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BLOW-UP BEHAVIOR IN DEGENERATE NONLINEAR REACTION-DIFFUSION EQUATIONS: THEORY AND NUMERICAL SIMULATION

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Abstract In this paper, we investigate the blow-up dynamics of solutions to a semilinear degenerate parabolic reaction–diffusion system, in which the degeneracy arises in the time derivative. Under suitable conditions on the initial data, we establish the occurrence of finite-time blow-up. To complement the theoretical analysis, we design and implement a finite-difference numerical scheme in MATLAB to approximate the solution of the discretized system. The numerical simulations capture the development of sharp spatial gradients and pronounced peaks near the blow-up region, offering valuable insight into the formation and structure of the blow-up profile. This computational study constitutes a preliminary step toward the precise identification and characterization of the blow-up set.

1 Introduction

In recent decades, considerable attention has been devoted to the study of finite-time blow-up phenomena in solutions to nonlinear and degenerate parabolic equations and systems. Blow-up refers to the situation where a solution becomes unbounded in finite time, typically due to strong nonlinearities or degeneracies in the diffusion terms. This behavior has been widely observed in models from physics, chemistry, and biology, including heat conduction in heterogeneous media, chemical reactions, and population dynamics.

The mathematical analysis of blow-up phenomena began with classical studies such as Kaplan [16], who investigated the growth of solutions of quasilinear parabolic equations and provided foundational results on blow-up rates. Later, Samarskii et al. [27] developed a comprehensive theory for blow-up problems in quasilinear parabolic equations, focusing on both analytical and numerical aspects. Galaktionov and Vázquez [14] provided a detailed classification of blow-up behaviors and discussed self-similar structures arising in nonlinear parabolic systems. The monograph by Quittner and Souplet [24] offers an extensive treatment of superlinear parabolic problems, addressing global existence, blow-up, and steady states.

The theoretical framework for degenerate equations was further developed by Coddington and Levinson [11], who established fundamental results in the theory of ordinary differential equations that form the backbone of blow-up analysis. Chipot et al. [10] investigated stationary solutions and blow-up for semilinear parabolic equations under nonlinear boundary conditions, linking boundary effects to finite-time singularities. Boni [3] examined blow-up and asymptotic behavior for nonlinear parabolic equations with nonlinear boundary conditions, demonstrating how boundary nonlinearities affect blow-up rates.

In the context of degenerate systems, Du [12] studied blow-up for degenerate reaction–diffusion equations with localized sources and provided sharp conditions for finite-time blow-up. Shi et al.

[28] and Zhou et al. [30] extended these results to singular and nonlocal systems, revealing that blow-up strongly depends on the spatial degeneracy and nonlocal sources. Li et al. [20] considered global existence and blow-up in degenerate and singular parabolic systems with localized sources, showing the crucial influence of nonlinearity on solution profiles.

In particular, Belyacine [2] studied certain classes of degenerate semilinear parabolic systems and proposed a spatial truncation technique to remove degeneracy by restricting the domain to $(\varepsilon_1, 1-\varepsilon_2)$. Using sub- and supersolution methods, the existence and uniqueness of positive local solutions were established. Later, Nisse and Belyacine [22] analyzed the blow-up behavior of such systems and proved that under suitable initial data, blow-up occurs precisely at the degenerate boundary points.

Floater [13] considered the scalar degenerate parabolic problem

$$\begin{cases} x\,u_t(x,t) = u_{xx}(x,t) + u^p(x,t), & x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in [0,1], \end{cases} \tag{1.1}$$

and proved that for $1 < p \leq 2$ and concave, sufficiently large initial data, u blows up in finite time at the degenerate boundary $x = 0$. This work was later generalized by Kouche [18], who examined a coupled degenerate system and established that blow-up occurs under similar conditions at $x = 0$.

The importance of such studies lies in their ability to describe singular behaviors that occur in physical processes governed by diffusion–reaction mechanisms. Du's analysis [12] and the extensions by Li et al. [20] and Shi et al. [28] provide a strong theoretical foundation for the degenerate parabolic system investigated here.

From a numerical standpoint, numerous authors have explored finite difference and adaptive time-stepping methods for approximating blow-up times and profiles. Abia et al. [1] developed explicit and implicit Euler finite difference schemes with adaptive time stepping for semilinear reaction–diffusion equations, showing that the numerical blow-up time can closely match the theoretical one. Cho and Okamoto [9] extended these methods to axisymmetric nonlinear heat equations, whereas Cho and Lu [5] proposed new schemes for coupled parabolic systems exhibiting blow-up. Similarly, Rasheed et al. [25] and Rasheed et al. [26] analyzed the numerical estimation of blow-up times for semilinear heat equations with gradient terms, confirming convergence to analytical blow-up times. Khalil et al. [17] investigated numerical finite difference approximations of coupled parabolic systems with blow-up and demonstrated the accuracy of adaptive schemes. Polyanin and Shingareva [23] presented analytical and numerical approaches for nonlinear blow-up problems, including non-local transformations and exact solutions. In [4] Z. Chebana et al. are discussed the blow-up at finite-time of exact and numerical solutions for a class of semi-linear heat equation with nonlinear nonlocal conditions of second type. N. Yılmaz [29] proved that under suitable assumptions on the initial data, solutions with negative initial energy to a logarithmic Petrovsky equation with variable exponent experience finite-time blow-up.

Hamadneh et al. [15] extended the study of finite-time blow-up to fractional reaction–diffusion equations, combining analytical proofs with numerical simulations. Momani et al. [21] investigated weak solutions and blow-up phenomena in nonlinear reaction–diffusion equations with integral conditions, supported by numerical experiments. These contributions highlight the strong interaction between theoretical and computational aspects of blow-up dynamics.

The importance of blow-up analysis has recently expanded to nonlocal and viscoelastic models. Choucha et al. [6] investigated systems of nonlocal singular viscoelastic equations with sources and distributed delay terms, proving global existence and blow-up results. In a related study, Choucha and Boulaaras [7] analyzed a viscoelastic plate equation with logarithmic nonlinearity and variable exponents, establishing global existence, general decay, and blow-up of solutions. Furthermore, Choucha et al. [8] examined a coupled nonlinear viscoelastic Kirchhoff system with fractional boundary conditions and variable exponents, presenting results on blow-up, growth, and decay rates. More recently, Laribi et al. [19] studied the exponential decay of laminated beams with nonlinear time-varying delays and microtemperature effects, showing how dynamic boundary terms influence the overall stability.

Overall, these developments demonstrate that blow-up behavior remains a central challenge in the analysis of nonlinear parabolic systems. The combination of degeneracy, nonlinearity, and nonlocal effects requires sophisticated analytical and numerical tools to capture the underlying dynamics accurately.

The aim of this paper is to develop a numerical method to compute blow-up (amplification) solutions of a one-dimensional degenerate semilinear reaction–diffusion system. We consider the following initial–boundary value problem:

$$\begin{cases} x\,u_t(x,t) = u_{xx}(x,t) + v^p(x,t), & x \in (0,1),\ t>0, \\ (1-x)\,v_t(x,t) = v_{xx}(x,t) + u^p(x,t), & x \in (0,1),\ t>0, \\ u(x,t) = v(x,t) = 0, & x \in \{0,1\},\ t>0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in [0,1], \end{cases} \tag{1.2}$$

where $p > 1$, and the initial data u_0 and v_0 belong to an appropriate functional space. We say that the numerical solution u (resp. v) blows up in a finite time T if

$$\lim_{t \nearrow T} \left(\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \right) = +\infty.$$

The degeneracy in (1.2) appears in the time derivatives: the coefficients of u_t and v_t vanish when $x \to 0$ and $x \to 1$, respectively, which alters the parabolic nature of the problem. The theoretical and numerical approaches mentioned above motivate the development of robust methods capable of detecting and reproducing blow-up profiles in such degenerate systems.

