi-DCC on ideals if and only if R satisfies epi-DCC on ideals, $(1 - e)R$ is a radical ideal of R and every prime ideal of R containing $1 - e$ is maximal and finitely generated (Theorem 3.17). We prove that $R + X(eR)[[X]]$ is isoartinian if and only if $R + X(eR)[X]$ is isoartinian if and only if R is isoartinian, $(1 - e)R$ is a radical ideal of R and every prime ideal of R containing $1 - e$ is maximal and finitely generated (Theorem 3.18).

2 Power series rings with epi-DCC/ DCC_d on ideals

Throughout this paper, let R be a ring and X an indeterminate over R .

Proposition 2.1. *We have the following.*

- (i) *If R is isoartinian, then R satisfies epi-DCC on ideals.*
- (ii) *If R satisfies DCC_d on ideals, then R satisfies epi-DCC on ideals.*

Proof. 1) Trivial. 2) If $I \supseteq J$ are ideals of R such that $J = rI$ for some $r \in R$, then the map $I \rightarrow J$, which maps x to rx , is an epimorphism (i.e., a surjective homomorphism) of R -modules. □

The converse of Proposition 2.1-(2) is false. An example of a ring with epi-DCC on ideals (in fact isoartinian ring) but it does not satisfy DCC_d on ideals.

Example 2.2. Let F be a field and R the ring quotient of the algebra $F[x_1, x_2, \dots]$ modulo the relations $x_i x_j = 0$ for all positive integers i, j . Thus R is an isoartinian ring by [3, Example 3.1]. Then R satisfies epi-DCC on ideals. Note that $R = F \oplus \bigoplus_{i \geq 1} Fx_i$ is an F -vector space with an infinite countable basis $\{1, x_1, x_2, \dots\}$. For each positive integer j , let $I_j = \bigoplus_{i \geq j} Fx_i$ which is an ideal of R . Thus $I_1 \supseteq I_2 \supseteq \dots$ is a descending chain of ideals of R . Assume that there exists a positive integer n such that $I_{n+1} = rI_n$ for some $r \in R$. Since $x_i I_n = 0$ for all i , we can assume that $r = u \in F$. Since u is invertible in R , $I_{n+1} = I_n$ and so $x_n \in I_{n+1}$, a contradiction. Then $I_{j+1} \neq rI_j$ for all j and all $r \in R$. Hence R does not satisfy DCC_d on ideals.

The converse of Proposition 2.1-(1) is false. An example of a ring with epi-DCC on ideals but it is not isoartinian.

Example 2.3. Let K be a field, $(X_r)_{0 < r < 1}$ an uncountable set of algebraically independent indeterminates over K indexed by the real open interval $]0, 1[$ and $R = K[(X_r)_{0 < r < 1}]/I$ where I is the ideal of $K[(X_r)_{0 < r < 1}]$ generated by $(X_{r+t} - X_r X_t)_{r+t < 1}$ and $(X_r X_t)_{r+t \geq 1}$. Thus R is a chained ring with zero Krull dimension, with maximal ideal $M = (x_r)_{0 < r < 1}$, $\bar{M}^2 = M$ and every nonzero proper ideal $\neq M$ of R is principal generated by some x_r (the class of X_r modulo I) by [11, Example 3.13]. Then R satisfies DCC_d on ideals and so R satisfies epi-DCC on ideals by Proposition 1.2. By [11, Example 3.13], R does not satisfy DCC on annihilators. Hence R is not isoartinian by [6, Lemma 4.10].

R. Dastanpour and A. Ghorbani proved that an integral domain D satisfies DCC_d on ideals if and only if D is a PID [4, Corollary 4.6]. Also they proved that a perfect ring R satisfies DCC_d on ideals if and only if R is Artinian [4, Corollary 4.17].

Lemma 2.4. *Let D be an integral domain. The following statements are equivalent.*

- (i) D satisfies epi-DCC on ideals.
- (ii) D is isoartinian.
- (iii) D satisfies DCC_d on ideals.
- (iv) D is a PID.

Proof. i) \Leftrightarrow iv) By [5, Corollary 4.8].

ii) \Leftrightarrow iv) By [6, Corollary 4.5].

iii) \Leftrightarrow iv) By [4, Corollary 4.6]. □

It was proved that if I is an ideal of an isoartinian ring R such that I contains a regular element of R , then I is principal [3, Lemma 2.4]. In fact, this happens also in rings with epi-DCC on ideals as the following shows.

Lemma 2.5. *Assume that R satisfies epi-DCC on ideals and I an ideal of R . If I contains a regular element of R , then I is a principal ideal.*

Proof. Let $r \in I$ be a regular element of R . Thus $I \supseteq rR \supseteq rI \supseteq r^2R \supseteq r^2I \supseteq \dots$ is a descending chain of ideals of R and so there exist a positive integer n and an epimorphism $\varphi : r^n R \rightarrow r^n I$. Then $r^n I = \varphi(r^n R) = \varphi(r^n)R = r^n aR$ for some $a \in I$. Hence $I = aR$ because r^n is regular in R . □

For each $f \in R[[X]]$, $f^{(i)}$ denotes the coefficient of X^i in f .

