Palestine Journal of Mathematics

Vol 15(2)(2026) , 696–706

H-Supplemented property on the class of coclosed submodules

Enas Mustafa Kamil

Communicated by: Manoj Patel

MSC 2010 Classifications: Primary 16D10 ; Secondary 16D80.

Keywords and phrases: Coextending modules, coclosed submodules, H-supplemented modules

Corresponding Author: Enas Mustafa Kamil

Abstract We define a module S to be H-coextending in this article if, for each coclosed submodule B of S, there is a direct summand C of S such that both B and C are coessential submodules of the sum $B+C$ in S. We examine the structural characteristics of H-coextending modules and identify the relationships between the other lifting properties. Different characteristics of this category of modules are provided. A few prerequisites are given for the direct sum of H-coextending modules to be H-coextending.

Keywords and phrases: Coextending modules, coclosed submodules, H-supplemented modules.

2020 AMS Mathematics Subject Classification: 16D10, 16D80.

1 Introduction

All rings in this work are associative with unity, all modules are unital right, and R represents such a ring. We denote B as a submodule of S (an essential submodule, small submodule) using $B \leqslant S(B \underset{e}{\leqslant} S, B \ll S)$.

As dual concepts, extending modules and lifting modules are significant in the context of rings and categories of modules. In this paper, we will abbreviate the term"direct summand" as "d.s." to avoid plagiarism. Many authors have recently conducted extensive studies on extending modules and their generalizations, as seen in [1],[2],[3],[4] and [5]. Additionally, certain generalizations of lifting modules were introduced, such as [6], [7], and [8]. H-supplemented modules, introduced by Mohamed and Muller [9], are one of the significant generalizations of lifting modules. Let S be a module, S is termed H-supplemented if, for each submodule B of S, there exists a d.s. C of S such that $B+X=S$ "if and only if" $C+X=S$, for each submodule X of S. In keeping with [10], take into account the next relations on the set of submodules of a module S:

(i) $B\bar{\alpha}C$ "if and only if"there exists $X \leqslant S$ such that $\frac{B}{X} \ll \frac{S}{X}$ and $\frac{C}{X} \ll \frac{S}{X}$;

(ii) $B\overline{\beta}C$ "if and only if" $\frac{B+C}{B} \ll \frac{S}{B}$ and $\frac{B+C}{C} \ll \frac{S}{C}$, or equivalently, $B\overline{\beta}C$ "if and only if" $B+X=S$ implies $C+X=S$ and $C+Y=S$ implies that $B+Y=S$, for all $X, Y \leqslant S$).

According to [11], these relations are the dual notions of the relations α and β. Keep in mind that while $\bar{\alpha}$ is reflexive and symmetric, $\bar{\beta}$ constitutes an equivalent relation. A module S is a lifting module"if and only if"for each submodule B of S, there is a d.s. C of S such that $B\bar{\alpha}C$, as seen in [10]. A module S is H-supplemented"if and only if"for every submodule B of S, there is a d.s. C of S such that $B\bar{\beta}C$. It is evident that every lifting module is H-supplemented.

Another beneficial generalization of lifting modules is coextending modules. According to [12] and [13], a module S is named as coextending if every coclosed submodule of S is a d.s. of S. It is worth noting that a submodule B of S is referred to as coclosed in S if the condition $\frac{B}{X} \ll \frac{S}{X}$, implies $B=X$ holds It is indicated by $B \underset{cc}{\leqslant} S$, as detailed in [14].

We introduce and investigate a module condition involving the $\bar{\beta}$ relation on the collection of coclosed submodules of a module S. We call a module S is H-coextending if, for every

$B \underset{cc}{\leqslant} S$, there exists a d.s. C of S such that $B\bar{\beta}C$. The class of H-coextending modules evidently includes, as a proper subset, the category of coextending and H-supplemented modules.

Section 2 presents and examines the relationships among lifting modules, coextending, H-supplemented circumstances, and the H-coextending characteristic. Furthermore, we provide a few conditions for obtaining the fundamental characteristics and structural behavior of the class of H-supplemented modules where these ideas are equivalent. We examine the idea of radical projectivity, which is a generalization of the concept of projectivity, in section 3, when examining the structure of H-coextending (H-supplemented) modules, this generalization is quite helpful. The decomposition theory of the H-coextending is covered in Section 4. We deal with the situation where a direct sum of H-coextending is likewise H-coextending, since a direct sum of H-coextending modules does not always have to be H-coextending. Additionally, we look into the conditions that allow d.s. to inherit the H-coextending characteristic.

Let S be a module, and for each proper submodule K of S and $X \leqslant S, X$ is small in S if $S \neq X + K$, denoted by $X \ll S$. If $S = B + C$ and B is minimal with regard to this property, then B is a supplement of C in S . Equivalently, $S = B + C$ and $B \cap C \ll B$. If S contains a supplement for each of its submodules, S is said to be supplemented. If there is a supplement X of B such that $X \leqslant C$, for any submodules B and C of S with $S = B + C$, then S is said to be amply supplemented. A submodule B of S is coclosure of C if B is coclosed in S and $B \underset{ce}{\leqslant} C$ in S, and S is known as UCC-module provided that each submodule of S has unique coclosure in S. A Let S be any module, S is called D_3 - module if $S = B + X$, where each of B and X are both d.s. of S, then $B \cap X$ is a d.s. of S [14].