Organization of the paper. Section 2 recalls the theoretical results of [22] on finite-time blow-up for the continuous and semi-discrete versions of problem (1.2). In Section 3, we introduce the numerical discretization based on the finite difference θ-scheme. Section 4 presents numerical experiments illustrating the blow-up behavior and confirming the theoretical predictions obtained from the analytical framework.

2 Preliminaries

In this section, we recall several theoretical results obtained by Niss and Belyacine [22] concerning the finite-time blow-up of solutions to the continuous problem (1.2). We assume that a classical positive solution (u,v) of (1.2) exists on $[0,1] \times [0,T[$, that is,

$$u, v \in C([0,1] \times [0,T[) \cap C^{2,1}((0,1) \times (0,T)).$$

To study the blow-up behavior, the authors applied the eigenfunction technique (Fourier coefficient method), originally introduced by Kaplan [16], to system (1.2). Consider the associated spectral problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda\, x(1-x)\,\phi(x), & x \in (0,1), \\ \phi(0) = \phi(1) = 0, & \\ \phi(x) > 0, & x \in (0,1). \end{cases} \tag{2.1}$$

It is known (see [11]) that problem (2.1) has a unique positive eigenvalue $\lambda > 0$, associated with an eigenfunction $\phi \in C([0,1]) \cap C^2((0,1))$.

Let (u,v) be a solution of (1.2) on $[0,1] \times [0,T[$. We introduce the following weighted Fourier coefficients:

$$\begin{cases} \mathcal{U}(t) = \displaystyle\int_0^1 x\,\phi(x)\,u(x,t)\,dx, & t \in [0,T[, \\ \mathcal{V}(t) = \displaystyle\int_0^1 (1-x)\,\phi(x)\,v(x,t)\,dx, & t \in [0,T[, \end{cases}$$

and define

$$\mathcal{W}(t) = \mathcal{U}(t) + \mathcal{V}(t).$$

Lemma 2.1 ([22]). *Let $p > 1$. Then there exists a constant $C > 0$ such that*

$$\frac{d\mathcal{W}(t)}{dt} \geq -\lambda\,\mathcal{W}(t) + \frac{C}{2^{\,p-1}}\mathcal{W}^p(t), \qquad \forall\, t \in (0,T).$$

Definition 2.2 (Finite-time blow-up). A solution (u,v) of (1.2) is said to blow up in finite time if there exists a time $T_b < \infty$ such that

$$\|u(\cdot,t)\|_\infty < \infty, \qquad \|v(\cdot,t)\|_\infty < \infty, \qquad \forall\, t \in [0,T_b[,$$

but

$$\lim_{t\to T_b} \|u(\cdot,t)\|_\infty = \infty, \qquad \lim_{t\to T_b} \|v(\cdot,t)\|_\infty = \infty,$$

where $\|u(\cdot,t)\|_\infty = \sup_{0\leq x\leq 1} |u(x,t)|$, and similarly for v. The time T_b is called the *blow-up time*.

Definition 2.3 (Blow-up point). A point $x \in (0,1)$ is said to be a blow-up point of (1.2) if there exists a sequence (x_m,t_m) such that

$$t_m \to T_b, \qquad x_m \to x,$$

and

$$u(x_m,t_m) \to \infty, \qquad v(x_m,t_m) \to \infty \qquad \text{as } m \to \infty.$$

The following theorem provides a sufficient condition on the initial data that guarantees finite-time blow-up of the solution.

Theorem 2.4 ([22]). *Let $p > 1$ and let C be the constant appearing in Lemma 2.1. If the initial data satisfy*

$$\int_0^1 x\,\phi(x)\,u_0(x)\,dx \;+\; \int_0^1 (1-x)\,\phi(x)\,v_0(x)\,dx \;>\; 2\left(\frac{\lambda}{C}\right)^{\frac{1}{p-1}},$$

then the corresponding solution (u,v) of (1.2) blows up in a finite time.

3 The semidiscrete problem

In this section, we approximate system (1.2) by applying the online method (OLM) to discretize the spatial variable. Let N be a positive integer and define the spatial step size by $h = 1/(N+1)$. The grid points are given by $x_i = ih$, for $i = 0,\dots,N+1$, in the interval $]0,1[$.

To approximate the spatial derivatives, we employ finite difference techniques. In particular, we use the centered second–order finite difference operator for the second derivatives of u and v with respect to x. This leads to the following system of ordinary differential equations:

$$\begin{cases} \dfrac{\partial u_i}{\partial t}(t) = \dfrac{1}{x_i h^2}\,(u_{i+1} - 2u_i + u_{i-1}) + \dfrac{1}{x_i} v_i^p, & i = 1,\dots,N,\ t > 0, \\ \dfrac{\partial v_i}{\partial t}(t) = \dfrac{1}{(1-x_i)h^2}\,(v_{i+1} - 2v_i + v_{i-1}) + \dfrac{1}{1-x_i} u_i^p, & i = 1,\dots,N,\ t > 0, \\ u_0(t) = u_{N+1}(t) = v_0(t) = v_{N+1}(t) = 0, & t > 0, \\ u_i(0) = u_0(x_i) \geq 0, \qquad v_i(0) = v_0(x_i) \geq 0. \end{cases} \tag{3.1}$$

Here, $u_i = u(x_i,t)$ and $v_i = v(x_i,t)$ for $1 \leq i \leq N$.

System (3.1) can be written compactly in matrix form as

$$\begin{cases} \dfrac{\partial \mathcal{U}}{\partial t}(t) = A_h\mathcal{U} + f(\mathcal{V}), & t > 0, \\ \dfrac{\partial \mathcal{V}}{\partial t}(t) = B_h\mathcal{V} + f(\mathcal{U}), & t > 0, \end{cases} \tag{3.2}$$

where the matrices A_h and B_h are defined by

$$A_h = \frac{1}{h^3}\begin{bmatrix} -\frac{2}{1} & 1 & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & -\frac{2}{2} & \frac{1}{2} & \ddots & & \vdots \\ 0 & \frac{1}{3} & -\frac{2}{3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{1}{N-1} \\ 0 & \cdots & \cdots & 0 & \frac{1}{N} & -\frac{2}{N} \end{bmatrix}$$

and

$$B_h = \frac{1}{h^3}\begin{bmatrix} -\frac{2}{N} & \frac{1}{N} & 0 & \cdots & \cdots & 0 \\ \frac{1}{N-1} & -\frac{2}{N-1} & \frac{1}{N-1} & \ddots & & \vdots \\ 0 & \frac{1}{N-2} & -\frac{2}{N-2} & \frac{1}{N-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{1} & -\frac{2}{1} \end{bmatrix}.$$

