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COMMON FIXED POINT THEOREMS ON TRICOMPLEX VALUED b-METRIC SPACES

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Abstract. In this paper, we introduce the concept of tricomplex valued b-metric space and prove common fixed point theorems. Our results mainly focus on generalize and expand some recent well-known results. Finally, we explain an application of our main result to a certain type of non linear Fredholm integral equation.

1 Introduction

The development of special commutative hypercomplex algebras was pioneered by Serge [1]. His work was instrumental in systematically generalizing complex numbers into bicomplex, tricomplex, and other higher-order number systems, forming an infinite sequence of algebras. Subsequent contributions to this field were made by several authors [2-4]. However, the subject experienced a period of limited advancement for approximately five decades thereafter. A significant revival occurred when Price [5] advanced the theory of bicomplex numbers and their function theory. Recently, there has been a resurgence of interest in [6], this area due to its profound applications across various domains of mathematical sciences, as well as in science and technology . Notable foundational work on the elementary functions of bicomplex numbers was developed by Luna-Elizarrarás, Shapiro, Struppa, and Vajiac [7]. Within the framework of fixed point theory, several key results have been established in these novel metric spaces. Choi et al. [8] proved a common fixed point theorem (CFPT) for two weakly compatible mappings in bicomplex-valued metric spaces. Jebril [9] established a CFPT for a pair of maps under rational-type contraction conditions in the same setting. Beg, Datta et al. [10] further contributed to the fixed point theory in bicomplex-valued metric spaces. In a bicomplex valued metric space of rational contractions of two variables, Tassaddique, Ahmad et al. [11] proved common fixed point theorems. Choi, Datta et al. [12] established certain common fixed point theorems for a pair of weakly compatible mappings satisfying (CLRg) (or (E.A)) property in the bicomplex valued metric spaces. Abdalla, Mostefaoui et al. [13] demonstrated the existence of unique common fixed point theorems of contracting mappings that satisfies the concept of weak compatibility in bicomplex-valued metric spaces. Datta, Pal et al. [14] introduced bicomplex valued b-metric spaces with some rational inequality on a pair of self contracting mappings and established common fixed point theorems. Datta, Pal et al. [15] studied common fixed point theorems in bicomplex valued b-metric space that fulfilled some rational inequalities of two pairs of weakly compatible self contracting mappings. Extending this research to even higher dimensions, Gunaseelan et al. [16] recently proved a common fixed point theorem in tricomplex-valued metric spaces in 2022. Some fixed point theorems in tricomplex valued b-metric space were proved by Singh et al. [17] in 2024. Ramasamy et al. [18] established fixed point theorems on tricomplex valued metric space for a pair of contractive mappings. The concept of tricomplex valued parametric metric space is introduced by Poornavel et al. [19] during 2024. Tiwari and Rajput [20] introduced tricomplex valued fuzzy metric spaces and proved fixed point theorems. Siva and Rasikannan [21] initiate a novel tricomplex valued bipolar metric space concept and proved some fixed point theorems in 2022. Ramasamy et al. [22] proposed the tricomplex controlled metric space and established fixed point theorems. Tricomplex partial metric space is introduced by Mostefaoui and colleagues [23] also they proved fixed point theorems with applications. In this paper, we prove some CFPT on tricomplex valued b-metric space (shortly TC_bMS).

2 Preliminaries

Throughout this paper, we denote $\mathbb{C}_0$, $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_3$ are the family of real, complex, bicomplex and tricomplex numerals respectively. The bicomplex numeral define by Price [5] as following:

$$\varrho = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 i_1 i_2,$$

where $\rho_1, \rho_2, \rho_3, \rho_4 \in \mathbb{C}_0$, and independent elements i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we means that the family of bicomplex numerals $\mathbb{C}_2$ is explain as:

$$\mathbb{C}_2 = \{\varrho : \varrho = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 i_1 i_2, \rho_1, \rho_2, \rho_3, \rho_4 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_2 = \{\varrho : \varrho = \varkappa_1 + i_2 \varkappa_2, \varkappa_1, \varkappa_2 \in \mathbb{C}_1\},$$

where $\varkappa_1 = \rho_1 + \rho_2 i_1 \in \mathbb{C}_1$ and $\varkappa_2 = \rho_3 + \rho_4 i_1 \in \mathbb{C}_1$. Price [5] defined the tricomplex numeral as:

$$\mathit{\Omega} = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 j_1 + \rho_5 i_3 + \rho_6 j_2 + \rho_7 j_3 + \rho_8 i_4,$$

where $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8 \in \mathbb{C}_0$, and independent units $i_1, i_2, i_3, i_4, j_1, j_2, j_3$ are such that $i_1^2 = i_4^2 = -1, i_4 = i_1 j_3 = i_1 i_2 i_3, j_2 = i_1 i_3 = i_3 i_1, j_2^2 = 1, j_1 = i_1 i_2 = i_2 i_1$ and $j_1^2 = 1$, we means the family of tricomplex numerals $\mathbb{C}_3$ is defined as:

$$\begin{aligned}\mathbb{C}_3 = \{\mathit{\Omega} : \mathit{\Omega} = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 j_1 + \rho_5 i_3 + \rho_6 j_2 + \rho_7 j_3 + \rho_8 i_4,\\ \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8 \in \mathbb{C}_0\},\end{aligned}$$

i.e.,

$$\mathbb{C}_3 = \{\mathit{\Omega} : \mathit{\Omega} = \varrho_1 + i_3 \varrho_2, \varrho_1, \varrho_2 \in \mathbb{C}_2\},$$

where $\varrho_1 = \varkappa_1 + \varkappa_2 i_2 \in \mathbb{C}_2$ and $\varrho_2 = \varkappa_3 + \varkappa_4 i_2 \in \mathbb{C}_2$.

If $\mathit{\Omega} = \varrho_1 + i_3 \varrho_2$ and $\wp = \mu_1 + i_3 \mu_2$ be any two tricomplex numerals then the sum is

$$\mathit{\Omega} \pm \wp = (\varrho_1 + i_3 \varrho_2) \pm (\mu_1 + i_3 \mu_2) = \varrho_1 \pm \mu_1 + i_3(\varrho_2 \pm \mu_2)$$

and the product is

$$\mathit{\Omega}.\wp = (\varrho_1 + i_3 \varrho_2)(\mu_1 + i_3 \mu_2) = (\varrho_1 \mu_1 - \varrho_2 \mu_2) + i_3(\varrho_1 \mu_2 + \varrho_2 \mu_1).$$

There are four idempotent elements in $\mathbb{C}_3$, they are $0, 1, {}_1 = \frac{1+j_3}{2}, {}_2 = \frac{1-j_3}{2}$. hence, ${}_1$ and ${}_2$ are nontrivial such that ${}_1 + {}_2 = 1$ and ${}_{12} = 0$.

Every tricomplex numeral $\varrho_1 + i_3 \varrho_2$ is uniquely be convey as the union of ${}_1$ and ${}_2$, namely

$$\mathit{\Omega} = \varrho_1 + i_3 \varrho_2 = (\varrho_1 - i_2 \varrho_2)_1 + (\varrho_1 + i_2 \varrho_2)_2.$$

This notation of $\mathit{\Omega}$ is representation for idempotent on the tricomplex numeral and the complex coefficients $\mathit{\Omega}_1 = (\varrho_1 - i_2 \varrho_2)$ and $\mathit{\Omega}_2 = (\varrho_1 + i_2 \varrho_2)$ are familiar as idempotent parts of the bicomplex numeral $\mathit{\Omega}$.

An element $\mathit{\Omega} = \varrho_1 + i_3 \varrho_2 \in \mathbb{C}_3$ is invertible if there exists $\wp \in \mathbb{C}_3$ such that $\mathit{\Omega}\wp = 1$ and $\wp$ is inverse (multiplicative) of $\mathit{\Omega}$.

Consequently $\mathit{\Omega}$ is inverse (multiplicative) of $\wp$. It has an inverse in $\mathbb{C}_3$ is said to be the non-singular element of $\mathbb{C}_3$ and it has not an inverse in $\mathbb{C}_3$ is said to be the singular element of $\mathbb{C}_3$. Here, $\mathit{\Omega} = \varrho_1 + i_3\varrho_2 \in \mathbb{C}_3$ is non-singular iff $|\varrho_1^2 + \varrho_2^2| \neq 0$ and singular iff $|\varrho_1^2 + \varrho_2^2| = 0$.