2 Basic Results.

H-coextending property based on $\bar{\beta}$ relation and coclosed submodules. We derive some findings about coextending modules to those mentioned in [12], in this part. H-coextending modules and a few additional module features are also located.

We will examine the subsequent relations on the collection of submodules of an R module S.

(i) $K\bar{\alpha}X$ "if and only if" there exists $C \leqslant S$ such that $\frac{K}{C} \ll \frac{S}{C}$ and $\frac{X}{C} \ll \frac{S}{C}$.

(ii) $K\bar{\beta}X$ "if and only if" $\frac{K+X}{X} \ll \frac{S}{X}$ and $\frac{K+X}{K} \ll \frac{S}{K}$.

From [10], $\bar{\alpha}$ is symmetric and reflexive but it may not be transitive, however, $\bar{\beta}$ is an equivalent relation. In case S is amply supplemented, $\bar{\alpha}$ is transitive. Observe that if $B\alpha X$, then $B\bar{\beta}X$. When S is amply supplemented, it is easy to see that $\bar{\alpha} = \bar{\beta}$.

Theorem 2.1. *([15], Theorem 2.3) Let S represent a module, with B and C as its submodules. The following properties are equivalent.*

(i) $B\bar{\beta}C$;

(ii) $B \underset{ce}{\leqslant} B + C$ in S and $C \underset{ce}{\leqslant} B + C$ in S;

(iii) For any submodule D of S with $B + C + D = S$, the subsequent statements are true $B + D = S$ and $C + D = S$;

(iv) If $X \leqslant S$ such that $B + X = S$, then $C + X = S$, and if $G \leqslant S$ such that $C + G = S$, then $B + G = S$.

Theorem 2.2. *([15], Theorem 2.6) Let B and C be submodules of a module S. If $B\bar{\beta}C$, then.*

(i) $B \ll S$ "if and only if" $C \ll S$.

(ii) B has a (weak) supplement X in S "if and only if" X is a (weak) supplement of C in S.

We then provide certain coextending module characterizations that are not present in [12].

Proposition 2.3. *A module S is coextending "if and only if" there exists a d.s. B of S such that $X\bar{\alpha}B$, $X \underset{cc}{\leqslant} S$.*

Proof. The proof is standard. □

Theorem 2.4. *Let S be a module. The statements that follow are equivalent.*

(i) *S is coextending;*

(ii) *For every $B \underset{cc}{\leqslant} S$, there is a d.s. C of S such that $S = C \oplus C'$, $C \leqslant B$, $C' \leqslant S$ and $B \cap C' \ll C'$;*

(iii) *It is possible to express $B = C \oplus X$ for each $B \underset{cc}{\leqslant} S$, where C is a d.s. of S and $X \ll S$.*

Proof. $(i) \Rightarrow (ii)$ Assume that S is coextending module and let $B \underset{cc}{\leqslant} S$, then B is a d.s. of S, and the result is obtained.
$(ii) \Rightarrow (iii)$ Let $B \underset{cc}{\leqslant} S$. By (ii), there is a d.s. C of S such that $C \leqslant B$, $S = C \oplus C', C' \leqslant S$ and $B \cap C' \ll C'$. Observe that $B = B \cap (C \oplus C') = C \oplus (B \cap C')$. So, the result is obtained.
$(iii) \Rightarrow$ (i) Let $B \underset{cc}{\leqslant} S$. By (iii), $B = C \oplus X$, where C is a d.s. of S and $X \ll S$. One can easily show that $\frac{B}{C} \ll \frac{S}{C}$. As B is coclosed in $S, B = C$. Thus, S is coextending. □

Definition 2.5. A module S is H-coextending "if and only if" there is a d.s. C of S such that $B\bar{\beta}C$ for all $B \underset{cc}{\leqslant} S$.

Proposition 2.6. *Let S represent a module, and consider the conditions below.*

(i) *S is lifting,*

(ii) *S is H-supplemented,*

(iii) *S is $\oplus$-supplemented,*

(iv) *S is coextending,*

(v) *S is H-coextending,*

(vi) *Every coclosed submodule of S has a supplement which is a d.s.*

Then $(i) \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}, (i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi), (ii) \Rightarrow (v)$ *and* $(iii) \Rightarrow (vi)$

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$. See ([10], Proposition 2.5).
(i) $\Rightarrow$ (iv) See [12].
(iv) $\Rightarrow$ (v) It is clear from the fact that the relation $\bar{\alpha}$ yields the relation $\bar{\beta}$.
$(v) \Rightarrow (vi)$ Let $B \underset{cc}{\leqslant} S$. Then there exists a d.s. C of S such that $B\bar{\beta}C$. Take $S = C \oplus C', C' \leqslant S$, then $S = B + C'$. Let $C' = (B \cap C') + K, K \leqslant C'$, hence $S = C \oplus [(B \cap C') + K] = C \oplus K$ and hence $K = C'$. Thus, $B \cap C' \ll C'$, which means C' is a supplement of B in S.
$(ii) \Rightarrow (v)$ and $(iii) \Rightarrow (vi)$ Straightforward.
$(ii) \not\Rightarrow (i)$ The $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is an H-supplemented while it is not lifting.
$(iv) \not\Rightarrow$ (i)$\mathbb{Z}$ as $\mathbb{Z}$-module is coextending since the trivial submodules of $\mathbb{Z}$ are the only coclosed in $\mathbb{Z}$, while it is not a lifting module. Also, it is an example to show that $(v) \not\Rightarrow$ (ii).
$(v) \not\Rightarrow (iv)$ Let $S = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$-module. As S is H-supplemented, S is H-coextending while the submodule $\langle(\bar{1}, \bar{2})\rangle$ is coclosed in S which is not a d.s, so S is not coextending module.
$(vi) \not\Rightarrow (iii)$ Trivially, $\mathbb{Z}$ as $\mathbb{Z}$-module. □

The following example also shows that coextending module is a proper generalization of the lifting module.