3.1 Properties of the matrices $\mathcal{A}_h$ and $\mathcal{B}_h$

Following the approach in [22], the matrix $\mathcal{A}_h$ can be decomposed as

$$\mathcal{A}_h = \mathcal{A}_h^s + \mathcal{T}_h^1,$$

where $\mathcal{A}_h^s$ is a symmetric tridiagonal matrix and $\mathcal{T}_h^1$ is a strictly upper triangular matrix with nonnegative entries. They are explicitly given by

$$\mathcal{A}_h^s = \frac{1}{h^3}\begin{bmatrix} -2 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & -\frac{2}{2} & \frac{1}{3} & \ddots & & \vdots \\ 0 & \frac{1}{3} & -\frac{2}{3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{1}{N} \\ 0 & \cdots & \cdots & 0 & \frac{1}{N} & -\frac{2}{N} \end{bmatrix}, \qquad \mathcal{T}_h^1 = \frac{1}{h^3}\begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \frac{1}{6} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \frac{1}{N(N-1)} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}.$$

Analogously, the matrix $\mathcal{B}_h$ can be written as

$$\mathcal{B}_h = \mathcal{B}_h^s + \mathcal{T}_h^2,$$

where $\mathcal{B}_h^s$ is a symmetric tridiagonal matrix and $\mathcal{T}_h^2$ is a strictly lower triangular matrix with nonnegative elements:

$$\mathcal{B}_h^s = \frac{1}{h^3}\begin{bmatrix} -\frac{2}{N} & \frac{1}{N} & 0 & \cdots & \cdots & 0 \\ \frac{1}{N} & -\frac{2}{N-1} & \frac{1}{N-1} & \ddots & & \vdots \\ 0 & \frac{1}{N-1} & -\frac{2}{N-2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & \frac{1}{2} & -\frac{2}{1} \end{bmatrix}, \qquad \mathcal{T}_h^2 = \frac{1}{h^3}\begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1}{N(N-1)} & 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \frac{1}{2} & 0 \end{bmatrix}.$$

Lemma 3.1. *The matrix $\mathcal{A}_h^s$ has N real, distinct, and strictly negative eigenvalues*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N,$$

where λ_1 denotes the largest eigenvalue.

Lemma 3.2. *The eigenvector associated with the eigenvalue λ_1 has strictly positive components.*

Remark 3.3. Let $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ be the eigenvector corresponding to λ_1, normalized so that

$$\sum_{i=1}^{N} \phi_i = 1.$$

Then $\phi_i > 0$ for every $i = 1, \dots, N$.

Remark 3.4. The vector

$$\omega = (\phi_N, \phi_{N-1}, \dots, \phi_1)^T = (\phi_{N-i+1})_{1 \le i \le N}$$

is an eigenvector of $\mathcal{B}_h^s$ associated with the same eigenvalue λ_1.

3.2 Blow-up time estimate

To establish the finite-time blow-up of the solution to system (3.2), we apply the Kaplan method. Define

$$\begin{cases} \varphi_h(t) = \langle \mathcal{U}_h(t), \phi \rangle, & t \in]0, T[, \\ \psi_h(t) = \langle \mathcal{V}_h(t), \omega \rangle, & t \in]0, T[, \end{cases} \tag{3.3}$$

where T denotes the maximal existence time of the solution $(\mathcal{U}_h, \mathcal{V}_h)$.

We further introduce the functional

$$\begin{cases} \Phi_h(t) = \varphi_h(t) + \psi_h(t), \\ \Phi_h(0) = \varphi_h(0) + \psi_h(0). \end{cases}$$

Theorem 3.5 ([22]). *Assume $p > 1$, set $\mu = -\lambda_1$, and define*

$$c = \min_{1 \le i \le N} \left(\frac{\phi_i}{ih\, \phi_{N-i+1}} \right) = \min_{1 \le i \le N} \left(\frac{\phi_{N-i+1}}{(N-i+1)h\, \phi_i} \right).$$

If the initial condition satisfies

$$\Phi_h(0) > 2\, c^{\frac{1}{1-p}}\, \mu^{\frac{1}{p-1}},$$

then the corresponding solution blows up in finite time, and the maximal blow-up time T satisfies

$$T \le \frac{1}{\mu(p-1)} \log \left(\frac{2^{c/(p-1)}}{2^{c/(p-1)} - \mu\, \Phi_h(0)^{\,1-p}} \right).$$

3.3 Properties of the solution to the semi-discrete problem

From system (3.1), we introduce the vector

$$\mathcal{W}_h(t) = (\omega_1(t), \dots, \omega_N(t), \omega_{N+1}(t), \dots, \omega_{2N}(t))^T,$$

where, for $t \ge 0$,

$$\omega_j(t) = \begin{cases} u_j(t), & 1 \le j \le N, \\ v_{j-N}(t), & N+1 \le j \le 2N, \end{cases}$$

and each component satisfies the nonlinear ordinary differential equation

$$\frac{d\omega_j(t)}{dt} - \delta_h^2(\omega_j(t)) = F_j(\omega_j(t)), \qquad 1 \le j \le 2N, \tag{3.4}$$

where

$$\delta_h^2(\omega_j(t)) = \begin{cases} \dfrac{1}{jh^3}\left(u_{j+1} - 2u_j + u_{j-1}\right), & 1 \leq j \leq N, \\ \dfrac{1}{(2N+1-j)h^3}\left(v_{j+1-N} - 2v_{j-N} + v_{j-1-N}\right), & N+1 \leq j \leq 2N, \end{cases}$$

and the nonlinear source term is defined by

$$F_j(\omega_j(t)) = \begin{cases} \dfrac{\omega_{j+N}^p}{jh}, & 1 \leq j \leq N, \\ \dfrac{\omega_{j-N}^p}{(2N+1-j)h}, & N+1 \leq j \leq 2N. \end{cases}$$

The initial conditions are

$$\omega_j(0) = u_0(x_j) \quad \text{for } 1 \leq j \leq N, \qquad \omega_j(0) = v_0(x_j) \quad \text{for } N+1 \leq j \leq 2N.$$

For convenience, we also set $\omega_0(t) = \omega_{2N+1}(t) = 0$. Since the nonlinear term is locally Lipschitz continuous, the system (3.4) admits a unique solution on its maximal interval of existence.

Proposition 3.6 ([2]). *Let $\mathcal{W}_h(t) \in C^1([0,T[,\mathbb{R}^N)$ be a solution of (3.4) on $]0,T[$, and assume that*

$$\omega_j(0) \geq 0, \qquad 1 \leq j \leq 2N.$$

Then, for all $t \in]0,T[$,

$$\omega_j(t) \geq 0, \qquad 1 \leq j \leq 2N.$$

3.4 Convergence of the Semi-Discrete Solution

In this subsection, we prove that for every fixed interval $[0,T]$ on which the continuous functions u and v are defined, the semi-discrete approximation $(\mathcal{U}_h, \mathcal{V}_h)$ obtained from (3.1) converges to the exact solution (u,v) as $h \to 0$.