Theorem 2.6. *The following statements are equivalent.*

- (i) $R[[X]]$ satisfies epi-DCC on ideals.
- (ii) $R[[X]]$ satisfies DCC_d on ideals.
- (iii) $R[[X]]$ is an isoartinian ring.
- (iv) R is a reduced Artinian ring (\Leftrightarrow isomorphic to a finite product of fields).

Proof. iv) \Rightarrow ii) If $R \cong K_1 \times \dots \times K_m$ (a ring isomorphism) where K_1, \dots, K_m are fields, then $R[[X]] \cong K_1[[X]] \times \dots \times K_m[[X]]$. Each $K_i[[X]]$ is a PID and so satisfies DCC_d on ideals by Lemma 2.4. Note that a finite product of rings with DCC_d on ideals satisfies DCC_d on ideals [4, Example 4.2-(b)]. Then $R[[X]]$ satisfies DCC_d on ideals.

iv) \Rightarrow iii) Similar argument as in the previous implication. Each $K_i[[X]]$ is a PID and so is an isoartinian ring by Lemma 2.4. In [6, Lemma 4.1], the authors proved that a finite product of isoartinian rings is isoartinian. Then $R[[X]]$ is isoartinian.

iii) \Rightarrow i) By Proposition 2.1.

i) \Rightarrow iv) First, we prove that each ideal of R is principal generated by an idempotent. Let I be an ideal of R . Then $I + XR[[X]]$ is an ideal of $R[[X]]$ and it contains the regular element X of $R[[X]]$. By Lemma 2.5, $I + XR[[X]]$ is a principal ideal of $R[[X]]$. Let so $f \in R[[X]]$ such that $I + XR[[X]] = fR[[X]]$. Thus $I = f^{(0)}R$. There exists $g \in R[[X]]$ such that $X = fg$. Then

$$f^{(0)}g^{(0)} = 0 \quad \text{and} \quad 1 = f^{(0)}g^{(1)} + f^{(1)}g^{(0)}$$

Thus $f^{(0)} = (f^{(0)})^2g^{(1)}$ and so $e = e^2$ where $e = f^{(0)}g^{(1)}$. Therefore, $I = eR$. Each ideal of R is idempotent and so R is a von-Neumann regular ring by [1, Theorem 5]. Then R is a reduced ring with zero Krull dimension by [1, Theorem 2]. Since each ideal of R is principal, R is a Noetherian ring and so is Artinian. It is well known that a reduced Artinian ring is isomorphic to a finite product of fields. In fact, $R \cong \prod_P R/P$ where P range over prime ideals of R .

ii) \Rightarrow i) By Proposition 2.1. □

Example 2.2 provided an example of an isoartinian ring that is not Artinian (in fact it has zero Krull dimension but it is not Noetherian). The following provide an example of a one-dimensional Noetherian ring that is not isoartinian.

Example 2.7. An example of a one-dimensional Noetherian ring that is not isoartinian (in fact does not satisfy epi-DCC on ideals). Let K be a field, Y an indeterminate over K and consider the quotient ring $R = K[Y]/(Y^2)$. Let $T = R[[X]]$. Then R is a zero-dimensional Noetherian ring (so Artinian). Then T is Noetherian. By [2, Theorem 2], T has Krull dimension one. Since R is not reduced, T is not isoartinian by Theorem 2.6.

Let $\mathcal{A} = (A_i)_{i \geq 0}$ be an ascending chain of rings, $A = \bigcup_{i \geq 0} A_i$ and X an indeterminate over A . Let

$$\mathcal{A}[[X]] = \left\{ \sum_{i \geq 0} a_i X^i \in A[[X]] \quad \mid \quad \text{each } a_i \in A_i \right\}$$

be the ring of power series with coefficient of degree i in A_i

$$\text{and} \quad \mathcal{A}[X] = \{ a_0 + a_1 X + \dots + a_n X^n \in A[X] \quad \mid \quad n \in \mathbb{N}, \quad \text{each } a_i \in A_i \}$$

the ring of polynomial with coefficient of degree i in A_i . Clearly, $\mathcal{A}[[X]]$ is a subring of $A[[X]]$ and containing $\mathcal{A}[X]$. For more information and fundamental results on the rings $\mathcal{A}[[X]]$ and $\mathcal{A}[X]$, readers are referred to [8, 9].

Corollary 2.8. *The following statements are equivalent.*

- (i) $\mathcal{A}[[X]]$ satisfies epi-DCC on ideals.
- (ii) $\mathcal{A}[[X]]$ satisfies DCC_d on ideals.
- (iii) $\mathcal{A}[[X]]$ is an isoartinian ring.
- (iv) For all i , $A_i = A_0$ is a reduced Artinian ring.