Example 2.7. Let S be a unique composition series uniserial module such that $0 \neq X \subset Y \subset S$. Then $S \oplus \frac{Y}{X}$ is coextending which is not lifting.

The following propositions give conditions under which H-coextending and coextending are equivalent.

Proposition 2.8. *Let S be a module, then.*

(i) If S is indecomposable, then S is H-coextending "if and only if" it is coextending.

(ii) If S is UCC-module, then S is H-coextending "if and only if" it is coextending.

(iii) Let End (S_R) be Abelian such that, if $B \leqslant S$, then $B = \sum_{i\in I}\varphi_i(S)$, where each $\varphi_i \in \operatorname{End}(S_R)$. Then S is H-coextending "if and only if" it is coextending.

Proof. (i) It is straightforward.

(ii) Let S be an H-coextending module, let B be coclosed in S, there exists a d.s. C of S such that $B\bar{\beta}C$, so $B \underset{ce}{\leqslant} B + C$ in S and $C \underset{ce}{\leqslant} B + C$ in S, then B and C are both coclosures of $B + C$. However, S is $UCC, B = C$. Thus, S is coextending.

(iii) Let $B \underset{cc}{\leqslant} S$. By hypothesis, $B = \sum_{i\in L}\varphi_i(S)$, where $\varphi_i \in \operatorname{End}(S_R)$. As S is H-coextending, there exists $c^2 = c \in \operatorname{End}(S_R)$ such that cS is a supplement of B. Since End (S_R) is Abelian, $(1 - c)S \leqslant B$. Hence $\frac{B}{(1-c)S} \ll \frac{S}{(1-c)S}$. As B is coclosed in $S, B = (1 - c)S$. So, S is coextending.

□

A module S is called completely distributive, if for every collection S_i submodules of S and every submodule B of S, the relation $B + \cap_{i\in\Lambda}S_i = \cap_{i\in\Lambda}(B + S_i)$ hold, for more details see [16].

Proposition 2.9. *Let S be an H-coextending. If S is completely distributive, then S is coextending.*

Proof. It follows from ([12] Lemma 4.7]). □

Remember that the coessential intersection property, or CEIP, is a feature of module S , if for each submodules B_1, B_2, B_3, and B_4 of S, if $S_1 \underset{ce}{\leqslant} S_2$ in S and $S_3 \underset{ce}{\leqslant} S_4$ in S, then $S_1 \cap S_3 \underset{ce}{\leqslant} S_2 \cap S_4$ in S, [17].

The following proposition gives equivalent conditions to the CEIP.

Proposition 2.10. *The following statements are equivalent for a module S.*

(i) S has CEIP;

(ii) For every submodules K and V of S, if $K \underset{ce}{\leqslant} K + V$ in S, then $K \cap V \underset{ce}{\leqslant} V$ in S;

(iii) If $K \underset{ce}{\leqslant} V$ in S and $L \leqslant S$, then $K \cap L \underset{ce}{\leqslant} V \cap L$ in S.

Proof. It's simple to check. □

Proposition 2.11. *Let S be an H-coextending with the CEIP. Then S is coextending.*

Proof. Let $K \underset{cc}{\leqslant} S$. There is a d.s. C of S such that $K \underset{ce}{\leqslant} K + C$ in S and $C \underset{ce}{\leqslant} K + C$ in S. As S has CEIP, $K \cap C \underset{ce}{\leqslant} K + C$ in S.But $K \cap C \subseteq C \subseteq K + C$, hence $K \cap C \underset{ce}{\leqslant} C$ in S. As C is coclosed in $S, K \cap C = C$. However, K is coclosed in S, therefore $K = C$. Thus S is coextending □

Proposition 2.12. *Let S be an R-module. If every submodule of S is coclosed, then.*

(i) S is semismple "if and only if" it is coextending;

(ii) S is H-supplemented"if and only if"it is H-coextending.

Proof. Straightforward. □

Corollary 2.13. *Let S be a module over a V-ring, then.*

(i) S is semismple "if and only if" it is coextending;

(ii) S is H-supplemented"if and only if"it is H-coextending.

Proof. Consequent from ([18] Proposition 2.1). □

The following proposition gives a condition under which H-coextending and H-supplemented conditions are equivalent.

Proposition 2.14. *Let S be a module such that every submodule of S has a coclosure, then S is H-coextending "if and only if" it is H-supplemented.*

Proof. Let S be H-coextending such that every submodule of S has a coclosure, let B be a submodule of S, then B has a coclosure say L in S, hence L is coclosed in S and $\frac{B}{L} \ll \frac{S}{L}$. As S is H-coextending, there is a d.s. C of S such that $L\bar{\beta}C$. Now, $L\bar{\beta}B$ and $L\bar{\beta}C$, yield that $B\bar{\beta}C$. Thus, S is H-supplemented. The converse is clear. □

The next corollary is simple to demonstrate.