Define the error vectors

$$\tau_h(t) = \mathcal{U}_h(t) - u_h(t), \qquad \varrho_h(t) = \mathcal{V}_h(t) - v_h(t),$$

where

$$u_h(t) = \big(u(x_1,t),\ldots,u(x_N,t)\big), \qquad v_h(t) = \big(v(x_1,t),\ldots,v(x_N,t)\big).$$

Lemma 3.7. *Let $z(x,t)$ and $y(x,t)$ be non-negative functions and let $\mathcal{C}_1'', \mathcal{C}_2'' > 0$. Then, for all indices j,*

$$-\mathcal{C}_1''\left(y(x_j,t) - |\varrho_h(t)|\right) \leq \mathcal{C}_1''\,|\varrho_h(t) - y(x_j,t)|,$$

$$-\mathcal{C}_2''\left(z(x_j,t) - |\tau_h(t)|\right) \leq \mathcal{C}_2''\,|\tau_h(t) - z(x_j,t)|.$$

Theorem 3.8. *Assume that problem (1.2) admits a classical solution $(u,v) \in (C^{4,1}([0,1] \times [0,T]))^2$ and that the initial data of (3.1) satisfy*

$$\big\|\mathcal{W}_h(0) - \omega_h(0)\big\| = o(1) \qquad \text{as } h \to 0, \tag{3.5}$$

where

$$\omega_h(t) = \big(u(x_1,t),\ldots,u(x_N,t),v(x_1,t),\ldots,v(x_N,t)\big).$$

Then, for h sufficiently small, the semi-discrete problem (3.1) admits a unique solution $\mathcal{W}_h \in C^1([0,T];\mathbb{R}^{2N})$ such that

$$\max_{0\leq t\leq T} \big\|\mathcal{W}_h(t) - \omega_h(t)\big\|_\infty = O\Big(\big\|\mathcal{W}_h(0) - \omega_h(0)\big\|_\infty + h^2\Big) \qquad \text{as } h \to 0. \tag{3.6}$$

Proof. Choose $\alpha, \beta > 0$ such that

$$\|u(\cdot,t)\|_\infty < \alpha, \qquad \|v(\cdot,t)\|_\infty < \beta, \qquad \forall t \in [0,T]. \tag{3.7}$$

For each h, system (3.1) admits a unique solution $(\mathcal{U}_h(t), \mathcal{V}_h(t)) \in C^1([0, T_b^h[; \mathbb{R}^{2N})$. Let $t(h)$ denote the maximal value satisfying $t(h) < \min\{T, T_b^h\}$ and

$$\|\mathcal{U}_h(t) - u_h(t)\|_\infty < 1, \qquad \|\mathcal{V}_h(t) - v_h(t)\|_\infty < 1, \qquad \forall t \in [0, t(h)]. \tag{3.8}$$

The existence of $t(h) > 0$ follows from (3.5).

By the triangle inequality,

$$\|\mathcal{U}_h(t)\|_\infty \le \|u(\cdot,t)\|_\infty + \|\mathcal{U}_h(t) - u_h(t)\|_\infty, \qquad \|\mathcal{V}_h(t)\|_\infty \le \|v(\cdot,t)\|_\infty + \|\mathcal{V}_h(t) - v_h(t)\|_\infty,$$

which implies

$$\|\mathcal{U}_h(t)\|_\infty \le \alpha + 1, \qquad \|\mathcal{V}_h(t)\|_\infty \le \beta + 1, \qquad \forall t \in [0, t(h)]. \tag{3.9}$$

Using Taylor expansion, for $t \in (0, t(h))$,

$$\begin{cases} jh\,\dfrac{d\tau_j(t)}{dt} - \delta^2\tau_j(t) = p\,\eta^{p-1}\varrho_j(t) + o(h^2), \\ (N+1-j)h\,\dfrac{d\varrho_j(t)}{dt} - \delta^2\varrho_j(t) = p\,\xi^{p-1}\tau_j(t) + o(h^2), \end{cases} \tag{3.10}$$

where

$$\xi_h(t) = \lambda_t \mathcal{U}_h(t) + (1-\lambda_t)u_h(t), \qquad \eta_h(t) = \lambda_t \mathcal{V}_h(t) + (1-\lambda_t)v_h(t), \qquad \lambda_t \in (0,1).$$

From (3.7)–(3.9), there exist constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{K} > 0$ such that

$$\begin{cases} jh\,\dfrac{d\tau_j(t)}{dt} - \delta^2\tau_j(t) \le \mathcal{C}_1|\varrho_j(t)| + \mathcal{K}h^2, \\ (N+1-j)h\,\dfrac{d\varrho_j(t)}{dt} - \delta^2\varrho_j(t) \le \mathcal{C}_2|\tau_j(t)| + \mathcal{K}h^2. \end{cases} \tag{3.11}$$

Define comparison functions

$$z(x,t) = \exp\left((L+1)t + \mathcal{C}x^2\right)\left(\|\mathcal{U}_h(0) - u_h(0)\|_\infty + \mathcal{Q}h^2\right),$$

$$y(x,t) = \exp\left((L+1)t + \mathcal{C}x^2\right)\left(\|\mathcal{V}_h(0) - v_h(0)\|_\infty + \mathcal{Q}h^2\right),$$

where $L, \mathcal{C}, \mathcal{Q} > 0$ are fixed constants. Substitution into (1.2) yields

$$\begin{cases} xz_t(x,t) - z_{xx}(x,t) = \big(x(L+1) - 2\mathcal{C} - 4\mathcal{C}^2x^2\big)z(x,t), \\ (1-x)y_t(x,t) - y_{xx}(x,t) = \big((1-x)(L+1) - 2\mathcal{C} - 4\mathcal{C}^2x^2\big)y(x,t). \end{cases} \tag{3.12}$$

Standard comparison arguments complete the proof of (3.6). By setting

$$z_j(t) = z(x_j, t), \qquad y_j(t) = y(x_j, t),$$

and using the second-order finite difference approximation, we have

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_j = \frac{z_{j-1}(t) - 2z_j(t) + z_{j+1}(t)}{h^2} + o(h^2),$$

$$\left(\frac{\partial^2 y}{\partial x^2}\right)_j = \frac{y_{j-1}(t) - 2y_j(t) + y_{j+1}(t)}{h^2} + o(h^2).$$

Using (3.12), we obtain the discrete system

$$\begin{cases} x_j \dfrac{dz_j(t)}{dt} - \delta^2 z_j(t) = \big(x_j(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_j^2\big) z_j(t) + o(h^2), \\[2ex] (1-x_j)\dfrac{dy_j(t)}{dt} - \delta^2 y_j(t) = \big((1-x_j)(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_j^2\big) y_j(t), \\[2ex] z_j(0) = e^{\mathcal{C}x_j^2}\left(\|\mathcal{U}_h(0) - u_h(0)\|_\infty + \mathcal{Q}h^2\right), \\[2ex] y_j(0) = e^{\mathcal{C}x_j^2}\left(\|\mathcal{V}_h(0) - v_h(0)\|_\infty + \mathcal{Q}h^2\right). \end{cases} \tag{3.13}$$

From

$$z(x,t) = \left(\frac{\|\mathcal{U}_h(0) - u_h(0)\|_\infty + \mathcal{Q}h^2}{\|\mathcal{V}_h(0) - v_h(0)\|_\infty + \mathcal{Q}h^2}\right) y(x,t), \qquad y(x,t) = \left(\frac{\|\mathcal{V}_h(0) - v_h(0)\|_\infty + \mathcal{Q}h^2}{\|\mathcal{U}_h(0) - u_h(0)\|_\infty + \mathcal{Q}h^2}\right) z(x,t),$$

we can choose parameters $L, \mathcal{C}, \mathcal{Q} > 0$ sufficiently large so that

$$\begin{cases} x_j \dfrac{dz_j(t)}{dt} - \delta^2 z_j(t) \geq \big(x_j(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_j^2\big) y_j(t) + \mathcal{K}h^2, \\[2ex] (1-x_j)\dfrac{dy_j(t)}{dt} - \delta^2 y_j(t) \geq \big((1-x_j)(L+1) - 2\mathcal{C} - 4\mathcal{C}^2 x_j^2\big) z_j(t) + \mathcal{K}h^2, \\[2ex] z_j(0) > \tau_j(0), \qquad y_j(0) > \varrho_j(0). \end{cases} \tag{3.14}$$

Thus we obtain

$$\begin{cases} jh\dfrac{dz_j(t)}{dt} - \delta^2 z_j(t) \geq \mathcal{C}_1' y_j(t) + \mathcal{K}h^2, \\[2ex] (N+1-j)h\dfrac{dy_j(t)}{dt} - \delta^2 y_j(t) \geq \mathcal{C}_2' z_j(t) + \mathcal{K}h^2, \\[2ex] z_j(0) > \tau_j(0), \qquad y_j(0) > \varrho_j(0), \end{cases} \tag{3.15}$$

where $\mathcal{C}_1' \geq \mathcal{C}_1$ and $\mathcal{C}_2' \geq \mathcal{C}_2$.