Proof. i) \Rightarrow iv) Let n be a positive integer and $\mathcal{A}_n = (A_i)_{i \geq n}$. Thus

$$\mathcal{A}_n[[X]] = \left\{ \sum_{i \geq 0} a_i X^i \in A[[X]] \quad \mid \quad \text{each } a_i \in A_{i+n} \right\}$$

Note that $\mathcal{A}[[X]]$ is a subring of $\mathcal{A}_n[[X]]$ and $X^n \mathcal{A}_n[[X]] \subseteq \mathcal{A}[[X]]$. Thus $X^n \mathcal{A}_n[[X]]$ is an ideal of $\mathcal{A}[[X]]$. Note that $X^n \mathcal{A}_n[[X]]$ contains the regular element X^n of $\mathcal{A}[[X]]$. By Lemma 2.5, $X^n \mathcal{A}_n[[X]]$ is a principal ideal of $\mathcal{A}[[X]]$. Let so $f \in \mathcal{A}_n[[X]]$ such that $X^n \mathcal{A}_n[[X]] = X^n f \mathcal{A}[[X]]$. Thus $\mathcal{A}_n[[X]] = f \mathcal{A}[[X]]$. So f is invertible in the ring $\mathcal{A}_n[[X]]$ and thus $\mathcal{A}_n[[X]] = \mathcal{A}[[X]]$. Hence $A_n = A_0$. Therefore $A_0[[X]] = \mathcal{A}[[X]]$ and so A_0 is reduced and Artinian by Theorem 2.6.

iv) \Rightarrow iii) By Theorem 2.6.

iv) \Rightarrow ii) By Theorem 2.6.

ii)/iii) \Rightarrow i) By Proposition 2.1. □

The same argument given in Theorem 2.6 works also for the polynomial ring. We deduce that

Theorem 2.9. *The following statements are equivalent.*

- (i) $R[X]$ satisfies epi-DCC on ideals.
- (ii) $R[X]$ satisfies DCC_d on ideals.
- (iii) $R[X]$ is an isoartinian ring.
- (iv) R is a reduced Artinian ring.

The argument given in Corollary 2.8 works also for the polynomial ring. We deduce the following.

Corollary 2.10. *The following statements are equivalent.*

- (i) $\mathcal{A}[X]$ satisfies epi-DCC on ideals.
- (ii) $\mathcal{A}[X]$ satisfies DCC_d on ideals.
- (iii) $\mathcal{A}[X]$ is an isoartinian ring.
- (iv) For all i , $A_i = A_0$ is a reduced Artinian ring.

Corollary 2.11. *The ring $R[[X, Y]]$ (respectively $R[X, Y]$) never satisfies epi-DCC on ideals.*

Proof. For a ring A , $\dim A$ denotes the Krull dimension of A . It is well known that $\dim R[[X]] \geq 1 + \dim R \geq 1$ and so $R[[X]]$ is never Artinian because an Artinian ring has zero Krull dimension. Then $R[[X, Y]]$ never satisfies epi-DCC on ideals by Theorem 2.6. □

3 When $R + XI[[X]]$ satisfies epi-DCC/ DCC_d on ideals

Let I be an ideal of R . Now we turn to study when the ring $R + XI[[X]]$ satisfies epi-DCC/ DCC_d on ideals and when it is isoartinian. We proved that $R + XI[[X]]$ (respectively $R + XI[X]$) is isonoetherian if and only if (i) R is isonoetherian, (ii) I is idempotent and (iii) each ideal of R contained in I is finitely generated [11, Theorem 2.5]. Also we proved that $R + XI[[X]]$ satisfies ACC_d on ideals if and only if (i) R satisfies ACC_d on ideals, (ii) I is idempotent and (iii) each ideal of R contained in I is finitely generated [11, Theorem 3.8]. Note that in this case, I must be a principal ideal generated by an idempotent element.

Lemma 3.1. *If $R + XI[[X]]$ (respectively $R + XI[X]$) satisfies epi-DCC on ideals and r is a regular element of R , then $I = rI$.*