Corollary 2.15. *Let S be a UCC-module, the following are identical.*

(i) S is lifting;

(ii) S is H-supplemented;

(iii) S is H-coextending;

(iv) S is coextending.

Proposition 2.16. *Let S be an amply supplemented module, then S is H-coextending "if and only if" it is H-supplemented.*

Proof. Let B be any submodule of S. Since S is amply supplemented, there exists a $L \underset{cc}{\leqslant} S$ such that $\frac{B}{L} \ll \frac{S}{L}$, that implies $B\bar{\beta}L$. Since S is H-coextending there is a d.s. C of S such that $L\bar{\beta}C$. As $\bar{\beta}$ is transitive, $B\bar{\beta}C$, S is H-coextending module. □

Next, we give various characterizations of H-coextending modules.

Theorem 2.17. *For a module S, the assertions below are equivalent.*

(i) S is H-coextending;

(ii) For every coclosed submodule B of S, there exists a submodule X of S and a d.s. C of S such that $B \underset{ce}{\leqslant} X$ in S and C *$\underset{ce}{\leqslant} X$ in S.*

(iii) For every $B \underset{cc}{\leqslant} S$, there exists a supplement K of B and a supplement L of K such that K is a d.s. of S and $B\bar{\beta}L$ and every homomorphism $\theta : S \to \frac{S}{(L\cap K)}$ liftable to a homomorphism ψ :$S\to$ S.

Proof. $(i) \Rightarrow (ii)$ Put $X = B + C$, we get the result.
$(ii) \Rightarrow (iii)$ Let $B \underset{cc}{\leqslant} S$, there exists a d.s. C of S such that $\frac{B+C}{B} \ll \frac{S}{B}$ and $\frac{B+C}{C} \ll \frac{S}{C}$. Let $S = C \oplus C'$ and take $C' = K$ and $C = L$. Then $S = C' + B$ and B $\cap\, C' \ll C'$. Hence C' is a supplement of B in S.
$(iii) \Rightarrow (i)$ From ([19] Lemma 2.2), L is a d.s. of S . So S is H-coextanding. □

The next theorem gives another characterization of H-coextending modules.

Theorem 2.18. *Consider any R-module S. The circumstances listed below are equivalent.*

(i) S is H-coextending;

(ii) For every $B \underset{cc}{\leqslant} S$, there is a d.s. C of S such that $S = C \oplus C'$ and $(B + C) \cap C' \ll C'$;

(iii) For every $B \underset{cc}{\leqslant} S$, there is a d.s. C of S such that $B + C = C\oplus X, X \ll S$.

Proof. The proof's steps are comparable to those of the Theorem 2.4. □

3 Radical projectivity

In this part, we study a concept that generalized the definition of an N-projective module S. When it comes to examining the structure of H-coextending modules, this generalization proves highly beneficial, [20].

Let S and N be R-modules, S is N-radical projective if for every submodule B of N and any homomorphism $\varphi : S \to \frac{N}{B}$, there exists a homomorphism $\psi : S \to N$ such that $\mathrm{Im}(\pi\circ \psi - \varphi) \ll \frac{N}{B}$, where $\pi : N \to \frac{N}{B}$ is the natural epimorphism. See the following diagram.

$$\begin{tikzcd}
& S \arrow[d, shift right, "P"'] \arrow[ddl, "h"'] \\
& K \arrow[u, shift right, "i"'] \arrow[d, "\varphi"] \arrow[dl, dashed, "\psi"'] \\
N \arrow[r, "\pi"'] & \frac{N}{B}
\end{tikzcd}$$

We say that S_1 and S_2 are relatively radical projective if S_i is S_j-radical projective, $\forall i, j = 1, 2, i \neq j$, [20]

Next, we study some properties of radical propjectivity. It is easy to see that if S_1 is S_2 -projective, then S_1 is S_2-radical projective.

Lemma 3.1. *If N is a semisimple module, then every module S is N-projective.*

Proof. This proof is standard procedure. □

Proposition 3.2. *Let S and N be R-modules. If S is N-radical projective, then K is N radical projective, for all d.s K of S.*

Proof. Let B be a submodule of N, and let $\varphi : K \to \frac{N}{B}$ be a homomorphism and consider the following diagram.

Where i is the inclusion map, P is the projection map and π is the canonical. Since S is N-radical projective, there is a homomorphism $h : S \to N$ such that, $\mathrm{Im}(\pi \circ h - \varphi \circ P) \ll \frac{N}{B}$. Put $\psi = h \circ i$, observe that $\mathrm{Im}[(\pi \circ \psi) - \varphi] = \mathrm{Im}[(\pi \circ h \circ i) - \varphi] \subseteq \mathrm{Im}[(\pi \circ h) - (\varphi \circ P)] \ll \frac{N}{B}$. Thus, K is N-radical projective. □

$$\begin{tikzcd}
& S \arrow[d, "\varphi"] \arrow[dl, dashed, "\psi"'] \\
N \arrow[r, "\pi"'] & \frac{N}{B}
\end{tikzcd}$$

Proposition 3.3. *Let S and N be R-modules such that $S = S_1 \oplus S_2$. If S_1 is N-radical projective and S_2 is N-radical projective, then S is N-radical projective.*

Proof. Let B be a submodule of N and let $\varphi : S \to \frac{N}{B}$ be a homomorphism. Consider the following diagram.