Now define the error functions

$$e_h(t) = z_h(t) - \tau_h(t), \qquad g_h(t) = y_h(t) - \varrho_h(t).$$

Combining (3.11) with (3.15), we deduce

$$\begin{cases} jh\dfrac{de_j(t)}{dt} - \delta^2 e_j(t) - \mathcal{C}_1' y_j(t) + \mathcal{C}_1|\varrho_j(t)| \geq 0, \\[2ex] (N+1-j)h\dfrac{dg_j(t)}{dt} - \delta^2 g_j(t) - \mathcal{C}_2' z_j(t) + \mathcal{C}_2|\tau_j(t)| \geq 0, \\[2ex] e_j(0) > 0, \qquad g_j(0) > 0. \end{cases} \tag{3.16}$$

Using Lemma 3.7, this becomes

$$\begin{cases} jh\dfrac{de_j(t)}{dt} - \delta^2 e_j(t) + \mathcal{C}_1'' \left|\varrho_j(t) - y_j(t)\right| \geq 0, \\[2ex] (N+1-j)h\dfrac{dg_j(t)}{dt} - \delta^2 g_j(t) + \mathcal{C}_2'' \left|\tau_j(t) - z_j(t)\right| \geq 0, \\[2ex] e_j(0) > 0, \qquad g_j(0) > 0. \end{cases}$$

where $\mathcal{C}_1'' = \min\{\mathcal{C}_1, \mathcal{C}_1'\}$ and $\mathcal{C}_2'' = \min\{\mathcal{C}_2, \mathcal{C}_2'\}$. Thus, system (3.16) becomes

$$\begin{cases} jh\,\dfrac{de_j(t)}{dt} - \delta^2 e_j(t) + \mathcal{C}_1''|g_j(t)| \geq 0, \\ (N+1-j)h\,\dfrac{dg_j(t)}{dt} - \delta^2 g_j(t) + \mathcal{C}_2''|e_j(t)| \geq 0, \\ e_j(0) > 0, \quad g_j(0) > 0, \end{cases} \tag{3.17}$$

for $1 \leq j \leq N$. Equivalently,

$$\begin{cases} \dfrac{de_j(t)}{dt} - \delta_h^2 e_j(t) + \dfrac{\mathcal{C}_1''}{jh}|g_j(t)| \geq 0, \\ \dfrac{dg_j(t)}{dt} - \delta_h^2 g_j(t) + \dfrac{\mathcal{C}_2''}{(N+1-j)h}|e_j(t)| \geq 0, \\ e_j(0) > 0, \quad g_j(0) > 0, \end{cases}$$

where δ_h^2 denotes the discrete Laplacian operator introduced in Proposition 3.6.

Define

$$\mathcal{G}_h(t) = \big(e_1(t), \ldots, e_N(t), g_1(t), \ldots, g_N(t)\big),$$

and the nonlinear mapping

$$\mathcal{H}\big(|\mathcal{G}_j(t)|\big) = \begin{cases} \dfrac{\mathcal{C}_1''}{jh}|g_j(t)|, & 1 \leq j \leq N, \\ \dfrac{\mathcal{C}_2''}{(N+1-j)h}|e_j(t)|, & 1 \leq j \leq N. \end{cases}$$

Hence, we obtain the differential inequality

$$\begin{cases} \dfrac{d\mathcal{G}_j(t)}{dt} - \delta_h^2 \mathcal{G}_j(t) + \mathcal{H}\big(|\mathcal{G}_j(t)|\big) \geq 0, \\ \mathcal{G}_j(0) > 0, \end{cases} \tag{3.18}$$

and, by Proposition 3.6, it follows that

$$\begin{aligned} z_j(t) &\geq \tau_j(t), \\ y_j(t) &\geq \varrho_j(t), \end{aligned} \qquad \text{for } t \in (0, t(h)),\ 1 \leq j \leq N. \tag{3.19}$$

Similarly, one also proves that

$$\begin{aligned} z_j(t) &> -\tau_j(t), \\ y_j(t) &> -\varrho_j(t), \end{aligned} \qquad \text{for } t \in (0, t(h)),\ 1 \leq j \leq N. \tag{3.20}$$

Since

$$-\tau_h = -\mathcal{U}_h(t) + u_h(t), \qquad -\varrho_h = -\mathcal{V}_h(t) + v_h(t),$$

we obtain

$$\begin{aligned} z_j(t) &> |\tau_j(t)|, \\ y_j(t) &> |\varrho_j(t)|, \end{aligned} \qquad \text{for } t \in (0, t(h)),\ 1 \leq j \leq N.$$

Consequently,

$$\begin{aligned} |\mathcal{U}_h(t) - u_h(t)|_\infty &\leq \exp\big((L+1)t + \mathcal{C}\big)\big(|\mathcal{U}_h(0) - u_h(0)|_\infty + \mathcal{Q}h^2\big), \\ |\mathcal{V}_h(t) - v_h(t)|_\infty &\leq \exp\big((L+1)t + \mathcal{C}\big)\big(|\mathcal{V}_h(0) - v_h(0)|_\infty + \mathcal{Q}h^2\big), \end{aligned} \qquad t \in (0, t(h)).$$

Thus,

$$|\mathcal{W}_h(t)-\omega_h(t)|_\infty \le \exp\big((L+1)t+\mathcal{C}\big)\big(|\mathcal{W}_h(0)-\omega_h(0)|_\infty+\mathcal{Q}h^2\big), \qquad t\in(0,t(h)).$$

Finally, assume by contradiction that $T>t(h)$. Using (3.8) we have

$$\begin{aligned} 1 &= |\mathcal{W}_h(t(h))-\omega_h(t(h))|_\infty \\ &\le \exp\big((L+1)T+\mathcal{C}\big)\big(|\mathcal{W}_h(0)-\omega_h(0)|_\infty+\mathcal{Q}h^2\big). \end{aligned}$$

Since

$$\lim_{h\to 0}\exp\big((L+1)T+\mathcal{C}\big)\big(|\mathcal{W}_h(0)-\omega_h(0)|_\infty+\mathcal{Q}h^2\big)=0,$$

we obtain $1\le 0$, a contradiction. Hence, $t(h)=T$. □

Theorem 3.9 ([2]). *Assume that the hypotheses of Theorem 3.8 are satisfied. Let T_b and T_h denote, respectively, the finite blow-up time of the continuous problem (1.2) and of its semi-discrete approximation (3.1). Suppose that there exists a continuous and non-decreasing function*

$$F:[0,\infty)\to\mathbb{R}$$

such that

$$\|\mathcal{W}_h\|_\infty \le F\left(\frac{1}{T_h-t}\right), \qquad t<T_h.$$

Then, the blow-up times satisfy the convergence property

$$\lim_{h\to 0}T_h=T_b.$$

4 Finite Difference Schemes

In this section, we present the fully discrete θ-scheme finite difference method for problem (1.2). The time derivative in (1.1) is approximated by a convex combination of the forward and backward finite differences.