Proof. Let r be a regular element of R (so of $\mathcal{R} := R + XI[[X]]$) and $\mathcal{I} = rR + XI[[X]]$ which is an ideal of \mathcal{R} containing r . By Lemma 2.5, \mathcal{I} is principal and so $\mathcal{I} = f\mathcal{R}$ for some $f \in \mathcal{I}$. There exists $h \in \mathcal{R}$ such that $r = fh$ and so $r = f^{(0)}h^{(0)}$. Since $f \in \mathcal{I}$, $f^{(0)} \in rR$ and so $r \in rh^{(0)}R$. Since r is regular, $h^{(0)}$ is a unit of R . By induction, we show that $f^{(i)} \in rI$ for all $i \geq 1$. Since $r = fh$, $0 = f^{(0)}h^{(1)} + f^{(1)}h^{(0)}$ and so $f^{(1)} = -f^{(0)}h^{(1)}(h^{(0)})^{-1} \in rI$ because $h^{(1)} \in I$. Assume that $f^{(i)} \in rI$ for all $1 \leq i < n$. Since $0 = f^{(0)}h^{(n)} + \dots + f^{(n)}h^{(0)}$, $-f^{(n)} = (h^{(0)})^{-1}(f^{(0)}h^{(n)} + \dots + f^{(n-1)}h^{(1)}) \in rI$. Thus $f = rg$ for some $g \in R + XI[[X]]$. Therefore, $\mathcal{I} = r\mathcal{R}$. For all $a \in I$, there exists $\rho \in \mathcal{R}$ such that $aX = r\rho$ and so $a = r\rho^{(1)} \in rI$. Similar argument for the polynomial case. □

Corollary 3.2. *Let R be an integral domain and I a proper ideal of R . The ring $R + XI[[X]]$ (respectively $R + XI[X]$) satisfies epi-DCC on ideals if and only if $I = 0$ and R is a PID.*

Proof. (\Rightarrow) Assume that I contains a nonzero element r . By Lemma 3.1, $r \in rI$ and so $1 \in I$. Thus $I = R$, a contradiction. Then $I = 0$ and $R + XI[[X]] = R$. By Lemma 2.4, R is a PID. (\Leftarrow) If $I = 0$, then $R + XI[[X]] = R$ which is a PID and so satisfies epi-DCC on ideals by Lemma 2.4. \square

Corollary 3.3. *Let R be an integral domain and I a proper ideal of R and $\mathcal{R} = R + XI[[X]]$ or $R + XI[X]$. The ring \mathcal{R} satisfies DCC_d on ideals (respectively is isoartinian) if and only if $I = 0$ and R is a PID.*

Proof. By Corollary 3.2, Proposition 2.1 and Lemma 2.4. \square

Lemma 3.4. *If $H \supseteq J$ are two nonzero ideals of R and J contains a regular element of R , then each epimorphism $H \rightarrow J$ is an isomorphism.*

Proof. Let $\varphi : H \rightarrow J$ be an epimorphism and $r \in H$ such that $\varphi(r) = 0$. Let $c \in J$ a regular element of R and $a \in H$ such that $c = \varphi(a)$. Thus $cr = \varphi(a)r = \varphi(ar) = a\varphi(r) = a0 = 0$. Then $r = 0$. Hence φ is an isomorphism. \square

Remark 3.5. It follows that if R is an integral domain, then R satisfies epi-DCC on ideals if and only if R is isoartinian (this gives us a direct proof of $1) \Leftrightarrow 2)$ in Lemma 2.4).

Lemma 3.6. *Let $H \supseteq J \supset P$ be ideals of R such that P is a prime ideal of R . If there is an epimorphism $H \rightarrow J$, then $H/P \cong_{R/P} J/P$ (i.e., isomorphism of (R/P) -modules).*

Proof. Let $\varphi : H \rightarrow J$ be an epimorphism and $a \in J \setminus P$. For each $r \in P$, $a\varphi(r) = \varphi(ar) = r\varphi(a) \in P$ and so $\varphi(r) \in P$. Since $\varphi(P) \subseteq P$, the map $\psi : H/P \rightarrow J/P$, defined by $\psi(\bar{x}) = \overline{\varphi(x)}$, is well defined. It is straightforward to see that ψ is an epimorphism. Since $J \neq P$, ψ is an isomorphism by Lemma 3.4. \square

Theorem 3.7. *If R satisfies epi-DCC on ideals, then R/P is a PID for each prime ideal P of R (in particular, R has Krull dimension ≤ 1).*

Proof. Let $J_1 \supseteq J_2 \supseteq \dots \supseteq P$ be a descending chain of ideals of R containing P . There exists a positive integer k such that for all $i \geq k$, there is an epimorphism $J_i \rightarrow J_{i+1}$. By Lemma 3.6, $J_i/P \cong_{R/P} J_{i+1}/P$ for all $i \geq k$. Thus $J_i/P \cong_{R/P} J_k/P$ for all $i \geq k$. Then R/P is an isoartinian ring. By Lemma 2.4, R/P is a PID. For each minimal prime ideal P of R , R/P has Krull dimension ≤ 1 and so is R . \square

Corollary 3.8. *If $R + XI[[X]]$ satisfies epi-DCC on ideals, then I is an SFT ideal of R .*