Where i_1, i_2 are inclusion maps, P_1, P_2 are projections and $\pi : N \to \frac{N}{B}$ is the natural epimorphism. As S_1 and S_2 are both N-radical projective, there exists homomorphisms $\psi_1 : S_1 \to N$ and $\psi_2 : S_2 \to N$ such that $\mathrm{Im}\,[(\pi \circ \psi_1) - (\varphi_0 i_1)] \ll \frac{N}{B}$, and $\mathrm{Im}[(\pi\circ\ \psi_2) - (\varphi_0 i_2)] \ll$

$$\begin{tikzcd}
 & S_2 \arrow[ddl, bend right, "\psi_2"'] \arrow[dr, shift left, "i_2"] & & S_1 \arrow[dl, shift left, "i_1"] \arrow[ddlll, bend right=60, "\psi_1"'] \\
 & & S \arrow[ul, shift left, "p_2"] \arrow[ur, shift left, "p_1"] \arrow[dll, dashed, "\psi"'] \arrow[d, "\varphi"] & \\
N \arrow[rr, "\pi"'] & & \frac{N}{B} &
\end{tikzcd}$$

$\frac{N}{B}$. Define $\psi : S \to N$ by $\psi = (\psi_1, \psi_2)$. Let $x \in S$, then $x = (x_1, x_2), x_1 \in S_1$ and $x_2 \in S_2, (\pi \circ \psi)(x) - \varphi(x) = \pi(\psi_1(x_1), \psi_2(x_2)) - \varphi(x_1, x_2) = ((\pi \circ \psi_1)(x_1), (\pi \circ \psi_2)) - ((\varphi \circ i_1)(x_1), (\varphi \circ i_2)(x_2)) = ((\pi \circ \psi_1)(x_1) - (\varphi \circ i_1)(x_1), (\pi \circ \psi_2)(x_2) - (\varphi \circ i_2)(x_2))$. Since $\operatorname{Im}[(\pi \circ \psi_1) - (\varphi_0 i_1)] \ll \frac{N}{B}$ and $\operatorname{Im}[(\pi \circ \psi_2) - (\varphi_0 i_2)] \ll \frac{N}{B}$ then $\operatorname{Im}[(\pi \circ \psi_1) - (\varphi_0 i_1)] \oplus \operatorname{Im}[(\pi \circ \psi_2) - (\varphi_0 i_2)] \ll \frac{N}{B}$. So, $\operatorname{Im}[(\pi \circ \psi) - \varphi] \ll \frac{N}{B}$. Thus, S is N-radical projective. □

Corollary 3.4. *Let $S = \bigoplus_{i=1}^{n} S_i$ and N be R-modules, then S_i is N-radical projective "if and only if" S is N-radical projective.*

Proof. From Proposition 3.2 and Proposition 3.3. □

Proposition 3.5. *Let S and N be R-modules. If S is N-radical projective, then S is $\frac{N}{K}$ radical projective, for every submodule K of N.*

Proof. Let $\frac{X}{K} \leqslant \frac{N}{K}$ and $\varphi : S \to \frac{\frac{N}{X}}{\frac{X}{K}}$ be R-homomorphism, and let $g : \frac{N}{\frac{X}{K}} \to \frac{N}{X}$ be an isomorphism. Consider the following diagram.

Where π, π_1, π_2 are natural epimorphisms. Since S is N-radical projective, there exists a

$$\begin{tikzcd}
 & S \arrow[dl, dashed, "h"'] \arrow[d, "\varphi"] & \\
N \arrow[r, "g"'] \arrow[dr, "\pi"'] & N' \arrow[d, "\psi"] & \\
 & \frac{N}{Ker g} \arrow[r] & 0
\end{tikzcd}$$

homomorphism $\psi : S \to N$ such that $\operatorname{Im}[(\pi_1 \circ \psi) - (g \circ \varphi)] \ll \frac{N}{X}$. Take $h : S \to \frac{N}{K}$ be defined by $h = \pi_2 \circ \psi$. Note that $\operatorname{Im}[(\pi \circ h) - \varphi] = \operatorname{Im}[(g \circ \pi \circ \pi_2 \circ \psi) - (g \circ \varphi)] \subseteq \operatorname{Im}[(\pi_1 \circ \psi) - (\varphi)] \ll \frac{N}{X}, g$ is isomorphism. Thus, S is $\frac{N}{K}$-radical projective. □

Theorem 3.6. *Let S and N be R-modules, then S is N-radical projective "if and only if" for every epimorphism $g : N \to N'$ and $\varphi : S \to N'$, there is a homomorphism $h : S \to N$ such that* $\operatorname{Im}[(g \circ h) - \varphi] \ll N'$*,*

Proof. Take a look at the next diagram.

$$\begin{tikzcd}
& S \arrow[dl, dashed, "h"'] \arrow[d, "\varphi"] \arrow[ddl, bend right=60, "\psi"'] \\
\frac{N}{K} \arrow[r, "\pi"] & \frac{\frac{N}{K}}{\frac{X}{K}} \arrow[d, "g"] \\
N \arrow[u, "\pi_2"] \arrow[r, "\pi_1"'] & \frac{N}{X}
\end{tikzcd}$$

Where $\psi : N' \to \frac{N}{\mathrm{Ker}g}$ isomorphism and $\pi : N \to \frac{N}{\mathrm{Ker}g}$ is the natural epinorphism. Since S is N-radical projective, there exists a homomorphism $h : S \to N$ such that $\mathrm{Im}[(g \circ h) - \varphi] \ll N'$. The converse is clear. □

Recall that If $S = S_1 \oplus S_2, S_1$ is said to be S_2-sjective if for every submodule B of S such that $S = B + S_2$, there exists $L \leqslant S$ such that $S = L \oplus S_2$ and $\frac{B+L}{B} \ll \frac{S}{B}$, see [10].