Let $\mathcal{U}_i^n$ and $\mathcal{V}_i^n$ denote approximations of $u(x_i,t_n)$ and $v(x_i,t_n)$, respectively. We define a uniform spatial grid by

$$x_i=ih, \qquad 0\le i\le N+1, \qquad h=\frac{1}{N+1},$$

and a nonuniform time partition

$$t_{n+1}=t_n+k_n, \qquad n=0,1,\dots.$$

Thus, h is the spatial step and k_n the time step.

To derive the θ-scheme, the spatial second derivatives $\delta_x^2\mathcal{U}(t)$ and $\delta_x^2\mathcal{V}(t)$ are approximated at the mesh points using

$$\begin{aligned} \delta_x^2\mathcal{U}_i(t_n) &= \theta\,\delta_x^2\mathcal{U}_i(t_{n+1})+(1-\theta)\,\delta_x^2\mathcal{U}_i(t_n), \\ \delta_x^2\mathcal{V}_i(t_n) &= \theta\,\delta_x^2\mathcal{V}_i(t_{n+1})+(1-\theta)\,\delta_x^2\mathcal{V}_i(t_n), \end{aligned}$$

while the nonlinear terms are taken explicitly at time t_n.

This yields the linear θ-scheme

$$\begin{aligned} x_i\,\frac{\mathcal{U}_i^{n+1}-\mathcal{U}_i^n}{k_n} &= \theta\,\delta_x^2\mathcal{U}_i^{n+1}+(1-\theta)\,\delta_x^2\mathcal{U}_i^n+(\mathcal{V}_i^n)^p, \\ (1-x_i)\,\frac{\mathcal{V}_i^{n+1}-\mathcal{V}_i^n}{k_n} &= \theta\,\delta_x^2\mathcal{V}_i^{n+1}+(1-\theta)\,\delta_x^2\mathcal{V}_i^n+(\mathcal{U}_i^n)^p, \end{aligned} \tag{4.1}$$

for $i=1,\dots,N$, where

$$\delta_x^2\mathcal{U}_i^n=\frac{\mathcal{U}_{i+1}^n-2\mathcal{U}_i^n+\mathcal{U}_{i-1}^n}{h^2}, \qquad \delta_x^2\mathcal{V}_i^n=\frac{\mathcal{V}_{i+1}^n-2\mathcal{V}_i^n+\mathcal{V}_{i-1}^n}{h^2}.$$

The boundary and initial conditions are discretized as:

$$\mathcal{U}_0^n = \mathcal{U}_{N+1}^n = \mathcal{V}_0^n = \mathcal{V}_{N+1}^n = 0,$$

$$\mathcal{U}_i^0 = u_0(x_i), \qquad \mathcal{V}_i^0 = v_0(x_i), \qquad i = 0, \dots, N+1.$$

In matrix form, scheme (4.1) becomes

$$\begin{cases} \mathcal{A}\mathcal{U}^{n+1} = \mathcal{A}\mathcal{A}\,\mathcal{U}^n + k_n F_{v,p}^n, & n = 0, \dots, M-1, \\ \mathcal{B}\mathcal{V}^{n+1} = \mathcal{B}\mathcal{B}\,\mathcal{V}^n + k_n F_{u,p}^n, & n = 0, \dots, M-1, \end{cases} \tag{4.2}$$

where

$$\mathcal{U}^0 = (u_0(x_1), \dots, u_0(x_N))^T, \qquad \mathcal{V}^0 = (v_0(x_1), \dots, v_0(x_N))^T,$$

and the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}\mathcal{A}$, and $\mathcal{B}\mathcal{B}$ are given by

$$\mathcal{A} = \begin{bmatrix} x_1 + 2\theta\gamma_1 & -\theta\gamma_1 & 0 & \cdots & 0 \\ -\theta\gamma_2 & x_2 + 2\theta\gamma_2 & -\theta\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\theta\gamma_{N-1} \\ 0 & \cdots & 0 & -\theta\gamma_N & x_N + 2\theta\gamma_N \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} (1-x_1) + 2\theta\gamma_1 & -\theta\gamma_1 & 0 & \cdots & 0 \\ -\theta\gamma_2 & (1-x_2) + 2\theta\gamma_2 & -\theta\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\theta\gamma_{N-1} \\ 0 & \cdots & 0 & -\theta\gamma_N & (1-x_N) + 2\theta\gamma_N \end{bmatrix}$$

$$\mathcal{A}\mathcal{A} = \begin{bmatrix} x_1 - 2(1-\theta)\gamma_1 & (1-\theta)\gamma_1 & 0 & \cdots & 0 \\ (1-\theta)\gamma_2 & x_2 - 2(1-\theta)\gamma_2 & (1-\theta)\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (1-\theta)\gamma_{N-1} \\ 0 & \cdots & 0 & (1-\theta)\gamma_N & x_N - 2(1-\theta)\gamma_N \end{bmatrix}$$

$$\mathcal{B}\mathcal{B} = \begin{bmatrix} (1-x_1) - 2(1-\theta)\gamma_1 & (1-\theta)\gamma_1 & 0 & \cdots & 0 \\ (1-\theta)\gamma_2 & (1-x_2) - 2(1-\theta)\gamma_2 & (1-\theta)\gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & (1-\theta)\gamma_{N-1} \\ 0 & \cdots & 0 & (1-\theta)\gamma_N & (1-x_N) - 2(1-\theta)\gamma_N \end{bmatrix}$$

with

$$\gamma_n = \frac{k_n}{h^2}.$$

Finally, for $p = 2$, the variable time-step ensuring stability of the θ-scheme is chosen as

$$k_n = \min\left(\frac{h^2}{2}, \frac{h^\alpha}{\|\mathcal{U}_h^n\|_\infty^p}, \frac{h^\alpha}{\|\mathcal{V}_h^n\|_\infty^p}\right), \qquad \alpha \geq 1.$$

4.1 Local Truncation Error Estimate

Assume that the exact solution $(u,v)(x,t)$ of (1.2) is sufficiently smooth, namely

$$u, v \in C^{4,2}(\Omega \times [0, T_b)),$$

and remains bounded for all $t < T_b$ (i.e., no blow-up before T_b). Under these assumptions, we can estimate the local truncation error of the θ-scheme.