The inverse of $\mathit{\Omega}$ is defined as

$$\mathit{\Omega}^{-1} = \wp = \frac{\varrho_1 - i_3\varrho_2}{\varrho_1^2 + \varrho_2^2}.$$

The positive real valued mapping is $||.||$ of $\mathbb{C}_3$ and $||.|| : \mathbb{C}_3 \to \mathbb{C}_0^+$ is explain by

$$\begin{aligned} ||\mathit{\Omega}|| = ||\varrho_1 + i_3\varrho_2|| &= \{|\varrho_1|^2 + |\varrho_2|^2\}^{\frac{1}{2}} \\ &= \left[\frac{|(\varrho_1 - i_2\varrho_2)|^2 + |(\varrho_1 + i_2\varrho_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= (\rho_1^2 + \rho_2^2 + \rho_1^2 + \rho_3^2 + \rho_4^2 + \rho_5^2 + \rho_6^2 + \rho_7^2 + \rho_8^2)^{\frac{1}{2}}, \end{aligned}$$

where $\mathit{\Omega} = \rho_1 + \rho_2 i_1 + \rho_3 i_2 + \rho_4 j_1 + \rho_5 i_3 + \rho_6 j_2 + \rho_7 j_3 + \rho_8 i_4 = \varrho_1 + i_3\varrho_2 \in \mathbb{C}_3$.

The linear space $\mathbb{C}_3$ with respect to explain norm is a norm linear space, also $\mathbb{C}_3$ is complete, therefore $\mathbb{C}_3$ is the Banach space. If $\mathit{\Omega}, \wp \in \mathbb{C}_3$ then

$$||\mathit{\Omega}\wp|| \leq 2||\mathit{\Omega}||||\wp||$$

satisfies instead of $||\mathit{\Omega}\wp|| \leq ||\mathit{\Omega}||||\wp||$, therefore $\mathbb{C}_3$ is not the Banach space.
The partial order relation $\preceq_{i_3}$ on $\mathbb{C}_3$ is explain as:

Let $\mathbb{C}_3$ be the family of tricomplex numerals and $\mathit{\Omega} = \varrho_1 + i_3\varrho_2$, $\wp = \mu_1 + i_3\mu_2 \in \mathbb{C}_3$ then $\mathit{\Omega} \preceq_{i_3} \wp$ if and only if $\varrho_1 \preceq_{i_2} \mu_1$ and $\varrho_2 \preceq_{i_2} \mu_2$, that is, $\mathit{\Omega} \preceq_{i_3} \wp$ if one of the below axioms are full filled:

(i) $\varrho_1 = \mu_1$, $\varrho_2 = \mu_2$,

(ii) $\varrho_1 \prec_{i_2} \mu_1$, $\varrho_2 = \mu_2$,

(iii) $\varrho_1 = \mu_1$, $\varrho_2 \prec_{i_2} \mu_2$, and

(iv) $\varrho_1 \prec_{i_2} \mu_1$, $\varrho_2 \prec_{i_2} \mu_2$,

Particularly, we will write $\mathit{\Omega} \lnsim_{i_3} \wp$ if $\mathit{\Omega} \preceq_{i_3} \wp$ and $\mathit{\Omega} \neq \wp$ that is, one of (ii),(iii) and (iv) is full filled and we will write $\mathit{\Omega} \prec_{i_3} \wp$ if only (iv) is full filled.
For any two tricomplex numerals $\mathit{\Omega}, \wp \in \mathbb{C}_3$ we can check the followings:

(1) $\mathit{\Omega} \preceq_{i_3} \wp$ iff $||\mathit{\Omega}|| \leq ||\wp||$,

(2) $||\mathit{\Omega} + \wp|| \leq ||\mathit{\Omega}|| + ||\wp||$,

(3) $||\rho\mathit{\Omega}|| = \rho||\mathit{\Omega}||$, where ρ is a non negative real number,

(4) $||\mathit{\Omega}\wp|| \leq 2||\mathit{\Omega}||||\wp||$ and the equality satisfies only when at least one of $\mathit{\Omega}$ and $\wp$ is non-singular,

(5) $||\mathit{\Omega}^{-1}|| = ||\mathit{\Omega}||^{-1}$ if $\mathit{\Omega}$ is a non-singular,

(6) $||\frac{\mathit{\Omega}}{\wp}|| = \frac{||\mathit{\Omega}||}{||\wp||}$, if $\wp$ is a non-singular.

In example 6 [16], we take $\eth(\ell, \propto) = |\ell - \propto|^2(1 + i_3)$ then it is not satisfied triangular inequality, motivated this idea, we introduce the notion of TC_bMS.

Definition 2.1. Let $\mho$ be a non-void set and let $s \geq 1$. Suppose that the mapping $\eth : \mho \times \mho \to \mathbb{C}_3$, satisfies the below conditions:

(A1) $0 \preceq_{i_3} \eth(\ell, \propto)$, for all $\ell, \propto \in \mho$ and $\eth(\ell, \propto) = 0$ if and only if $\ell = \propto$;

(A2) $\eth(\ell, \propto) = \eth(\ell, \propto)$ for all $\ell, \propto \in \mho$;

(A3) $\eth(\ell, \propto) \preceq_{i_3} s(\eth(\ell, \wp) + \eth(\wp, \propto))$ for all $\ell, \propto, \wp \in \mho$.

Then $\eth$ is said to be the TC_bM on $\mho$, and $(\mho, \eth)$ is said to be the TC_bMS.

Example 2.1. Let $\mho = [0,1]$ and $\eth : \mho \times \mho \to \mathbb{C}_3$ be defined by $\eth(\ell, \propto) = |\ell - \propto|^2 e^{i_3 \frac{\pi}{6}}$. Then $(\mho, \eth)$ is a TC_bMS.

Now, let us recall some Definitions and Lemmas, which will be used in the sequel.

Definition 2.2. Let $(\mho, \eth)$ be a TC_bMS and $\mathcal{J} \subseteq \mho$,

(B1) $\rho \in \mathcal{J}$ is said to be an interior point of a family $\mathcal{J}$ if there is $0 \prec_{i_3} \flat \in \mathbb{C}_3$ such that

$$\mathcal{N}(\rho, \flat) \subseteq \mathcal{J},$$

where $\mathcal{N}(\rho, \flat) = \{\ell \in \mho : \eth(\rho, \ell) \preceq_{i_3} \flat\}$.

(B2) A point $\ell \in \mho$ is said to be a limit point of $\mathcal{J}$if there is for each $0 \prec_{i_3} \flat \in \mathbb{C}_3$,

$$\mathcal{N}(\ell, \flat) \cup (\mathcal{J} - \mho) \neq \emptyset.$$

(B3) A subset $\mathcal{Y} \subseteq \mho$ is said to be a open if there is every element of $\mathcal{Y}$ is an interior point of $\mathcal{Y}$. A subset $\mathcal{J} \subseteq \mho$ is said to be a closed if there is every limit point of $\mathcal{J}$ belongs to $\mathcal{J}$. The set

$$\mathcal{V} = \{\mathcal{N}(\ell, \flat) : \ell \in \mho, 0 \prec_{i_3} \flat\}$$

is a sub-basis for a topology on $\mho$. We means that tricomplex topology by $\varGamma_{tc}$. Indeed, the topology $\varGamma_{tc}$ is Hausdorff.

Definition 2.3. Let $(\mho, \eth)$ be a TC_bMS. A sequence $\{\ell_\varpi\}$ in $\mho$ is called a convergent and converges to $\ell \in \mho$ if for each $0 \prec_{i_3} \Upsilon \in \mathbb{C}_3$ there exists $\varpi_0 \in \mathbb{N}$ such that $\eth(\ell_\varpi, \ell) \prec_{i_3} \delta$, for all $\varpi \geq \varpi_0$ and it is denoted by $\lim\limits_{\varpi \to \infty} \ell_\varpi = \ell$.

Lemma 2.1. Consider $(\mho, \eth)$ be a TC_bMS. A sequence $\{\ell_\varpi\} \in \mho$ is converges to $\ell \in \mho$ if and only if $\lim\limits_{\varpi \to \infty} ||\eth(\ell_\varpi, \ell)|| = 0$.