Proof. Assume that I is not an SFT ideal of R and denote $\mathcal{R} = R + XI[[X]]$. Hizem ([10, the proof of Theorem 2.1]) proved that the ring \mathcal{R} contains an infinite ascending chain of prime ideals where she worked with the assumption that R is an integral domain. It seems that her argument works also in the case of a ring R with zero divisors. For the sake of completeness and clarity, we include the proof here. Arnold proved that if the ring R is not SFT, then $R[[X]]$ contains an infinite ascending chain of prime ideals (see the proof of [2, Theorem 1]). Since I is not SFT, we may choose a sequence $(a_i)_{i \geq 0}$ of elements of I such that $a_{k+1}^{k+1} \notin Ra_0 + \dots + Ra_k$ for all integers $k \geq 0$ ([2, page 300]). For each positive integer k , let $A_k = (a_0, \dots, a_k)$ be the finitely generated ideal of R generated by a_0, \dots, a_k . Let $A = \bigcup_k A_k$. For each positive integer m , Arnold choose a sequence $(a_{m,i})_{i \geq 0}$ defined inductively on m as follows. For $m = 1$, let

$a_{1,i} = a_i$ for all $i \geq 0$. Assume that $(a_{m,i})_{i \geq 0}$ is constructed for all $1 \leq m < n$. He defined $a_{n,i} = a_{n-1,i^2+1}$ for all $i \geq 0$. For each positive integer n , set

$$f_{(n)} = \sum_{i=0}^{\infty} a_{n,i} X^i$$

For $g = \sum_{i=0}^{\infty} b_i X^i \in R[[X]]$ (where $b_i \in R$) and n, m, μ, r integers such that $m \geq n \geq 1$ and $r \geq 0$, we say that the tuple (g, m, μ, r) has the property (n) if for all $i \geq r$, there exists an integer t_i such that the following hold

- $b_{t_i} = a_{m,i}^{\mu} + \alpha$ for some $\alpha \in A_{s_{i-1}}$ where $a_{m,i} = a_{n,k_i} = a_{1,s_i}$.
- $t_i \leq \mu k_i$.
- $b_i \in A_{s_{i-1}}$ for all $0 \leq j < t_i$.

For each positive integer n , we denote S_n to be the set of $g \in R[[X]]$ such that (g, m, μ, r) has the property (n) for some $m, \mu \geq 1$ and some $r \geq 0$. The set S_n is nonempty because it contains $f_{(n)}$ since $(f_{(n)}, n, 1, 0)$ satisfies the property (n) . Arnold proved that $S_1 \supset S_2 \supset \dots$ is a strictly descending chain of closed multiplicative subsets of $R[[X]]$ ([2, Lemma 2, Lemma 3 and page 302]). Also, he proved that $AR[[X]] \cap S_1 = \emptyset$ and there exists an infinite ascending chain $AR[[X]] \subseteq \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots$ of prime ideals of $R[[X]]$ such that $f_{(n)} \in \mathcal{P}_{n+1} \setminus \mathcal{P}_n$ for each n (see [2, the proof of Theorem 1]). Since $A \subseteq I$, each $f_{(n)} \in I[[X]] \subseteq \mathcal{R}$ which is a subring of $R[[X]]$. Thus $f_{(n)} \in (\mathcal{P}_{n+1} \cap \mathcal{R}) \setminus (\mathcal{P}_n \cap \mathcal{R})$ for each n . Then $\mathcal{P}_1 \cap \mathcal{R} \subset \mathcal{P}_2 \cap \mathcal{R} \subseteq \dots$ is an infinite ascending chain of prime ideals of \mathcal{R} , a contradiction because \mathcal{R} has Krull dimension ≤ 1 by Theorem 3.7. □

Lemma 3.9. *If $R + XI[[X]]$ (respectively $R + XI[X]$) satisfies DCC_d on ideals, then I is pure.*

Proof. We can assume that I is proper. Let $r \in I$. For each positive integer i , let \mathcal{H}_i be the ideal of $R + XI[[X]]$ generated by $(rX^j)_{j \geq i}$. Since $\mathcal{H}_1 \supseteq \mathcal{H}_2 \supseteq \dots$ is a descending chain of ideals of $R + XI[[X]]$, there exists a positive integer n and $f \in R + XI[[X]]$ such that $\mathcal{H}_{n+1} = f\mathcal{H}_n$. Since $frX^n \in \mathcal{H}_{n+1}$, there exist $g_1, \dots, g_m \in R + XI[[X]]$ such that $frX^n = rX^{n+1}g_1 + \dots + rX^{n+m}g_m$. Thus $rf^{(0)} = 0$. Since $rX^{n+1} \in f\mathcal{H}_n$, there exist $h_0, \dots, h_k \in R + XI[[X]]$ such that

$$rX^{n+1} = frX^n h_0 + \dots + frX^{n+k} h_k$$

Thus $r = (rfh_0)^{(1)} + (rfh_1)^{(0)} = rf^{(0)}h_0^{(1)} + rf^{(1)}h_0^{(0)} + rf^{(0)}h_1^{(0)} = rf^{(1)}h_0^{(0)} \in rI$ because $f^{(1)} \in I$. Similar argument for the polynomial case. □

Theorem 3.10. *The following statements are equivalent.*

- (i) $R + XI[[X]]$ satisfies DCC_d on ideals.
- (ii) The following hold
 - R satisfies DCC_d on ideals.
 - I is principal generated by an idempotent.
 - I is a reduced Artinian ring.