Theorem 4.1 (Local truncation error of the θ-scheme). *Let* $((\tau^u)_i^n, (\tau^v)_i^n)$ *denote the local truncation errors of scheme* (4.2) *at the grid point* (x_i, t_n). *Then there exist constants*

$$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \geq 0,$$

independent of h *and* k_n, *such that*

$$|(\tau^u)_i^n| \leq \mathcal{C}_1 k + \mathcal{C}_2 h^2, \qquad |(\tau^v)_i^n| \leq \mathcal{C}_3 k + \mathcal{C}_4 h^2,$$

where

$$k = \max_n |k_n|.$$

Proof. We substitute the exact solution $(u,v)(x_i,t_n)$ into the numerical scheme (4.2), which leads to the expressions

$$(\tau^u)_i^n = x_i \, \frac{u_i^{n+1} - u_i^n}{k_n} - \theta \delta_x^2 u_i^{n+1} - (1-\theta)\delta_x^2 u_i^n - (v_i^n)^p,$$

$$(\tau^v)_i^n = (1 - x_i) \, \frac{v_i^{n+1} - v_i^n}{k_n} - \theta \delta_x^2 v_i^{n+1} - (1-\theta)\delta_x^2 v_i^n - (u_i^n)^p.$$

Since (u,v) is assumed smooth, we use Taylor expansions in time:

$$\frac{u(x_i,t_{n+1}) - u(x_i,t_n)}{k_n} = \frac{\partial u}{\partial t}(x_i,t_n) + \frac{k_n}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_n) + O(k_n^2),$$

and for the second spatial derivative,

$$\frac{u(x_{i+1},t_n) - 2u(x_i,t_n) + u(x_{i-1},t_n)}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_i,t_n) + O(h^2).$$

Similarly, expanding in time at t_{n+1} yields

$$\frac{u(x_{i+1},t_{n+1}) - 2u(x_i,t_{n+1}) + u(x_{i-1},t_{n+1})}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_i,t_n) + k_n \frac{\partial^2 u}{\partial t^2}(x_i,t_n) - (v_i^n)^p + O(k_n + h^2).$$

Substituting these expansions into the truncation error expression, we obtain

$$(\tau^u)_i^n = x_i \frac{\partial u}{\partial t}(x_i,t_n) + x_i \frac{k_n}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_n) - \theta\Big[x_i \frac{\partial u}{\partial t}(x_i,t_n) + x_i k_n \frac{\partial^2 u}{\partial t^2}(x_i,t_n)\Big]$$
$$- (1-\theta)\frac{\partial^2 u}{\partial x^2}(x_i,t_n) + O(k_n + h^2).$$

Using the PDE satisfied by the exact solution,

$$x_i \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - v^p = 0,$$

we simplify and obtain

$$(\tau^u)_i^n = k_n \left(\frac{1}{2} - \theta\right) \frac{\partial^2 u}{\partial t^2}(x_i,t_n) + O(k_n + h^2).$$

By the same argument applied to $v(x,t)$, we obtain

$$(\tau^v)_i^n = k_n \left(\frac{1}{2} - \theta\right) \frac{\partial^2 v}{\partial t^2}(x_i,t_n) + O(k_n + h^2).$$

Thus, both truncation errors are bounded by first order in time and second order in space, concluding the proof. □

4.2 Stability and Convergence Before Blow–Up

Let the perturbation errors be defined by

$$e_i^n = u_i^n - \mathcal{U}_i^n, \qquad f_i^n = v_i^n - \mathcal{V}_i^n,$$

where (u_i^n, v_i^n) denotes the exact solution of (1.2) evaluated on the mesh, and $(\mathcal{U}_i^n, \mathcal{V}_i^n)$ is the fully discrete θ–scheme approximation.

Using standard discrete energy estimates (or maximum–norm arguments), one obtains the relation

$$\|e^{n+1}\| \le (1 + Ck)\,\|e^n\| + \mathcal{O}(k^2 + h^2),$$

which shows that, for $\theta \ge \frac{1}{2}$, the method is unconditionally stable in the linear case and remains stable for the nonlinear system up to the blow–up time T^*.

Before establishing stability, we recall the following notion.

Definition 4.2 (Stability). Let

$$(e_u)_i^n = u_i^n - \mathcal{U}_i^n, \qquad (e_v)_i^n = v_i^n - \mathcal{V}_i^n, \qquad i = 1, \dots, N,$$

be the numerical errors at the node (x_i, t_n). We say that the difference scheme is *stable* if there exists $\beta > 0$, independent of h and k, such that for all $n \ge 1$:

$$\|\mathcal{E}_u^n\| \le \beta \max\{\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|\}, \qquad \|\mathcal{E}_v^n\| \le \beta \max\{\|\mathcal{E}_u^0\|, \|\mathcal{E}_v^0\|\},$$

where

$$\|\mathcal{E}_u^n\| = \max_{1\le i\le N} |(e_u)_i^n|, \qquad \|\mathcal{E}_v^n\| = \max_{1\le i\le N} |(e_v)_i^n|.$$

Since the blow–up occurs near the endpoints $x = 0$ and $x = 1$, the analysis is carried out on the interior subinterval

$$(\varepsilon_1,\, 1 - \varepsilon_2), \qquad \varepsilon_1, \varepsilon_2 > 0.$$

Theorem 4.3. *The θ–scheme (4.1) is stable provided that*

$$1 \ge 2(1-\theta)\frac{k}{h^2}, \qquad k = \max_{1\le n\le N} k_n.$$

Proof. Consider the first equation in scheme (4.1). Subtracting the discrete equations satisfied by (u_i^n, v_i^n) and $(\mathcal{U}_i^n, \mathcal{V}_i^n)$, we obtain

$$x_i \frac{(e_u)_i^{n+1} - (e_u)_i^n}{k_n} = \theta\delta_x^2 (e_u)_i^{n+1} + (1-\theta)\delta_x^2 (e_u)_i^n + \big((v_i^n)^p - (\mathcal{V}_i^n)^p\big).$$

After rearranging, this leads to

$$(x_i + 2\theta\gamma_n)(e_u)_i^{n+1} = \theta\gamma_n\big[(e_u)_{i+1}^{n+1} + (e_u)_{i-1}^{n+1}\big] + (x_i - 2(1-\theta)\gamma_n)(e_u)_i^n$$
$$+ (1-\theta)\gamma_n\big[(e_u)_{i+1}^n + (e_u)_{i-1}^n\big] + k_n\big((v_i^n)^p - (\mathcal{V}_i^n)^p\big).$$

Taking the maximum over i, and using the local Lipschitz continuity of $s \mapsto s^p$ (with constant L_1), we get

$$\|\mathcal{E}_u^{n+1}\| \le \|\mathcal{E}_u^n\| + \frac{k_n L_1}{\varepsilon_1}\|\mathcal{E}_v^n\|.$$

Repeating the same argument on the second equation of (4.1) gives

$$\|\mathcal{E}_v^{n+1}\| \le \|\mathcal{E}_v^n\| + \frac{k_n L_2}{\varepsilon_2}\|\mathcal{E}_u^n\|.$$

Iterating over n yields

$$\|\mathcal{E}_u^{n+1}\| \le \Big(1 + \frac{kL_1}{\varepsilon_1}\Big)^{n+1} \max\{\|\mathcal{E}_u^0\|,\, \|\mathcal{E}_v^0\|\},$$

$$\|\mathcal{E}_v^{n+1}\| \le \Big(1 + \frac{kL_2}{\varepsilon_2}\Big)^{n+1} \max\{\|\mathcal{E}_u^0\|,\, \|\mathcal{E}_v^0\|\}.$$

Thus the scheme is stable under the stated condition. □

Theorem 4.4 (Convergence). *Under the same condition*

$$1 \geq 2(1-\theta)\frac{k}{h^2},$$

the θ–scheme (4.1) *is convergent.*

Proof. The argument mirrors the previous stability proof, replacing the rounding errors with the local truncation error. Combining consistency and stability yields the convergence result. □

5 Numerical Results

In this section, we present numerical experiments. All numerical computations were carried out using MATLAB (R2020a). Our goal is to verify that the theoretical blow-up behavior described in Theorem 2.4 is observed in the numerical simulation. We apply the θ-scheme with $\theta \geq \left(1 - \frac{h^2}{2k}\right)$ and the initial conditions $u_0(x) = \sin(\pi x)$ and $v_0(x) = \sin(\pi x)$.