Proof. Let $\{\ell_\varpi\}$ is a convergent sequence and converges to a point ℓ and let $\Upsilon > 0$ be any real numeral. Suppose

$$\delta = \frac{\Upsilon}{\sqrt{8}} + i_1 \frac{\Upsilon}{\sqrt{8}} + i_2 \frac{\Upsilon}{\sqrt{8}} + j_1 \frac{\Upsilon}{\sqrt{8}} + i_3 \frac{\Upsilon}{\sqrt{8}} + j_2 \frac{\Upsilon}{\sqrt{8}} + j_3 \frac{\Upsilon}{\sqrt{8}} + i_4 \frac{\Upsilon}{\sqrt{8}}.$$

Then $0 \prec_{i_3} \delta \in \mathbb{C}_3$ and for this δ there exists $\varpi_0 \in \mathbb{N}$ such that $\eth(\ell_\varpi, \ell) \prec_{i_3} \delta$ for all $\varpi \geq \varpi_0$. Therefore

$$||\eth(\ell_\varpi, \ell)|| < ||\delta|| = \Upsilon, \ \forall \varpi \geq \varpi_0.$$

Hence $\lim\limits_{\varpi \to \infty} ||\eth(\ell_\varpi, \ell)|| = 0$.
Conversely, let $\lim\limits_{\varpi \to \infty} ||\eth(\ell_\varpi, \ell)|| = 0$. Then for each $0 \prec_{i_3} \delta \in \mathbb{C}_3$, there exists $\Upsilon > 0$ such that $\forall\, \varOmega \in \mathbb{C}_3$,

$$||\varOmega|| < \Upsilon \Rightarrow \varOmega \prec_{i_3} \delta.$$

Then, for this $\Upsilon > 0$, there exists $\varpi_0 \in \mathbb{N}$ such that

$$||\eth(\ell_\varpi, \ell)|| < \Upsilon, \ \forall\ \varpi \geq \varpi_0.$$

Therefore,

$$\eth(\ell_\varpi, \ell) \prec_{i_3} \delta, \ \forall\ \varpi \geq \varpi_0.$$

Hence $\{\ell_\varpi\}$ converges to a point ℓ. □

Definition 2.4. Let $(\mho, \eth)$ be a TC_bMS. A sequence $\{\ell_\varpi\} \in \mho$ is called a Cauchy sequence in $(\mho, \eth)$ if for any $0 \prec_{i_3} \Upsilon \in \mathbb{C}_3$, there exists $\mathfrak{g} \in \mathbb{N}$ such that $\eth(\ell_\varpi, \ell_{\varpi+\aleph}) \prec_{i_3} \Upsilon$ for all $\varpi, \aleph \in \mathbb{N}$ and $\varpi, \aleph \geq \mathfrak{g}$.

Definition 2.5. Let $(\mho, \eth)$ be a TC_bMS. Let a sequence $\{\ell_\varpi\} \in \mho$. Then, if every Cauchy sequence in $\mho$ is convergent in $\mho$ then $(\mho, \eth)$ is called a complete TC_bMS.

Lemma 2.2. Consider $(\mho, \eth)$ be a TC_bMS and a sequence $\{\ell_\varpi\}$ in $\mho$. Then $\{\ell_\varpi\}$ is a Cauchy sequence in $\mho$ if and only if $\lim_{\varpi \to \infty} ||\eth(\ell_\varpi, \ell_{\varpi+\aleph})|| = 0$.

Proof. Let $\{\ell_\varpi\}$ is a Cauchy sequence in $\mho$. Let $\Upsilon > 0$ be any real numeral. Suppose

$$\delta = \frac{\Upsilon}{\sqrt{8}} + i_1\frac{\Upsilon}{\sqrt{8}} + i_2\frac{\Upsilon}{\sqrt{8}} + j_1\frac{\Upsilon}{\sqrt{8}} + i_3\frac{\Upsilon}{\sqrt{8}} + j_2\frac{\Upsilon}{\sqrt{8}} + j_3\frac{\Upsilon}{\sqrt{8}} + i_4\frac{\Upsilon}{\sqrt{8}}.$$

Then $0 \prec_{i_3} \delta \in \mathbb{C}_3$ and for this δ there exists $\varpi_0 \in \mathbb{N}$ such that $\eth(\ell_\varpi, \ell_{\varpi+\aleph}) \prec_{i_3} \delta$, for all $\varpi > \varpi_0$. Therefore,

$$||\eth(\ell_\varpi, \ell_{\varpi+\aleph})|| < ||\delta|| = \Upsilon, \text{ for all } \varpi > \varpi_0.$$

And this implies,

$$\lim_{\varpi \to \infty} ||\eth(\ell_\varpi, \ell_{\varpi+\aleph})|| = 0.$$

Conversely, let $\lim_{\varpi \to \infty} ||\eth(\ell_\varpi, \ell_{\varpi+\aleph})|| = 0$. Then, for each $0 \prec_{i_3} \delta \in \mathbb{C}_3$, there exists $\Upsilon > 0$ such that $\forall\ \varOmega \in \mathbb{C}_3$,

$$||\varOmega|| < \Upsilon \Rightarrow \varOmega \prec_{i_3} \delta.$$

Then, for this $\Upsilon > 0$, there exists a natural numeral $\varpi_0 \in \mathbb{N}$ such that

$$||\eth(\ell_\varpi, \ell_{\varpi+\aleph})|| < \Upsilon,\ \forall\ \varpi > \varpi_0.$$

Therefore,

$$\eth(\ell_\varpi, \ell_{\varpi+\aleph}) \prec_{i_3} \delta,\ \forall\ \varpi > \varpi_0.$$

Hence $\{\ell_\varpi\}$ is a Cauchy sequence. □

Definition 2.6. Let $\mathcal{P}$ and $\mathcal{G}$ be self mappings of nonempty set $\mho$. A point $\ell \in \mho$ is said to be a common fixed point of $\mathcal{P}$ and $\mathcal{G}$ if $\ell = \mathcal{P}\ell = \mathcal{G}\ell$.

3 Main Results

Throughout, we discuss the deal with rational type contraction conditions using the CFPT on TC_bMS.

Theorem 3.1. Let $(\mho, \eth)$ be a complete TC_bMS with the coefficient $s \geq 1$. Let $\mathcal{P}$ and $\mathcal{G}$ are self-mappings defined on $\mho$, satisfying the condition

$$\eth(\mathcal{P}\ell, \mathcal{G}\propto) \preceq_{i_3} \nu\eth(\ell, \propto) + \frac{\tau\eth(\ell, \mathcal{P}\ell)\eth(\propto, \mathcal{G}\propto) + \omega\eth(\propto, \mathcal{P}\ell)\eth(\ell, \mathcal{G}\propto)}{1 + \eth(\ell, \propto)}$$

for all $\ell, \propto \in \mho$ where ν, τ, ω are non-negative reals with $\nu + \sqrt{2}\tau + \sqrt{2}\omega < \frac{1}{s}$, then $\mathcal{P}$ and $\mathcal{G}$ have a unique common fixed point.

Proof. Consider an arbitrary point ℓ_0 in $\mho$ and explain $\ell_{2\varpi+1} = \mathcal{P}\ell_{2\varpi}$, $\ell_{2\varpi+2} = \mathcal{G}\ell_{2\varpi+1}, \varpi = 0, 1, 2, \cdots$. Then

$$\begin{aligned}\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2}) &= \eth(\mathcal{P}\ell_{2\varpi}, \mathcal{G}\ell_{2\varpi+1}) \preceq_{i_3} \nu\eth(\ell_{2\varpi}, \ell_{2\varpi+1}) \\ &+ \frac{\tau\eth(\ell_{2\varpi}, \mathcal{P}\ell_{2\varpi})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1}) + \omega\eth(\ell_{2\varpi}, \mathcal{G}\ell_{2\varpi+1})\eth(\ell_{2\varpi+1}, \mathcal{P}\ell_{2\varpi})}{1 + \eth(\ell_{2\varpi}, \ell_{2\varpi+1})}.\end{aligned}$$

Since $\ell_{2\varpi+1} = \mathcal{P}\ell_{2\varpi}$ implies $\eth(\ell_{2\varpi+1}, \mathcal{P}\ell_{2\varpi}) = 0$, therefore

$$\begin{aligned}\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2}) &\preceq_{i_3} \nu\eth(\ell_{2\varpi}, \ell_{2\varpi+1}) \\ &\qquad + \frac{\tau\eth(\ell_{2\varpi}, \ell_{2\varpi+1})\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})}{1+\eth(\ell_{2\varpi}, \ell_{2\varpi+1})},\end{aligned}$$

which implies that

$$\begin{aligned}||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})|| &\leq \nu||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})|| \\ &\qquad + \frac{\sqrt{2}\tau||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})||||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||}{||1+\eth(\ell_{2\varpi}, \ell_{2\varpi+1})||}.\end{aligned}$$

Since $||1+\eth(\ell_{2\varpi}, \ell_{2\varpi+1})|| > ||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})||$, therefore

$$||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})|| \leq \nu||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})|| + \sqrt{2}\tau||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||,$$

so that

$$||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})|| \leq \frac{\nu}{1-\sqrt{2}\tau}||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})||.$$

Also,

$$\begin{aligned}&\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3}) = \eth(\mathcal{G}\ell_{2\varpi+1}, \mathcal{P}\ell_{2\varpi+2}) \\ &= \eth(\mathcal{P}\ell_{2\varpi+2}, \mathcal{G}\ell_{2\varpi+1}) \\ &\preceq_{i_3} \nu\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1}) \\ &+ \frac{\tau\eth(\ell_{2\varpi+2}, \mathcal{P}\ell_{2\varpi+2})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1}) + \omega\eth(\ell_{2\varpi+1}, \mathcal{P}\ell_{2\varpi+2})\eth(\ell_{2\varpi+2}, \mathcal{G}\ell_{2\varpi+1})}{1+\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})}.\end{aligned}$$