Proposition 3.7. *Let $S = S_1 \oplus S_2$ be R-module. If S_1 is S_2-radical projective, then S_1 is S_2-sjective. The converse is true when S is amply supplemented.*

Proof. See ([17], Theorem 3.5]). □

4 Decompositions

In actuality, H-coextending modules are not closed under direct sums, as we see in Example 4.1. In this section, we demonstrate that the category of H -coextending modules is not closed under direct sums, as illustrated by these modules S. Furthermore, the author has not yet been able to ascertain whether the category of H-coextending modules is closed under d.s. We explore different criteria when the direct sum of H-coextending modules is itself H-coextending, as well as when a d.s. of an H-coextending module qualifies as H-coextending.

Example 4.1. The $\mathbb{Z}$-modules, $\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}$ are both H-coextending, while $S = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ is not, where there is a coclosed submodule $B = n\mathbb{Z} \oplus 0, n > 1$ which is not related with any direct summand of S by $\bar{\beta}$.

Theorem 4.2. *[10] Let $S = S_1 \oplus S_2$ be a direct sum of modules.*

(i) if S_1 is S_2-sjective and each of S_1 and S_2 are H-supplemented, then S is H-supplemented.

(ii) If S_1 is S_2-projective and S is H-supplemented, then S_2 is H-supplemented.

Corollary 4.3. *Let $S = S_1 \oplus S_2$ be a direct sum of H-supplemented modules. If S_2 is S_2 radical projective, then S is H-supplemented.*

Proof. It is a direct consequence of Theorem 4.2. □

Proposition 4.4. *Let $S = S_1 \oplus S_2$, where S_2 is coextending. If S_1 is S_2-projective, then every coclosed submodule B of S with $S = B + S_2$ is a d.s of S.*

Proof. Let $B \underset{cc}{\leqslant} S$ with $S = B + S_2$. As S_1 is S_2-projective, there exists $S_3 \leqslant B$ such that $S = S_2 \oplus S_3$, by ([9], Lemma 4.47). Since $\frac{S}{S_3} \cong S_2$, then $\frac{S}{S_3}$ is coextending. Observe that $\frac{B}{S_3}$ is coclosed in $\frac{S}{S_3}$, hence $\frac{B}{S_3}$ is a d.s. of $\frac{S}{S_3}$, that is, there is $S_3 \leqslant K \leqslant S$ such that $S = B + K$ and $B \cap K = S_3$. Then $K = K \cap S = K \cap (S_3 + S_2) = S_3 + (K \cap S_2)$, hence $S = B + K = B + S_3 + (K \cap S_2) = B + (K \cap S_2)$ and $(B \cap K) \cap S_2 = S_3 \cap S_2 = 0$. Thus, B is a d.s. of S. □

From ([14], 20.26), it is easy to conclude the next result.

Proposition 4.5. *Let $S = S_1 \oplus S_2$ be a direct sum of supplemented modules, then S is coextending"if and only if"for every $B \underset{cc}{\leqslant} S$, if $S = B + S_1$ or $B + S_2 = S$, then B is a d.s. of S.*

Proposition 4.6. *Let $S = S_1 \oplus S_2$ be a direct sum of supplemented modules. If S_1 and S_2 are coextending and are relatively projective, then S is coextending.*

Proof. Let $B \underset{cc}{\leqslant} S$ with $S = B + S_1$. By ([9], Lemma 4.47), there exists a submodule B' of B such that $S = B' \oplus S_1$. Since $\frac{B}{B'}$ is coclosed in $\frac{S}{B'}$ and $\frac{S}{B'}$ is coextending, then it is a d.s. of $\frac{S}{B'}$. Then $S = B + X$ with $B \cap X = B'$. Now, $S = B + X = B + B' + (X \cap S_1) = B + (X \cap S_1)$, where $B \cap (X \cap S_1) = (B \cap X) \cap S_1 = 0$. Therefore, B is a d.s. of S. □

We can immediately conclude the The subsequent corollary.

Corollary 4.7. *Let $S = S_1 \oplus S_2$ be a direct sum of supplemented module. If S_1 is semisimple and S_2 projective, then S is coextending.*

Proposition 4.8. *Let $S = S_1 \oplus S_2$ be a direct sum of H-coextending modules. If $\operatorname{ann} S_1 + \operatorname{ann} S_2 = R$, then S is H-coextending.*

Proof. Let $B \underset{cc}{\leqslant} S$, then $B = B_1 \oplus B_2$, where B_1 and B_2 are coclosed submodules of S_1 and S_2 respectively, by ([12], Lemma 3.3). As S_1 and S_2 are both H-coextending, there are d.s. C_1 of S_1 and C_2 of S_2 such that $B_1 \bar{\beta} C_1$ and $B_2 \bar{\beta} C_2$, hence $(B_1 \oplus B_2)\, \bar{\beta}\, (C_1 \oplus C_2)$. Thus, S is H-coextending. □