Figures 1 and 2 display the evolution of the numerical solutions $\mathcal{U}^n$ and $\mathcal{V}^n$ of (4.2) with $p = 2$. The numerical solutions blow up at time $T_b \in (0, 1)$, at $x = 0$ for $\mathcal{U}^n$ and at $x = 1$ for $\mathcal{V}^n$.

We repeat the computations with a different initial data, $u_0(x) = x^3(1-x)^2$ and $v_0(x) = (1-x^3)\sin\left(\frac{\pi}{2}x\right)$. Figures 3 and 4 show that the blow-up behavior remains the same and occurs at the same spatial points.

The following table summarizes the numerical blow-up times and CPU execution times (in seconds) for increasing mesh refinements $N = 10, 20, 50, 100, 200, 300, 500$.

N	Blow-up time	CPU time (s)
10	0.0387315	0.070529
20	0.0149963	0.071728
50	0.0374250	0.072739
100	0.0430329	0.085657
200	0.0428022	0.175249
300	0.0408165	0.559521
500	0.0371345	2.784772

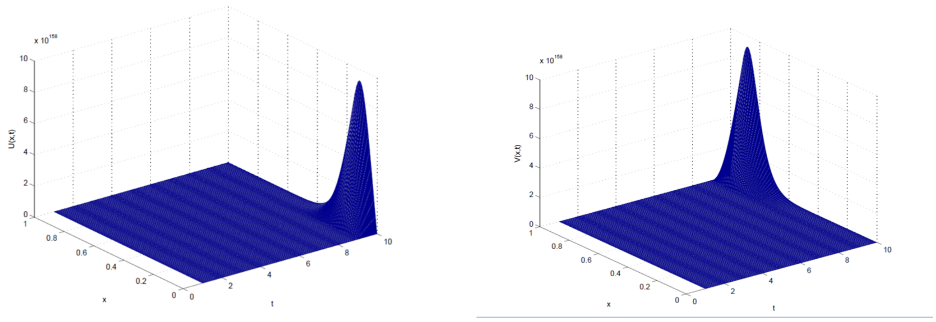

Figure 1. Numerical solution $(u, v)(x_i, t^n)$ using the θ-scheme, $p = 2$, $u_0(x) = v_0(x) = \sin(\pi x)$.

6 Conclusion and Perspectives

In this study, we established the numerical blow-up behavior of a degenerate nonlinear reaction-diffusion problem, thereby reinforcing and complementing the theoretical blow-up result obtained under suitable initial conditions. By implementing a finite difference discretization combined with the θ-scheme in MATLAB, we successfully reproduced the finite-time singularity

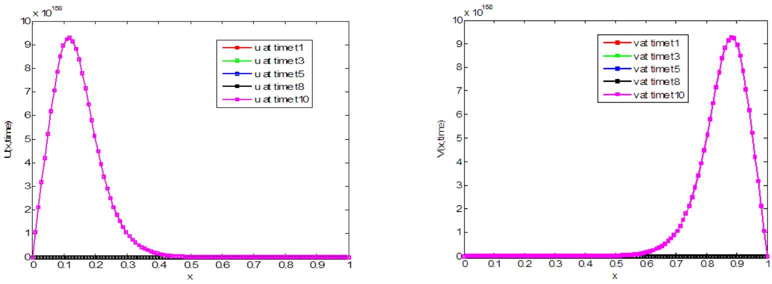

Figure 2. Numerical solution $(\mathcal{U}_h, \mathcal{V}_h)(t^n)$ using the θ-scheme, $p = 2$, $u_0(x) = v_0(x) = \sin(\pi x)$.

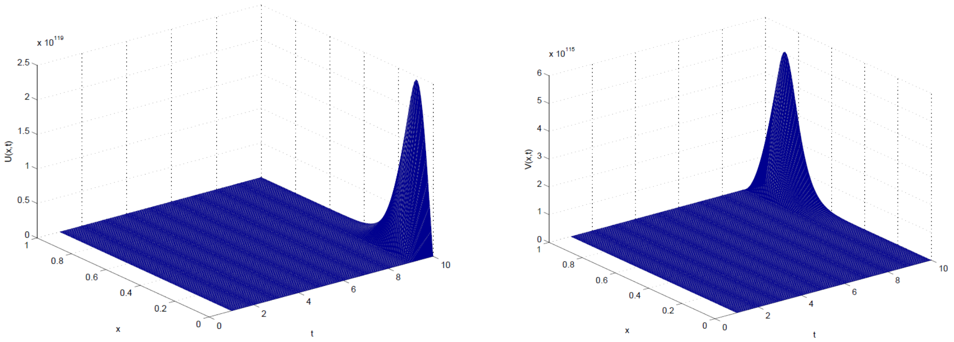

Figure 3. Numerical solution $(u, v)(x_i, t^n)$ using the θ-scheme, $p = 2$, $u_0(x) = x^3(1-x)^2$, $v_0(x) = (1 - x^3)\sin(\frac{\pi}{2}x)$.

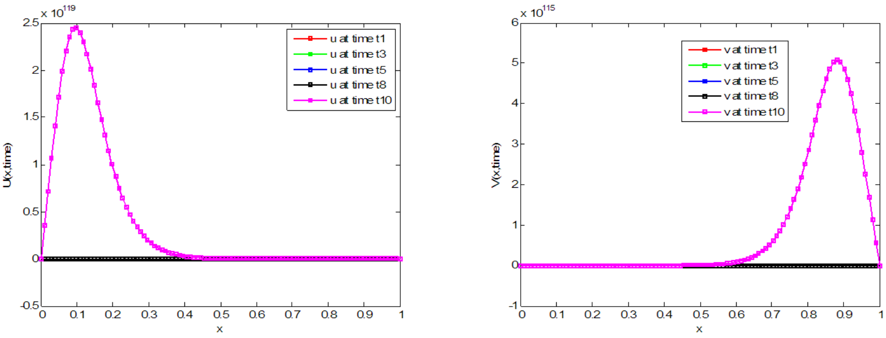

Figure 4. Numerical solution $(\mathcal{U}, \mathcal{V})(t^n)$ using the θ-scheme, $p = 2$, $\mathcal{U}_0(x) = x^3(1-x)^2$, $\mathcal{V}_0(x) = (1 - x^3)\sin(\frac{\pi}{2}x)$.

predicted by the analytical framework. The numerical evidence not only confirms the validity of the theoretical criteria but also provides a clearer picture of the solution's evolution as it approaches the blow-up time.

Moreover, the computational experiments highlight the sensitivity of the system to both discretization parameters and initial data, emphasizing the importance of carefully selecting numerical schemes when dealing with rapidly growing or unstable solutions. This work illustrates the effectiveness of numerical simulations as a means of validating theoretical results, especially in contexts where explicit analytical solutions are difficult or impossible to obtain.

Future extensions of this study may include applying higher-order schemes, exploring adaptive time stepping to better capture the singular behavior, or examining similar blow-up phenomena in more complex coupled or degenerate systems. Such developments would deepen our understanding of nonlinear dynamics and contribute to more reliable numerical approaches for analyzing finite-time explosion in partial differential equations.

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