Since $\ell_{2\varpi+2} = \mathcal{G}\ell_{2\varpi+1}$ implies $\eth(\ell_{2\varpi+2}, \mathcal{G}\ell_{2\varpi+1}) = 0$, therefore

$$\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3}) \preceq_{i_3} \nu\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1}) + \frac{\tau\eth(\ell_{2\varpi+2}, \mathcal{P}\ell_{2\varpi+2})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1})}{1+\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})},$$

which implies that

$$\begin{aligned}||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})|| &\leq \nu||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})|| \\ &\qquad + \frac{\sqrt{2}\tau||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})||||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||}{||1+\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})||}.\end{aligned}$$

As $||1+\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})|| > ||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})||$, therefore

$$||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})|| \leq \frac{\nu}{1-\sqrt{2}\tau}||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||.$$

Putting $\mathfrak{g} = \frac{\nu}{1-\sqrt{2}\tau}$, we have (for all ϖ)

$$||\eth(\ell_{\varpi}, \ell_{\mathfrak{t}+1})|| \leq \mathfrak{g}||\eth(\ell_{\mathfrak{t}-1}, \ell_{\varpi})|| \leq \mathfrak{g}^2||\eth(\ell_{\mathfrak{t}-2}, \ell_{\mathfrak{t}-1})|| \leq \cdots \leq \mathfrak{g}^{\varpi}||\eth(\ell_0, \ell_1)||.$$

Therefore, for any $\aleph > \varpi$, we have

$$\begin{aligned}||\eth(\ell_{\varpi}, \ell_{\aleph})|| &\leq ||\eth(\ell_{\varpi}, \ell_{\varpi+1})|| + ||\eth(\ell_{\varpi+1}, \ell_{\varpi+2})|| \\ &\quad + ||\eth(\ell_{\varpi+2}, \ell_{\varpi+3})|| \cdots + ||\eth(\ell_{\mathfrak{m}-1}, \ell_{\aleph})|| \\ &\leq [s\mathfrak{g}^{\varpi} + s^2\mathfrak{g}^{\varpi+1} + s^3\mathfrak{g}^{\varpi+2} + \cdots + s^{\aleph-\varpi}\mathfrak{g}^{\mathfrak{m}-1}]||\eth(\ell_0, \ell_1)|| \\ &\leq \left[\frac{(s\mathfrak{g})^{\varpi}}{1-s\mathfrak{g}}\right]||\eth(\ell_0, \ell_1)||,\end{aligned}$$

which implies that

$$||\eth(\ell_{\varpi},\ell_{\aleph})|| \leq \Big[\frac{(s\mathfrak{g})^{\varpi}}{1-s\mathfrak{g}}\Big]||\eth(\ell_0,\ell_1)|| \to 0 \text{ as } \varpi \to \infty.$$

by using Lemma 2.2, we conclude that a sequence $\{\ell_{\varpi}\}$ is Cauchy . Since $\mho$ is complete, there exists some $ł \in \mho$ such that $\ell_{\varpi} \to ł$ as $\varpi \to \infty$. On the contrary, let $ł \neq \mathcal{P}ł$ so that $0 \prec_{i_3} \wp = \eth(ł,\mathcal{P}ł)$ and henceforth we get

$$\begin{aligned}\wp = \eth(ł,\mathcal{P}ł) &\preceq_{i_3} s\eth(ł,\mathcal{G}\ell_{2\varpi+1}) + s\eth(\mathcal{G}\ell_{2\varpi+1},\mathcal{P}ł)\\ &\preceq_{i_3} s\eth(ł,\ell_{2\varpi+2}) + s\nu\eth(ł,\ell_{2\varpi+1})\\ &+ \frac{s\tau\eth(ł,\mathcal{P}ł)\eth(\ell_{2\varpi+1},\mathcal{G}\ell_{2\varpi+1}) + s\omega\eth(\ell_{2\varpi+1},\mathcal{P}ł)\eth(ł,\mathcal{G}\ell_{2\varpi+1})}{1+\eth(ł,\ell_{2\varpi+1})}\\ &\preceq_{i_3} s\eth(ł,\ell_{2\varpi+2}) + s\nu\eth(ł,\ell_{2\varpi+1})\\ &+ \frac{s\tau\wp\eth(\ell_{2\varpi+1},\mathcal{G}\ell_{2\varpi+1}) + s\omega\eth(\ell_{2\varpi+1},\mathcal{P}ł)\eth(ł,\mathcal{G}\ell_{2\varpi+1})}{1+\eth(ł,\ell_{2\varpi+1})}.\end{aligned}$$

Also, for every ϖ, we have

$$\begin{aligned}||\eth(ł,\mathcal{P}ł)|| \leq{}& ||s\eth(ł,\ell_{2\varpi+2})|| + s\nu||\eth(ł,\ell_{2\varpi+1})||\\ &+ \frac{s\tau\wp||\eth(\ell_{2\varpi+1},\ell_{2\varpi+2})|| + \sqrt{2}s\omega||\eth(\ell_{2\varpi+1},\mathcal{P}ł)||||\eth(ł,\ell_{2\varpi+2})||}{||1+\eth(ł,\ell_{2\varpi+1})||}.\end{aligned}$$

As $\varpi \to \infty$, we obtain

$$||\eth(ł,\mathcal{P}ł)|| = 0,$$

which is a contradiction so that $ł = \mathcal{P}ł$. Similarly, we also prove that $ł = \mathcal{G}ł$. Let $\alpha^* \in \mho$ be an another common fixed point of $\mathcal{P}$ and $\mathcal{G}$, that is $\alpha^* = \mathcal{P}\alpha^* = \mathcal{G}\alpha^*$. Then

$$\begin{aligned}\eth(ł,\alpha^*) = \eth(\mathcal{P}ł,\mathcal{G}\alpha^*) &\preceq_{i_3} \nu\eth(ł,\alpha^*) + \frac{\tau\eth(ł,\mathcal{P}ł)\eth(\alpha^*,\mathcal{G}\alpha^*) + \omega\eth(\alpha^*,\mathcal{P}ł)\eth(ł,\mathcal{G}\alpha^*)}{1+\eth(ł,\alpha^*)}\\ &= \nu\eth(ł,\alpha^*) + \frac{\omega\eth(\alpha^*,ł)\eth(ł,\alpha^*)}{1+\eth(ł,\alpha^*)},\end{aligned}$$

which implies

$$||\eth(ł,\alpha^*)|| \leq \nu||\eth(ł,\alpha^*)|| + \frac{\sqrt{2}\omega||\eth(\alpha^*,ł)||||\eth(ł,\alpha^*)||}{||1+\eth(ł,\alpha^*)||}.$$

Since $||1+\eth(ł,\alpha^*)|| > ||\eth(ł,\alpha^*)||$, therefore

$$||\eth(ł,\alpha^*)|| \leq (\nu+\sqrt{2}\omega)||\eth(ł,\alpha^*)||,$$

which is contradiction so that $ł = \alpha^*$ (as $\nu + \sqrt{2}\omega < 1$). □

By setting $\mathcal{P} = \mathcal{G}$ in Theorem 3.1, one concludes the following :

Corollary 3.2. Let $(\mho,\eth)$ be a complete TC_bMS with the coefficient $s \geq 1$. Let $\mathcal{G}$ be a self-mapping defined on $\mho$, satisfying the condition

$$\eth(\mathcal{G}\ell,\mathcal{G}\propto) \preceq_{i_3} \nu\eth(\ell,\propto) + \frac{\tau\eth(\ell,\mathcal{G}\ell)\eth(\propto,\mathcal{G}\propto) + \omega\eth(\propto,\mathcal{G}\ell)\eth(\ell,\mathcal{G}\propto)}{1+\eth(\ell,\propto)}$$

for all $\ell,\propto \in \mho$, where ν,τ,ω are non-negative reals with $\nu + \sqrt{2}\tau + \sqrt{2}\omega < 1$, then $\mathcal{G}$ has a unique fixed point.