Proof. i) \Rightarrow ii) Denote $\mathcal{R} = R + XI[[X]]$. Since \mathcal{R} satisfies epi- DCC on ideals, I is an SFT ideal of R by Corollary 3.8. There exist a finitely generated ideal F of R and a positive integer k such that $r^k \in F$ for all $r \in I$. Let $r \in I$. By Lemma 3.9, $r = ra$ for some $a \in I$. Thus $r = ra^k \in F$ and so $I = F$ is finitely generated. Since I is pure, I is idempotent (i.e., $I = I^2$). Gilmer proved that an idempotent and finitely generated ideal is principal generated by an idempotent element [7, Lemma 1]. Since I is idempotent, I is principal generated by an idempotent e by [7, Lemma 1]. Note that eR (respectively $\bar{e}R$) is a ring with identity element e (respectively \bar{e} where $\bar{e} = 1 - e$). Moreover $R = eR \oplus \bar{e}R \cong eR \times \bar{e}R$ (a ring isomorphism).

Also $\mathcal{R} = e\mathcal{R} \oplus \bar{e}\mathcal{R} \cong eR \times \bar{e}R$. Note that $e\mathcal{R} = eR + X(eI)[[X]] = I[[X]]$ because $I = eR$ and $\bar{e}\mathcal{R} = \bar{e}R$. Then $\mathcal{R} \cong I[[X]] \times \bar{e}R$. In [4, Example 4.2-(b) and (c)], the authors proved that a finite product of rings satisfies DCC_d on ideals if and only if each ring member of this product satisfies DCC_d on ideals. Then each of the rings $I[[X]]$ and $\bar{e}R$ satisfies DCC_d on ideals. By Theorem 2.6, I is a reduced Artinian ring. Then $R \cong I \times \bar{e}R$ satisfies DCC_d on ideals.

ii) \Rightarrow i) Let e be an idempotent of R such that $I = eR$. By Theorem 2.6, $I[[X]]$ satisfies DCC_d on ideals. Since $R = I \oplus \bar{e}R$, $\bar{e}R$ satisfies DCC_d on ideals. Hence $\mathcal{R} = I[[X]] \oplus \bar{e}R$ satisfies DCC_d on ideals. □

Example 3.11. Let \mathbb{Z} be the ring of integers, \mathbb{Q} the field of rational numbers, $R = \mathbb{Q} \times \mathbb{Z}$, $e = (1, 0)$ and $w = (0, 1)$. Then R satisfies DCC_d on ideals, e is an idempotent element of R and the ring $Re = \mathbb{Q} \times 0$ is reduced and Artinian. Thus $R + X(eR)[[X]]$ satisfies DCC_d on ideals by Theorem 3.10. Also w is an idempotent element of R but the ring $Rw = 0 \times \mathbb{Z}$ is not Artinian. Therefore, $R + X(wR)[[X]]$ does not satisfy DCC_d on ideals by Theorem 3.10.

Example 3.12. Let R be a ring, X, Y two indeterminates over R , I an ideal of R , $\mathcal{R} = R + XI[[X]]$ and $\mathcal{I} = I[[X]]$ which is an ideal of \mathcal{R} . Then

$$\mathcal{R} + Y\mathcal{I}[[Y]] = R + XI[[X]] + YI[[X, Y]]$$

If R satisfies DCC_d on ideals, $I = eR$ for some idempotent e of R and I is a reduced Artinian ring, then $R + XI[[X]] + YI[[X, Y]]$ does not satisfy DCC_d on ideals. Indeed, \mathcal{R} satisfies DCC_d on ideals by Theorem 3.10. Also $\mathcal{I} = I[[X]] = (eR)[[X]] = e\mathcal{R}$ but \mathcal{I} is not an Artinian ring because $\dim I[[X]] \geq 1 + \dim I \geq 1$. Hence $\mathcal{R} + Y\mathcal{I}[[Y]]$ does not satisfy DCC_d on ideals by Theorem 3.10.

Corollary 3.13. *The following statements are equivalent.*

(i) $R + XI[[X]]$ satisfies DCC_d on ideals.

(ii) *The following hold*

- R satisfies DCC_d on ideals.
- I is principal generated by an idempotent e .
- $\bar{e}R$ is a radical ideal of R where $\bar{e} = 1 - e$.
- Every prime ideal of R containing \bar{e} is maximal and finitely generated.