Proposition 4.9. *Let $S = \bigoplus_{i=1}^{n} S_i$ be a direct sum of H-coextending modules. If every coclosed submodule of S is a fully invariant, then S is H-coextending.*

Proof. Let $B \underset{cc}{\leqslant} S$, as B is fully invariant, $B = \bigoplus_{i=1}^{n} (B \cap S_i)$. Since B is coclosed in S, then $B \cap S_i$ is coclosed in $S_i, \forall i = 1, \dots, n$, by, ([12], Lemma 3.2). Hence there exists d.s. C_i of S_i such that $(B \cap S_i)\, \bar{\beta} C_i, \forall i = 1, \dots, n$. So, $B \bar{\beta} \left(\bigoplus_{i=1}^{n} C_i\right)$, so the result is obtained. □

A module S is called distributive, if for every submodules S_1, S_2 and S_3 of S, we have $S_1 \cap (S_2 + S_3) = (S_1 \cap S_2) + (S_1 \cap S_3)$, [21].

Proposition 4.10. *Let $S = S_1 \oplus S_2$ be a distributive module. If S_1 and S_2 are H-coextending, then S is H-coextending.*

Proof. Let $B \underset{cc}{\leqslant} S$, then $B = (B \cap S_1) \oplus (B \cap S_2)$, and each of $B \cap S_1$ and $B \cap S_2$ are coclosed submodules of S_1 and S_2 respectively, by ([12], Lemma 3.2). Hence there exists d.s. C_1 of S_1 and C_2 of S_2 such that $(B \cap S_1)\, \bar{\beta} C_1$ and $(B \cap S_2)\, \bar{\beta} C_2$, so $B = (B \cap S_1) \oplus (B \cap S_2)\, \bar{\beta}\, (C_1 \oplus C_2)$ Thus, S is H-coextending. □

Proposition 4.11. *Let S be H-coextending module and let B be a d.s. of S. If, for each decomposition $S = S_1 \oplus S_2$, we have $B = (B \cap S_1) \oplus (B \cap S_2)$, then B is H-coextending.*

Proof. Let $Y \underset{cc}{\leqslant} B$, hence Y is coclosed in S. There exists a d.s. C of S such that $C + A = S$ "if and only if" $Y + A = S$, for every $A \leqslant S$. Let $S = C \oplus C' = B \oplus B', B' \leqslant S, C' \leqslant S$. Since $B = (B \cap \mathrm{C}) \oplus (B \cap \mathrm{C}'), S = C \oplus C' = Y + C = B + C' = (C \cap B) \oplus (C' \cap B) + C' = (C \cap B) \oplus C'$ implies that $C \subseteq B$. Therefore, C is a d.s. of B. □

Theorem 4.12. *Let B be a projection invariant coclosed submodule of S.*

(i) If S is H-coextending, there is a decomposition $S = S_1 \oplus S_2$ with $S_2 \subseteq B$ and $\frac{B}{S_2} \ll \frac{S}{S_2}$.

(ii) If S is H-coextending, B has unique coclosure and every d.s. of B has a coclosure in S, then there is a decomposition $S = S_1 \oplus S_2$ with $S_2 \subseteq B$ and $\frac{B}{S_2} \ll \frac{S}{S_2}$, and S_1, S_2 are both H-coextending.

Proof. (i) Since S is H-coextending and B is coclosed in S, there is a d.s. C of S such that $\frac{B+C}{B} \ll \frac{S}{B}$ and $\frac{B+C}{C} \ll \frac{S}{C}$. Let $P : S \to C$ be the projection map and let $P(S) = C = S_2$ and $(1-P)(S) = S_1$ Since B is projection invariant, then $P(B) \subseteq B$, hence $B = P(B) \oplus (1-P)(B)$, $P(B) = B \cap P(S)$ and $(1-P)(B) = B \cap (1-P)(S)$. Observe that $S = P(S) \oplus (1-P)(S)$ yields that $S = B + (1-P)(S)$. So, $P(s) = P(B) = C \subseteq B$ and $\frac{P(S)+B}{P(S)} = \frac{B}{P(S)} \ll \frac{S}{P(S)}$.

(ii) From (i), $S = S_1 \oplus S_2$ such that $S_2 \subseteq B$ and $\frac{B}{S_2} \ll \frac{S}{S_2}$, where $e(S) = S_2, (1-e)(S) = S_1$ and $e : S \to S$ is idempotent. Let $S = X \oplus Y$, then $B = (B \cap X) \oplus (B \cap Y)$. By assumption, $B \cap X$ has a coclosure K in S and $B \cap Y$ has a coclosure K' in S, then $K \oplus K'$ is a coclosure of B in S. Since B has a unique coclosure in S, then $K \oplus K' = S_2$. Now, $X \cap (K \oplus K') = K \oplus (X \cap K')$ and $Y \cap (K \oplus K') = K' \oplus (K \cap Y)$. Hence $S_2 = K \oplus K' = (X \cap S_2) \oplus (Y \cap S_2)$. Since S_1 and S_2 are direct summards of S, then S_1 and S_2 are both H-coextending, by Proposition 4.11.