Theorem 3.3. Let $(\mho, \eth)$ be a complete TC_bMS with coefficient $s \geq 1$. Let $\mathcal{P}$ and $\mathcal{G}$ are self-maps defined on $\mho$, satisfying the condition

$$\eth(\mathcal{P}\ell, \mathcal{G}\propto) \preceq_{i_3} \begin{cases} \nu\eth(\ell,\propto) + \tau\frac{\eth(\ell,\mathcal{P}\ell)\eth(\propto,\mathcal{G}\propto)+\eth(\propto,\mathcal{P}\ell)\eth(\ell,\mathcal{G}\propto)}{\eth(\mathcal{P}\ell,\ell)+\eth(\mathcal{G}\propto,\propto)} \\ +\omega\frac{\eth(\ell,\mathcal{P}\ell)\eth(\ell,\mathcal{G}\propto)+\eth(\propto,\mathcal{P}\ell)\eth(\propto,\mathcal{G}\propto)}{\eth(\mathcal{P}\ell,\ell\propto)+\eth(\mathcal{G}\propto,\ell)}, & \text{if } \mathcal{I} \neq 0, \mathcal{I}_1 \neq 0 \\ 0, & \text{if } \mathcal{I} = 0 \text{ or } \mathcal{I}_1 = 0 \end{cases} \tag{3.1}$$

for all $\ell, \propto \in \mho$, where $\mathcal{I} = \eth(\mathcal{P}\ell, \ell) + \eth(\mathcal{G}\propto, \propto)$ and $\mathcal{I}_1 = \eth(\mathcal{P}\ell, \propto) + \eth(\mathcal{G}\propto, \ell)$ and ν, τ, ω are non-negative reals with $\nu + \sqrt{2}\tau + \omega < \frac{1}{s}$. Then $\mathcal{P}$, $\mathcal{G}$ have unique common fixed point.

Proof. Consider an arbitrary point $\ell_0 \in \mho$. Define $\ell_{2\varpi+1} = \mathcal{P}\ell_{2\varpi}$ and $\ell_{2\varpi+2} = \mathcal{G}\ell_{2\varpi+1}$, $\varpi = 0, 1, 2, \cdots$ Now, we consider two cases.

First, if (for $\varpi = 0, 1, 2, \cdots$)$\eth(\mathcal{P}\ell_{2\varpi}, \ell_{2\varpi}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi+1}) \neq 0$ and $\eth(\mathcal{P}\ell_{2\varpi}, \ell_{2\varpi+1}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi}) \neq 0$, then

$$\begin{aligned} \eth(\ell_{2\varpi+1}, \ell_{2\varpi+2}) &= \eth(\mathcal{P}\ell_{2\varpi}, \mathcal{G}\ell_{2\varpi+1}) \preceq_{i_3} \nu\eth(\ell_{2\varpi}, \ell_{2\varpi+1}) \\ &+ \tau\frac{\eth(\ell_{2\varpi}, \mathcal{P}\ell_{2\varpi})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1}) + \eth(\ell_{2\varpi+1}, \mathcal{P}\ell_{2\varpi})\eth(\ell_{2\varpi}, \mathcal{G}\ell_{2\varpi+1})}{\eth(\mathcal{P}\ell_{2\varpi}, \ell_{2\varpi}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi+1})} \\ &+ \omega\frac{\eth(\ell_{2\varpi}, \mathcal{P}\ell_{2\varpi})\eth(\ell_{2\varpi}, \mathcal{G}\ell_{2\varpi+1}) + \eth(\ell_{2\varpi+1}, \mathcal{P}\ell_{2\varpi})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1})}{\eth(\mathcal{P}\ell_{2\varpi}, \ell_{2\varpi+1}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi})}. \end{aligned}$$

Since $\ell_{2\varpi+1} = \mathcal{P}\ell_{2\varpi}$ and $\ell_{2\varpi+2} = \mathcal{G}\ell_{2\varpi+1}$, therefore

$$\begin{aligned} \eth(\ell_{2\varpi+1}, \ell_{2\varpi+2}) &\preceq_{i_3} \nu\eth(\ell_{2\varpi}, \ell_{2\varpi+1}) \\ &+ \tau\frac{\eth(\ell_{2\varpi}, \ell_{2\varpi+1})\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2}) + \eth(\ell_{2\varpi+1}, \ell_{2\varpi+1})\eth(\ell_{2\varpi}, \ell_{2\varpi+2})}{\eth(\ell_{2\varpi+1}, \ell_{2\varpi}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})} \\ &+ \omega\frac{\eth(\ell_{2\varpi}, \ell_{2\varpi+1})\eth(\ell_{2\varpi}, \ell_{2\varpi+2}) + \eth(\ell_{2\varpi+1}, \ell_{2\varpi+1})\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})}{\eth(\ell_{2\varpi+1}, \ell_{2\varpi+1}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi})}, \end{aligned}$$

or

$$\begin{aligned} \eth(\ell_{2\varpi+1}, \ell_{2\varpi+2}) \preceq_{i_3} \nu\eth(\ell_{2\varpi}, \ell_{2\varpi+1}) + \tau\frac{\eth(\ell_{2\varpi}, \ell_{2\varpi+1})\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})}{\eth(\ell_{2\varpi+1}, \ell_{2\varpi}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})} \\ + \omega\frac{\eth(\ell_{2\varpi}, \ell_{2\varpi+1})\eth(\ell_{2\varpi}, \ell_{2\varpi+2})}{\eth(\ell_{2\varpi+2}, \ell_{2\varpi})}, \end{aligned}$$

which implies that

$$\begin{aligned} ||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})|| &\leq \nu||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})|| \\ &+ \sqrt{2}\tau\frac{||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})||||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||}{||\eth(\ell_{2\varpi+1}, \ell_{2\varpi}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})||} \\ &+ \omega||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})||. \end{aligned}$$

Since $||\eth(\ell_{2\varpi+1}, \ell_{2\varpi}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})|| > ||\eth(\ell_{2\varpi+1}, \ell_{2\varpi})||$, therefore

$$\begin{aligned} ||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})|| \leq \nu||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})|| + \sqrt{2}\tau||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})|| \\ + \omega||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})||, \end{aligned}$$

so that

$$||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})|| \leq \frac{\nu + \omega}{1 - \sqrt{2}\tau}||\eth(\ell_{2\varpi}, \ell_{2\varpi+1})||.$$

Also

$$\begin{aligned} \eth(\ell_{2\varpi+2}, \ell_{2\varpi+3}) &= \eth(\mathcal{P}\ell_{2\varpi+2}, \mathcal{G}\ell_{2\varpi+1}) \preceq_{i_3} \nu\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1}) \\ &+ \tau\frac{\eth(\ell_{2\varpi+2}, \mathcal{P}\ell_{2\varpi+2})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1}) + \eth(\ell_{2\varpi+1}, \mathcal{P}\ell_{2\varpi+2})\eth(\ell_{2\varpi+2}, \mathcal{G}\ell_{2\varpi+1})}{\eth(\mathcal{P}\ell_{2\varpi+2}, \ell_{2\varpi+2}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi+1})} \\ &+ \omega\frac{\eth(\ell_{2\varpi+2}, \mathcal{P}\ell_{2\varpi+2})\eth(\ell_{2\varpi+2}, \mathcal{G}\ell_{2\varpi+1}) + \eth(\ell_{2\varpi+1}, \mathcal{P}\ell_{2\varpi+2})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1})}{\eth(\mathcal{P}\ell_{2\varpi+2}, \ell_{2\varpi+1}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi+2})}. \end{aligned}$$

Since $\ell_{2\varpi+3} = \mathcal{P}\ell_{2\varpi+2}$ and $\ell_{2\varpi+2} = \mathcal{G}\ell_{2\varpi+1}$, we get

$$\begin{aligned}\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3}) \preceq_{i_3}\ & \nu\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1}) \\ & + \tau\frac{\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2}) + \eth(\ell_{2\varpi+1}, \ell_{2\varpi+3})\eth(\ell_{2\varpi+2}, \ell_{2\varpi+2})}{\eth(\ell_{2\varpi+3}, \ell_{2\varpi+2}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})} \\ & + \omega\frac{\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})\eth(\ell_{2\varpi+2}, \ell_{2\varpi+2}) + \eth(\ell_{2\varpi+1}, \ell_{2\varpi+3})\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})}{\eth(\ell_{2\varpi+3}, \ell_{2\varpi+1}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+2})},\end{aligned}$$

or

$$\begin{aligned}\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3}) \preceq_{i_3}\ & \nu\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1}) \\ & + \tau\frac{\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})}{\eth(\ell_{2\varpi+3}, \ell_{2\varpi+2}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})} \\ & + \omega\frac{\eth(\ell_{2\varpi+1}, \ell_{2\varpi+3})\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})}{\eth(\ell_{2\varpi+1}, \ell_{2\varpi+3})},\end{aligned}$$

which implies that

$$\begin{aligned}||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})|| \leq\ & \nu||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})|| \\ & + \sqrt{2}\tau\frac{||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})||||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||}{||\eth(\ell_{2\varpi+3}, \ell_{2\varpi+2}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})||} \\ & + \omega||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||.\end{aligned}$$

Since $||\eth(\ell_{2\varpi+3}, \ell_{2\varpi+2}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})|| > ||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||$, therefore

$$\begin{aligned}||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})|| \leq\ & \nu||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})|| \\ & + \sqrt{2}\tau\frac{||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})||||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||}{||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})||} \\ & + \omega||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||,\end{aligned}$$

or

$$||\eth(\ell_{2\varpi+2}, \ell_{2\varpi+3})|| \leq \frac{\nu + \omega}{1 - \sqrt{2}\tau}||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||.$$