Proof. Artinian rings are exactly Noetherian rings with zero Krull dimension. Let e be an idempotent of R and $I = eR$. Note that $I \cong R/\bar{e}R$ (ring isomorphism). It follows that I is a reduced Artinian ring if and only if so is $R/\bar{e}R$ if and only if $\bar{e}R$ is a radical ideal of R and every prime ideal of R containing \bar{e} is maximal and finitely generated. □

A ring with a unique maximal ideal is called a quasi-local ring.

Corollary 3.14. *Assume that R is quasi-local. The following statements are equivalent.*

(i) $R + XI[[X]]$ satisfies DCC_d on ideals.

(ii) *Either $(I = 0$ and R satisfies DCC_d on ideals) or $I = R$ is a reduced Artinian ring.*

Proof. By Theorem 3.10, Theorem 2.6 and the fact that in a quasi-local ring the only idempotent elements are 0 and 1. □

Proposition 3.15. *Assume that I is a finitely generated ideal of R . The following statements are equivalent.*

- (i) $R + XI[X]$ satisfies DCC_d on ideals.
- (ii) The following hold
 - R satisfies epi-DCC on ideals.
 - I is principal generated by an idempotent e .
 - $\bar{e}R$ is a radical ideal of R .
 - Every prime ideal of R containing \bar{e} is maximal and finitely generated.

Proof. If $R + XI[X]$ satisfies epi-DCC on ideals, I is pure and so $I = I^2$ by Lemma 3.9. By [7, Lemma 1], I is principal generated by an idempotent. The same argument given in the proof of Theorem 3.10 works for the polynomial case. □

Lemma 3.16. *Let A, B be two rings. The product ring $A \times B$ satisfies epi-DCC on ideals if and only if each of the rings A and B satisfies epi-DCC on ideals.*

Proof. Note first that ideals of $A \times B$ are exactly $I \times J$ where I range over ideals of A and J range over ideals of B . (\Rightarrow) If $I_1 \supseteq I_2 \supseteq \dots$ is a descending chain of ideals of A , then $I_1 \times 0 \supseteq I_2 \times 0 \supseteq \dots$ is a descending chain of ideals of $A \times B$ and so there exists a positive integer n such that for each $i \geq n$, there exists $\varphi_i : I_i \times 0 \rightarrow I_{i+1} \times 0$ an epimorphism of $(A \times B)$ -modules. Clearly, there exists a map $\psi_i : I_i \rightarrow I_{i+1}$ such that $\varphi_i(x, 0) = (\psi_i(x), 0)$. It is straightforward to see that ψ_i is an epimorphism $I_i \rightarrow I_{i+1}$ of A -modules. Then A satisfies epi-DCC on ideals. With similar argument, B satisfies epi-DCC on ideals. (\Leftarrow) Let $I_1 \times J_1 \supseteq I_2 \times J_2 \supseteq \dots$ be a descending chain of ideals of $A \times B$. Since $I_1 \supseteq I_2 \supseteq \dots$, there exists a positive integer n such that for each $i \geq n$, there exists $f_i : I_i \rightarrow I_{i+1}$ an epimorphism of A -modules. Since $J_1 \supseteq J_2 \supseteq \dots$, there exists a positive integer m such that for each $i \geq m$, there exists $g_i : J_i \rightarrow J_{i+1}$ an epimorphism of B -modules. Hence $\varphi_i = (f_i, g_i) : I_i \times J_i \rightarrow I_{i+1} \times J_{i+1}$, that maps (x, y) to $(f_i(x), g_i(y))$, is an epimorphism of $(A \times B)$ -modules for each $i \geq \max\{n, m\}$. □

Theorem 3.17. *Let e be an idempotent element of R and $\bar{e} = 1 - e$. The following statements are equivalent.*

- (i) $R + X(eR)[[X]]$ satisfies epi-DCC on ideals.
- (ii) $R + X(eR)[X]$ satisfies epi-DCC on ideals.
- (iii) R satisfies epi-DCC on ideals and $R/\bar{e}R$ is a reduced Artinian ring.
- (iv) The following hold.
 - R satisfies epi-DCC on ideals.
 - $\bar{e}R$ is a radical ideal of R .
 - Every prime ideal of R containing \bar{e} is maximal and finitely generated.