□

Proposition 4.13. *Let S be H-coextending D_3-module. Then every d.s. of S is also H-coextending.*

Proof. Let $S = S_1 \oplus S_2$ and Y be a coclosed submodule of S_1, then Y is coclosed in S. Since S is H-coextending, there is a submodule X of S and idempotent $e : S \to S$ such that $e(S) \subseteq X$ and $\frac{X}{Y} \ll \frac{S}{Y}$, $\frac{X}{e(S)} \ll \frac{S}{e(S)}$. Since S has D_3-property, then $e(S) \cap S_1$ is a d.s. of S_1 and $\frac{X}{e(S) \cap S_1} \ll \frac{S}{e(S) \cap S_1}$. Thus, S_1 is H-coextending. □

Proposition 4.14. *Let S be module such that each coclosed submodule of S has a supplement which is a relatively projective d.s. of S, then S is H-coextending.*

Proof. Let B be a coclosed submodule of S, there exists a decomposition $S = C \oplus C'$ such that $S = B + C'$ and $B \cap C' \ll C'$, by hypothesis. Since C is C'-projective, then $S = X \oplus C'$, for some submodule X of B, by ([9], Lemma 4.47). Thus, S is H-coextending. □

Finally, the graphic in Figure 1. summarizes the connection between H-coextending and a few generalizations of extending property.

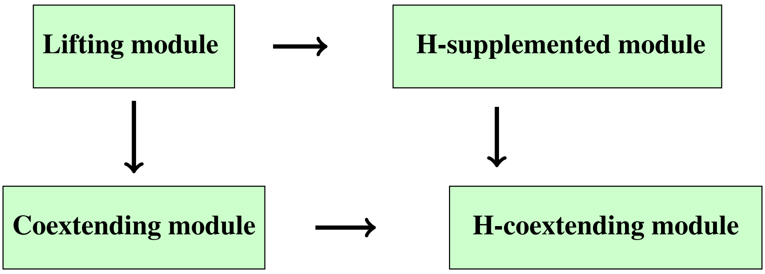

Figure 1. The relationship between H-coextending and different generalizations of extending property

References

[1] E. M. Kamil, B. Davvaz, 2025, Modules in which fully invariant z-closed submodules are direct summands, Palestine Journal of Mathematics, 14(2), 562–568.

[2] E. M. Kamil, B. Davvaz, 2025, A generalization of singular modules in terms of purely extending property, Annali dell'Università di Ferrara, 71, 7.

[3] E. M. Kamil, B. Davvaz, 2025, A generalization of ECS-modules via c-closed submodules, Afrika Matematika, 36, 64.

[4] E. M. Kamil, 2025, Summand intersection property on c-closed submodules, Annali dell'Università di Ferrara, 71, 44.

[5] E. M. Kamil, 2023, On CCLS modules, AIP Conference Proceedings, 2834, 080128.

[6] E. M. Kamil, W. Khalid, 2019, On μ-lifting modules, Iraqi Journal of Science, 60(21), 371–380.

[7] E. M. Kamil, W. Khalid, 2019, On H-μ-supplemented modules, Iraqi Journal of Science, 60(5), 1129–1137.

[8] Y. Talebi, A. R. M. Hamzekolaee, 2022, On a generalization of lifting modules via SSP-modules, Bulletin of the Iranian Mathematical Society, 48, 343–349.

[9] S. H. Mohamed, B. J. Müller, 1990, Continuous and discrete modules, London Mathematical Society Lecture Note Series, 147, Cambridge University Press, Cambridge.

[10] Y. Wang, D. Wu, 2012, On H-supplemented modules, Communications in Algebra, 40(10), 3679–3689.

[11] E. Akalan, G. F. Birkenmeier, A. Tercan, 2009, Goldie extending modules, Communications in Algebra, 37(2), 663–683.

[12] I. M. A. Hadi, M. A. Ahmed, 2014, Coextending modules, International Mathematical Forum, 9(13), 579–592.

[13] I. M. Ali and M. A. Ahmed, 2013, Couniform Modules, Baghdad Science Journal, 10, Article 28.

[14] J. Clark, C. Lomp, N. Vanaja, R. Wisbauer, 2006, Lifting modules, supplements and projectivity in module theory, Birkhäuser, Basel.

[15] G. Birkenmeier, F. Mutlu, C. Nebiev, N. Sokmez, A. Tercan, 2010, Goldie-supplemented modules, Glasgow Mathematical Journal, 52(A), 41–52.

[16] J. Abuhlial, 2015, Zariski topologies for coprime and second submodules, Algebra Colloquium, 22(1), 47–72.

[17] D. K. Tutuncu, M. J. Nematollahi, Y. Talebi, 2011, On H-supplemented modules, Algebra Colloquium, 18(Spec. 1), 915–924.

[18] L. Ganesan, N. Vanaja, 2002, Modules for which every submodule has a unique coclosure, Communications in Algebra, 30(5), 2355–2377.

[19] D. Keskin, 2002, Discrete and quasi-discrete modules, Communications in Algebra, 30(11), 5273–5282.

[20] M. C. Izurdiaga, 2004, Supplement submodules and a generalization of projective modules, Journal of Algebra, 277, 689–702.

[21] V. Erdogdu, 1987, Distributive modules, Canadian Mathematical Bulletin, 30, 248–254.

Author information

Enas Mustafa Kamil, Department of Mathematics, College of Science for Women, University of Baghdad, Iraq.
E-mail: `Inas.m@csw.uobaghdad.edu.iq`

Received: 2025-10-01
Accepted: 2026-02-04