Now, with $\mathfrak{g} = \frac{\nu+\omega}{1-2\tau}$, we have (for all ϖ)

$$||\eth(\ell_{\varpi}, \ell_{\varpi+1})|| \leq \mathfrak{g}||\eth(\ell_{\varpi-1}, \ell_{\varpi})|| \leq \cdots \leq \mathfrak{g}^{\varpi}||\eth(\ell_0, \ell_1)||.$$

So, for any $\aleph > \varpi$, we have

$$\begin{aligned}||\eth(\ell_{\varpi}, \ell_{\aleph})|| & \leq ||\eth(\ell_{\varpi}, \ell_{\varpi+1})|| + ||\eth(\ell_{\varpi+1}, \ell_{\varpi+2})|| \\ & \quad + ||\eth(\ell_{\varpi+2}, \ell_{\varpi+3})|| \cdots + ||\eth(\ell_{\mathfrak{m}-1}, \ell_{\aleph})|| \\ & \leq [s\mathfrak{g}^{\varpi} + s^2\mathfrak{g}^{\varpi+1} + s^3\mathfrak{g}^{\varpi+2} + \cdots + s^{\aleph-\varpi}\mathfrak{g}^{\mathfrak{m}-1}]||\eth(\ell_0, \ell_1)|| \\ & \leq \left[\frac{(s\mathfrak{g})^{\varpi}}{1 - s\mathfrak{g}}\right]||\eth(\ell_0, \ell_1)||,\end{aligned}$$

which implies that

$$||\eth(\ell_{\varpi}, \ell_{\aleph})|| \leq \left[\frac{(s\mathfrak{g})^{\varpi}}{1 - s\mathfrak{g}}\right]||\eth(\ell_0, \ell_1)|| \to 0 \text{ as } \varpi \to \infty.$$

On using Lemma 2.2, we concluded that a sequence $\{\ell_{\varpi}\}$ is Cauchy. Since $\mho$ is a complete, then we can find $\text{ł} \in \mho$ such that $\ell_{\varpi} \to \text{ł}$ as $\varpi \to \infty$. Now, we so that $\text{ł} = \mathcal{P}\text{ł}$, otherwise

$0 \prec_{i_3} \wp = \eth(\text{ł}, \mathcal{P}\text{ł})$ and we obtain

$$\begin{aligned}\wp = \eth(\text{ł}, \mathcal{P}\text{ł}) &\preceq_{i_3} s\eth(\text{ł}, \mathcal{G}\ell_{2\varpi+1}) + s\eth(\mathcal{G}\ell_{2\varpi+1}, \mathcal{P}\text{ł})\\ &\preceq_{i_3} s\eth(\text{ł}, \ell_{2\varpi+2}) + s\nu\eth(\text{ł}, \ell_{2\varpi+1})\\ &+ s\tau\frac{\eth(\text{ł}, \mathcal{P}\text{ł})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1}) + \eth(\ell_{2\varpi+1}, \mathcal{P}\text{ł})\eth(\text{ł}, \mathcal{G}\ell_{2\varpi+1})}{\eth(\mathcal{P}\text{ł}, \text{ł}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi+1})}\\ &+ s\omega\frac{\eth(\text{ł}, \mathcal{P}\text{ł})\eth(\text{ł}, \mathcal{G}\ell_{2\varpi+1}) + \eth(\ell_{2\varpi+1}, \mathcal{P}\text{ł})\eth(\ell_{2\varpi+1}, \mathcal{G}\ell_{2\varpi+1})}{\eth(\mathcal{P}\text{ł}, \ell_{2\varpi+1}) + \eth(\ell_{2\varpi+2}, \text{ł})},\end{aligned}$$

which implies that

$$\begin{aligned}||\wp|| = ||\eth(\text{ł}, \mathcal{P}\text{ł})|| &\leq ||s\eth(\text{ł}, \ell_{2\varpi+2})|| + \nu s||\eth(\text{ł}, \ell_{2\varpi+1})||\\ &+ s\tau\frac{\wp||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})|| + \sqrt{2}||\eth(\ell_{2\varpi+1}, \mathcal{P}\text{ł})||||\eth(\text{ł}, \ell_{2\varpi+2})||}{||\eth(\mathcal{P}\text{ł}, \text{ł}) + \eth(\ell_{2\varpi+2}, \ell_{2\varpi+1})||}\\ &+ s\omega\frac{\wp||\eth(\text{ł}, \ell_{2\varpi+2})|| + \sqrt{2}||\eth(\ell_{2\varpi+1}, \mathcal{P}\text{ł})||||\eth(\ell_{2\varpi+1}, \ell_{2\varpi+2})||}{||\eth(\mathcal{P}\text{ł}, \ell_{2\varpi+1}) + \eth(\ell_{2\varpi+2}, \text{ł})||},\end{aligned}$$

a contradiction, so that $||\wp|| = ||\eth(\text{ł}, \mathcal{P}\text{ł})|| = 0$, that is $\text{ł} = \mathcal{P}\text{ł}$. Similarly, we prove that $\text{ł} = \mathcal{G}\text{ł}$. Let $\alpha^* \in \mho$ be an another common fixed point of $\mathcal{P}$ and $\mathcal{G}$. Then

$$\mathcal{P}\alpha^* = \mathcal{G}\alpha^* = \alpha^*.$$

Since $\mathcal{I} = \eth(\mathcal{P}\text{ł}, \text{ł}) + \eth(\mathcal{G}\alpha^*, \alpha^*) = 0$, therefore by definition of contraction condition

$$\eth(\text{ł}, \alpha^*) = \eth(\mathcal{P}\text{ł}, \mathcal{G}\alpha^*) = 0,$$

so that $\text{ł} = \alpha^*$, which proves the uniqueness of common fixed point.

Second case, we assume that

$$\Big(\eth(\mathcal{P}\ell_{2\varpi}, \ell_{2\varpi}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi+1})\Big) \times \Big(\eth(\mathcal{P}\ell_{2\varpi}, \ell_{2\varpi+1}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi})\Big) = 0.$$

for any ϖ implies

$$\eth(\mathcal{P}\ell_{2\varpi}, \mathcal{G}\ell_{2\varpi+1}) = 0.$$

Now, if $\eth(\mathcal{P}\ell_{2\varpi}, \ell_{2\varpi}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi+1}) = 0$ then

$$\ell_{2\varpi} = \mathcal{P}\ell_{2\varpi} = \ell_{2\varpi+1} = \mathcal{G}\ell_{2\varpi} = \ell_{2\varpi+2}.$$

Thus, we have

$$\ell_{2\varpi+1} = \mathcal{P}\ell_{2\varpi} = \ell_{2\varpi},$$

so there exist ϖ_1 and $\aleph_1$ such that $\varpi_1 = \mathcal{P}\aleph_1 = \aleph_1$.

Using for this arguments, one can also prove that there exist ϖ_2 and $\aleph_2$ such that $\varpi_2 = \mathcal{G}\aleph_2 = \aleph_2$. As $\eth(\mathcal{P}\aleph_1, \aleph_1) + \eth(\mathcal{G}\aleph_2, \aleph_2) = 0$, (due to definition) implies

$$\eth(\mathcal{P}\aleph_1, \mathcal{G}\aleph_2) = 0.$$

So that

$$\varpi_1 = \mathcal{P}\aleph_1 = \mathcal{G}\aleph_2 = \varpi_2,$$

which give that

$$\varpi_1 = \mathcal{P}\aleph_1 = \mathcal{P}\varpi_1.$$

Similarly, we have $\varpi_2 = \mathcal{G}\varpi_2$. As $\varpi_1 = \varpi_2$, implies

$$\mathcal{P}\varpi_1 = \mathcal{G}\varpi_1 = \varpi_1.$$

Therefore $\varpi_1 = \varpi_2$, is a common fixed point of $\mathcal{P}$ and $\mathcal{G}$. Let $\varpi_1^* \in \mho$ be an another common fixed point of $\mathcal{P}$ and $\mathcal{G}$. Then

$$\mathcal{P}\varpi_1^* = \mathcal{G}\varpi_1^* = \varpi_1^*.$$

Since $\mathcal{I} = \eth(\mathcal{P}\varpi_1, \varpi_1) + \eth(\mathcal{G}\varpi_1^*, \varpi_1^*) = 0$, therefore

$$\eth(\varpi_1, \varpi_1^*) = \eth(\mathcal{P}\varpi_1, \mathcal{G}\varpi_1^*) = 0.$$

This implies that $\varpi_1^* = \varpi_1$.