Proof. Note first that eR (respectively $\bar{e}R$) is a ring with identity element e (respectively \bar{e}). Moreover $R = eR \oplus \bar{e}R \cong eR \times \bar{e}R$ (a ring isomorphism). Denote $\mathcal{R} = R + X(eR)[[X]]$. With similar argument, $\mathcal{R} = e\mathcal{R} \oplus \bar{e}\mathcal{R} \cong e\mathcal{R} \times \bar{e}\mathcal{R}$. Note also that $e\mathcal{R} = eR + X(eR)[[X]] = I[[X]]$ where $I = eR$ and $\bar{e}\mathcal{R} = \bar{e}R$ because $\bar{e}e = 0$. Then $\mathcal{R} \cong I[[X]] \times \bar{e}R$. By Lemma 3.16, \mathcal{R} satisfies epi-DCC on ideals if and only if each of the rings $I[[X]]$ and $\bar{e}R$ satisfies epi-DCC on ideals. By Theorem 2.6, $I[[X]]$ satisfies epi-DCC on ideals if and only if I is a reduced Artinian ring (in particular, I satisfies epi-DCC on ideals). Again by Lemma 3.16, $R \cong I \times \bar{e}R$ satisfies epi-DCC on ideals if and only if each of the rings I and $R/\bar{e}R$ satisfies epi-DCC on ideals. Since $I = eR \cong R/\bar{e}R$ (a ring isomorphism), (i) \Leftrightarrow (iii). With similar argument, (ii) \Leftrightarrow (iii). For the equivalence (iii) \Leftrightarrow (iv), see the proof of Corollary 3.13. □

The argument given in Theorem 3.17 works also for the isoartinian property. So we have the following.

Theorem 3.18. *Let e be an idempotent element of R and $\bar{e} = 1 - e$. The following statements are equivalent.*

- (i) $R + X(eR)[[X]]$ is isoartinian.
- (ii) $R + X(eR)[X]$ is isoartinian.
- (iii) R is isoartinian and $R/\bar{e}R$ is a reduced Artinian ring.
- (iv) The following hold.
 - R is isoartinian.
 - $\bar{e}R$ is a radical ideal of R .
 - Every prime ideal of R containing \bar{e} is maximal and finitely generated.

4 Conclusion remarks

The study concludes with the presentation of several open questions for future research.

(Q_1) If $R + XI[X]$ satisfies DCC_d on ideals, is I finitely generated ?

(Q_2) If $R + XI[[X]]$ satisfies epi-DCC on ideals or is isoartinian, is I principal generated by an idempotent ?

(Q_3) If $R + XI[X]$ satisfies epi-DCC on ideals or is isoartinian, is I principal generated by an idempotent ?

As we said before, an idempotent and finitely generated ideal is necessarily principal generated by an idempotent element. So, if the question (Q_1) has an affirmative answer, then we can generalize Proposition 3.15 without the assumption " I is finitely generated ". Also, if the question (Q_3) has an affirmative answer, then the ring $R + XI[X]$ satisfies epi-DCC on ideals (respectively is isoartinian) if and only if the following hold (i) R satisfies epi-DCC on ideals (respectively is isoartinian), (ii) $\bar{e}R$ is a radical ideal of R and (iii) Every prime ideal of R containing \bar{e} is maximal and finitely generated. Also, if the question (Q_2) has an affirmative answer, then the ring $R + XI[[X]]$ satisfies epi-DCC on ideals (respectively is isoartinian) if and only if (i),(ii) and (iii) hold.

References

- [1] M. Arapovic, *Characterizations of the 0-dimensional rings*, Glasnik Mat. Serr III, **18**(38), 39–46 (1983).
- [2] J. T. Arnold, *Krull dimension in power series rings*, Trans. Amer. Math. Soc., **177**, 299–304 (1973).
- [3] A. Daneshvar and K. Divaani-Aazar, *Characterization and examples of commutative iso-artinian rings*, Publ. Mat., **69**(2), 403–414 (2025).
- [4] R. Dastanpour and A. Ghorbani, *Rings with chain divisibility on ideals*, Comm. Algebra, **45**(7), 2889–2898 (2017).
- [5] R. Dastanpour and A. Ghorbani, *Modules with epimorphisms on chains of submodules*, J. Algebra Appl., **16**(6), 1–18 (2017).
- [6] A. Facchini and Z. Nazemian, *Modules with chain conditions up to isomorphism*, J. Algebra, **453**, 578–601 (2016).
- [7] R. Gilmer, *An existence theorem for non-Noetherian rings*, Amer. Math. Monthly, **77**(6), 621–623 (1970).
- [8] A. Hamed and S. Hizem, *S-Noetherian Rings of the Forms $\mathcal{A}[X]$ and $\mathcal{A}[[X]]$* , Comm. Algebra, **43**(9), 3848–3856 (2015).
- [9] S. Hizem, *Power series over an ascending chain of rings*, Comm. Algebra, **40**(11), 4263–4275 (2012).

-
- [10] S. Hizem and A. Benhissi, *Integral domains of the form $A + XI[[X]]$: prime spectrum and Krull dimension*, *J. Alg. Appl.*, **4**(6), 599–611 (2005).
- [11] M. Khalifa, *Isonoetherian power series rings III*, *Comm. Algebra*, **50**(12), 5459–5467 (2022).

Author information

M. Khalifa, University of Sousse, Higher institute of transport and logistics, 4002, Sousse, Tunisia.
E-mail: khalifa.mohamed_cr@yahoo.fr

ORCID 0000-0003-0939-8716

Received: 2025-09-21

Accepted: 2026-02-08