If $\eth(\mathcal{P}\ell_{2\varpi}, \ell_{2\varpi+1}) + \eth(\mathcal{G}\ell_{2\varpi+1}, \ell_{2\varpi}) = 0$, implies

$$\eth(\mathcal{P}\ell_{2\varpi}, \mathcal{G}\ell_{2\varpi+1}) = 0.$$

Then also complete the proof on this preceding lines. □

By setting $\mathcal{P} = \mathcal{G}$, we obtain the following.

Corollary 3.4. Consider $(\mho, \eth)$ be a complete TC_bMS with the coefficient $s \geq 1$. Let $\mathcal{G}$ be a self-mapping defined on $\mho$, satisfying the condition

$$\eth(\mathcal{G}\ell, \mathcal{G}\propto) \preceq_{i_3} \begin{cases} \nu\eth(\ell,\propto) + \tau\frac{\eth(\ell,\mathcal{G}\ell)\eth(\propto,\mathcal{G}\propto)+\eth(\propto,\mathcal{G}\ell)\eth(\ell,\mathcal{G}\propto)}{\eth(\mathcal{G}\ell,\ell)+\eth(\mathcal{G}\propto,\propto)} \\ +\omega\frac{\eth(\ell,\mathcal{G}\ell)\eth(\ell,\mathcal{G}\propto)+\eth(\propto,\mathcal{G}\ell)\eth(\propto,\mathcal{G}\propto)}{\eth(\mathcal{G}\ell,\propto)+\eth(\mathcal{G}\propto,\ell)}, & \text{if } \mathcal{I} \neq 0, \mathcal{I}_1 \neq 0 \\ 0, & \text{if } \mathcal{I} = 0 \text{ or } \mathcal{I}_1 = 0 \end{cases} \tag{3.2}$$

for all $\ell, \propto \in \mho$, where $\mathcal{I} = \eth(\mathcal{G}\ell, \ell) + \eth(\mathcal{G}\ell, \propto)$ and $\mathcal{I}_1 = \eth(\mathcal{G}\ell, \propto) + \eth(\mathcal{G}\propto, \ell)$ and ν, τ, ω are non-negative reals with $\nu + \sqrt{2}\tau + \omega < \frac{1}{s}$. Then $\mathcal{G}$ has unique fixed point.

Example 3.1. Let $\mho = \{0, \frac{1}{3}, 3\}$, define a mapping $\eth : \mho \times \mho \to \mathbb{C}_3$ by $\eth(\ell, \propto) = (1+i_3)|\ell - \propto|^2$, for all $\ell, \propto \in \mho$, where $|.|$ is the real modulus. Then $(\mho, \eth)$ is a complete TC_bMS. Let define a self mapping $\mathcal{G} : \mho \to \mho$ by

$$\mathcal{G}(0) = 0,\ \mathcal{G}\left(\frac{1}{3}\right) = 0 \text{ and } \mathcal{G}(3) = \frac{1}{3}.$$

Consider $\nu = \frac{1}{6}$, $\tau = \frac{1}{5}$ and $\nu = \frac{1}{9}$, then $s(\nu + \sqrt{2}\tau + \sqrt{2}\omega) = s(\frac{1}{6} + \frac{2}{5} + \frac{2}{9}) < 1$. Then, clearly

$$0 = \eth(\mathcal{G}\ell, \mathcal{G}\propto) \preceq_{i_3} \nu\eth(\ell, \propto) + \frac{\tau\eth(\ell, \mathcal{G}\ell)\eth(\propto, \mathcal{G}\propto) + \omega\eth(\propto, \mathcal{G}\ell)\eth(\ell, \mathcal{G}\propto)}{1 + \eth(\ell, \propto)}.$$

Hence, the Corollary 3.2 conditions are fulfilled. Hence, 0 is the unique fixed point of $\mathcal{G}$.

Example 3.2. Consider $\mho = \mathcal{J}(0, \flat)$, $\flat > 1$, for all $\ell, \propto \in \mho$. Define $\eth : \mho \times \mho \to \mathbb{C}_3$ by:

$$\eth(\ell(), \propto ()) = \frac{i_3}{2\pi}\left| \int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{\propto ()}{} \right|^2,$$

a complete TC_bMS, where $\mathcal{O}$ is a closed path in $\mho$ containing a point in 0. We show that $\eth$ is a

TC_bMS. For this,

$$\begin{aligned}
\eth(\ell(), \propto ()) &= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{\propto ()}{}\right|^2 \\
&= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{\wp()}{} + \int_{\mathcal{O}} \frac{\wp()}{} - \int_{\mathcal{O}} \frac{\propto ()}{}\right|^2 \\
&\preceq_{i_3} \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{\wp()}{}\right|^2 + \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\wp()}{} - \int_{\mathcal{O}} \frac{\propto ()}{}\right|^2 \\
&\quad + 2\frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{\wp()}{}\right|\left|\int_{\mathcal{O}} \frac{\wp()}{} - \int_{\mathcal{O}} \frac{\propto ()}{}\right| \\
&\preceq_{i_3} \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{\wp()}{}\right|^2 + \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\wp()}{} - \int_{\mathcal{O}} \frac{\propto ()}{}\right|^2 \\
&\quad + \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{\wp()}{}\right|^2 + \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\wp()}{} - \int_{\mathcal{O}} \frac{\propto ()}{}\right|^2 \\
&= 2(\eth(\ell(), \wp()) + \eth(\wp(), \propto ())).
\end{aligned}$$

Consider the mapping $\mathcal{P}, \mathcal{G} : \mho \to \mho$ by:

$$\mathcal{P}\ell() = ,\ \ \mathcal{G} \propto () = e - 1.$$

The Cauchy integral formula using the mappings $\mathcal{P}$ and $\mathcal{G}$ are analytic, we have:

$$\begin{aligned}
\eth(\mathcal{P}\ell(), \mathcal{G} \propto ()) &= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \quad - \int_{\mathcal{O}} \frac{e-1}{}\right|^2 = 0, \\
\eth(\ell(), \propto ()) &= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{\propto ()}{}\right|^2, \\
\eth(\ell(), \mathcal{P}\ell()) &= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \quad \right|^2 \\
&= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{}\right|^2, \\
\eth(\propto (), \mathcal{G} \propto ()) &= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\propto ()}{} - \int_{\mathcal{O}} \frac{e-1}{}\right|^2 \\
&= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\propto ()}{}\right|^2,
\end{aligned}$$

$$\begin{aligned}
\eth(\ell(), \mathcal{P} \propto ()) &= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \quad \right|^2 \\
&= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{}\right|^2, \\
\eth(\ell(), \mathcal{G} \propto ()) &= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{} - \int_{\mathcal{O}} \frac{e-1}{}\right|^2 \\
&= \frac{i_3}{2\pi}\left|\int_{\mathcal{O}} \frac{\ell()}{}\right|^2.
\end{aligned}$$

Clearly,

$$\eth(\mathcal{P}\ell, \mathcal{G} \propto) \preceq_{i_3} \nu\eth(\ell, \propto) + \frac{\tau\eth(\ell, \mathcal{P}\ell)\eth(\propto, \mathcal{G} \propto) + \omega\eth(\propto, \mathcal{P}\ell)\eth(\ell, \mathcal{G} \propto)}{1 + \eth(\ell, \propto)},$$

for all $\ell, \propto \in \mho$,where ν, τ, ω are non-negative reals with $s(\nu + \sqrt{2}\tau + \sqrt{2}\omega) < 1$.

Therefore, the Theorem 3.1 conditions of all are satisfied, then the maps $\mathcal{P}$ and $\mathcal{G}$ have a unique common fixed point in $\mho$.

4 Applications

Here, we described given an application to the Theorem 3.1 and Corollary 3.2.

Consider $\mho = C[\leftthreetimes_1, \leftthreetimes_2]$ be a family of real continuous functions on $[\leftthreetimes_1, \leftthreetimes_2]$ provide with metric $\eth(\ell, \propto) = (1 + i_3)(|\ell(\mathcal{H}) - \propto(\mathcal{H})|^2)$ for all $\ell, \propto \in C[\leftthreetimes_1, \leftthreetimes_2]$ and $\mathcal{H} \in [\leftthreetimes_1, \leftthreetimes_2]$, where $|.|$ is the real modulus. Then, $(\mho, \eth)$ is a complete TC_bMS.

Now, we consider the non-linear Fredholm integral equation of system is

$$\ell(\mathcal{H}) = \curlywedge(\mathcal{H}) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1} \int_{\leftthreetimes_1}^{\leftthreetimes_2} \eta_1(\mathcal{H}, \varsigma, \ell(\varsigma)) d\varsigma$$

and

$$\ell(\mathcal{H}) = \curlywedge(\mathcal{H}) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1} \int_{\leftthreetimes_1}^{\leftthreetimes_2} \eta_2(\mathcal{H}, \varsigma, \ell(\varsigma)) d\varsigma,$$

where $\mathcal{H}, \varsigma \in [\leftthreetimes_1, \leftthreetimes_2]$. Assume that $\eta_1, \eta_2 : [\leftthreetimes_1, \leftthreetimes_2] \times [\leftthreetimes_1, \leftthreetimes_2] \times \mho \to \mathbb{R}$ and $\curlywedge : [\leftthreetimes_1, \leftthreetimes_2] \to \mathbb{R}$ continuous, where $\curlywedge(\mathcal{H})$ is a function in $\mho$. We define a partial order $\preceq_{i_3}$ in $\mathbb{C}_3$ as $\ell \preceq_{i_3} \propto$ if and only if $\ell \leq \propto$.

Theorem 4.1. Let $(\mho, \eth)$ is a complete TC_bMS equipped with metric $\eth(\ell, \propto) = (1 + i_3)(|\ell(\mathcal{H}) - \propto(\mathcal{H})|^2)$ for all $\ell, \propto \in \mho$, $\mathcal{H} \in [\leftthreetimes_1, \leftthreetimes_2]$ and $\mathcal{P}, \mathcal{G} : \mho \to \mho$ be continuous operator on $\mho$ defined by

$$\mathcal{P}\ell(\mathcal{H}) = \curlywedge(\mathcal{H}) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1} \int_{\leftthreetimes_1}^{\leftthreetimes_2} \eta_1(\mathcal{H}, \varsigma, \ell(\varsigma)) d\varsigma \tag{4.1}$$

and

$$\mathcal{G}\ell(\mathcal{H}) = \curlywedge(\mathcal{H}) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1} \int_{\leftthreetimes_1}^{\leftthreetimes_2} \eta_2(\mathcal{H}, \varsigma, \ell(\varsigma)) d\varsigma. \tag{4.2}$$

If there exists $\nu < 1$ such that for all $\ell, \propto \in \mho$ with $\ell \neq \propto$ and $\varsigma, \mathcal{H} \in [\leftthreetimes_1, \leftthreetimes_2]$ satisfying the following inequality

$$|\eta_1(\mathcal{H}, \varsigma, \ell(s)) - \eta_2(\mathcal{H}, \varsigma, \propto(\varsigma))| \leq \nu |\ell(\mathcal{H}) - \propto(\mathcal{H})|, \tag{4.3}$$

then the integral operators defined by (4.1) and (4.2) have a unique common solution.

Proof. Let

$$\begin{aligned}
(1 + i_3)(|\mathcal{P}\ell(\mathcal{H}) - \mathcal{G}\propto(\mathcal{H})|^2) &= \frac{(1 + i_3)}{|\leftthreetimes_2 - \leftthreetimes_1|} \left(\left| \int_{\leftthreetimes_1}^{\leftthreetimes_2} \eta_1(\mathcal{H}, \varsigma, \ell(\varsigma)) d\varsigma \right. \right. \\
&\quad \left. \left. - \int_{\leftthreetimes_1}^{\leftthreetimes_2} \eta_2(\mathcal{H}, \varsigma, \propto(\varsigma)) d\varsigma \right|^2 \right) \\
&\leq \frac{(1 + i_3)}{|\leftthreetimes_2 - \leftthreetimes_1|} \left(\int_{\leftthreetimes_1}^{\leftthreetimes_2} |\eta_1(\mathcal{H}, \varsigma, \ell(\varsigma)) \right. \\
&\quad \left. - \eta_2(\mathcal{H}, \varsigma, \propto(\varsigma))|^2 d\varsigma \right) \\
&\leq \frac{(1 + i_3)\nu}{|\leftthreetimes_2 - \leftthreetimes_1|} \left(\int_{\leftthreetimes_1}^{\leftthreetimes_2} |\ell(\mathcal{H}) - \propto(\mathcal{H})|^2 d\varsigma \right) \\
&\leq \frac{\nu}{|\leftthreetimes_2 - \leftthreetimes_1|} \int_{\leftthreetimes_1}^{\leftthreetimes_2} (1 + i_3)|\ell(\mathcal{H}) - \propto(\mathcal{H})|^2 d\varsigma \\
&\leq \frac{\nu(1 + i_3)|\ell(\mathcal{H}) - \propto(\mathcal{H})|^2}{|\leftthreetimes_2 - \leftthreetimes_1|} \int_{\leftthreetimes_1}^{\leftthreetimes_2} d\varsigma.
\end{aligned}$$

Therefore,

$$\eth(\mathcal{P}\ell, \mathcal{G} \propto)) \leq \nu\eth(\ell, \propto).$$

Hence, all the conditions of Theorem 3.1 are fulfilled with $\nu < 1$, $\tau = \omega = 0$ and so, the integral operators $\mathcal{P}$ and $\mathcal{G}$ defined by (4.1) and (4.2) have a unique common solution. □

Theorem 4.2. Let $\mho = \mathbb{C}^{\varpi}$ be a TC_bMS with the metric

$$\eth(\ell, \propto) = \sum_{i=1}^{\varpi} \left(|\ell_i - \propto_i|^2 + i_3|\ell_i - \propto_i|^2\right),$$

where $\ell, \propto \in \mho$. If

$$\sum_{j=1}^{\varpi} |\nu_{ij}|^2 \preceq_{i_3} \nu < 1, \ \forall\ i = 1, 2, \ldots \varpi,$$

then the linear system

$$\begin{cases} \mathsf{T}_1 = \rho_{11}\ell_1 + \rho_{12}\ell_2 + \cdots + \rho_{1\varpi}\ell_{\varpi} \\ \mathsf{T}_2 = \rho_{21}\ell_1 + \rho_{22}\ell_2 + \cdots + \rho_{2\varpi}\ell_{\varpi} \\ \quad\vdots \\ \mathsf{T}_{\varpi} = \rho_{\varpi 1}\ell_1 + \rho_{\varpi 2}\ell_2 + \cdots + \rho_{\varpi\varpi}\ell_{\varpi} \end{cases}$$

of ϖ linear equations with ϖ unknowns has a unique solution.

Proof. Define $\mathcal{G} : \mho \to \mho$ by

$$\mathcal{G}(\ell) = \mathcal{Y}\ell + \mathsf{T},$$

where $\ell = (\ell_1, \ell_2, \ell_3, \ldots.\ell_{\varpi}) \in \mathbb{C}^{\varpi}$, $\mathsf{T} = (\mathsf{T}_1, \mathsf{T}_2, \ldots.\mathsf{T}_{\varpi}) \in \mathbb{C}^{\varpi}$ and

$$\mathcal{Y} = \begin{pmatrix} \rho_{11} & \rho_{11} & \cdots & \rho_{1\varpi} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2\varpi} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{\varpi 1} & \rho_{\varpi 2} & \cdots & \rho_{\varpi\varpi} \end{pmatrix}.$$

Now,

$$\begin{aligned} \eth(\mathcal{G}(\ell), \mathcal{G}(\propto)) &= \sum_{j=1}^{\varpi} (|\nu_{ij}(\ell_j - \propto_j)|^2 + i_3|\nu_{ij}(\ell_j - \propto_j)|^2) \\ &\preceq_{i_3} \sum_{j=1}^{\varpi} |\nu_{ij}|^2 \left(\sum_{j=1}^{\varpi} (|(\ell_j - \propto_j)|^2 + i_3|(\ell_j - \propto_j)|^2) \right) \\ &\preceq_{i_3} \nu \sum_{j=1}^{\varpi} (|(\ell_j - \propto_j)|^2 + i_3|(\ell_j - \propto_j)|^2) \\ &= \nu\eth(\ell, \propto) \\ &= \nu\eth(\ell, \propto) + \frac{\tau\eth(\ell, \mathcal{G}\ell)\eth(\propto, \mathcal{G} \propto) + \omega\eth(\propto, \mathcal{G}\ell)\eth(\ell, \mathcal{G} \propto)}{1 + \eth(\ell, \propto)}. \end{aligned}$$

Hence, all the conditions of Corollary 3.2 are satisfied with $\nu = \frac{1}{6}, \tau = 0, \omega = 0$, $s(\nu + \sqrt{2}\tau + \sqrt{2}\omega) < 1$ and so, the unique solution for linear system of equation. □

5 Conclusion

This paper introduced tricomplex valued b-metric spaces and established novel common fixed point theorems, generalizing key results from simpler spaces. We demonstrated the practical utility of our theory by proving the existence and uniqueness of a solution to a non-linear Fredholm integral equation. Further, we plan to extend our results to find the solutions of fractional differential equations. Namdev, Tiwari et al. [24] introduced bicomplex parametric partial metric space and proved fixed point theorems. It is an open problem to define tricomplex parametric partial metric space and tricomplex parametric partial b-metric space and prove fixed point theorems.

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