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SPECIAL GRAPHS APPEARING IN HAMILTON–JACOBI FLOWS

Kotaro Matsuoka

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Corresponding Author: Kotaro Matsuoka

Abstract. In this paper, we investigate a phenomenon related to the Hamilton–Jacobi flow defined by the Hopf–Lax formula, in which a distinct graph associated with the corresponding Lagrangian appears. This phenomenon is closely related to the singularity of the initial data.

Our primary goal is to characterize the class of initial data for which this phenomenon occurs. We show that a global subgradient condition provides a complete characterization for all time, and we also establish a local counterpart describing the behavior for small time.

1 Introduction

In this paper, we consider the Cauchy problem for the Hamilton–Jacobi equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + H\big(Du(t,x)\big) = 0, & (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

where Du denotes the gradient of u with respect $x \in \mathbb{R}^n$. Throughout this paper, we assume that the Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions:

$$H \in C^1(\mathbb{R}^n), \tag{1.1}$$

$$H \text{ is strictly convex on } \mathbb{R}^n, \tag{1.2}$$

$$H \text{ is superlinear, i.e., } \lim_{|p|\to\infty} \frac{H(p)}{|p|} = \infty. \tag{1.3}$$

When $f \in C(\mathbb{R}^n)$ satisfies the condition

$$\liminf_{|z|\to\infty} \frac{f(z)}{|z|} > -\infty, \tag{1.4}$$

there exists a unique viscosity solution in the class of uniformly continuous functions on $[0,T)\times\mathbb{R}^n$ for each $T \in (0,\infty)$ (see [2, Theorem 2.1]). It follows from [10, Corollary 2.2] that the viscosity solution is given by the Hopf–Lax formula:

$$H_t f(x) = \inf_{z\in\mathbb{R}^n} q_f(t,x;z), \quad (t,x) \in (0,\infty)\times\mathbb{R}^n, \tag{1.5}$$

where $\{q_f(t,x;z)\}_{z\in\mathbb{R}^n}$ is given by

$$q_f(t,x;z) = f(z) + tL\Big(\frac{x-z}{t}\Big), \quad (t,x,z) \in (0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n,$$

and the Lagrangian L is defined as the Legendre transform of H (see Section 2 and [4, 3.3.2]). We call this flow $\{H_t f\}_{t>0}$ the Hamilton–Jacobi flow.

In [5, 8], Fujita et al. studied the Hamilton–Jacobi flow in the case $n = 1$ with $H(p) = p^2/2$, focusing on how properties of the initial data f are reflected in the behavior of the flow $H_t f$. They showed that, due to the singularity of f, in the sense of nowhere differentiability, and the fact that the associated Lagrangian is given by $L(v) = v^2/2$, $H_t f$ is piecewise quadratic on $\mathbb{R}$ for each $t > 0$. In particular, in [7, 6], they investigated the Hamilton–Jacobi flow starting from the Takagi function, the typical example of an everywhere continuous and nowhere differentiable function on $\mathbb{R}^n$.

These results suggest that the Hamilton–Jacobi flow retains fine structural information about the initial data, in particular reflecting its nowhere differentiability. This observation motivates the present work.

However, the focus of the present paper is different. While [5, 6, 7, 8] deal with initial data that are nowhere differentiable on $\mathbb{R}$ and analyze global features of the flow, we consider a local geometric problem associated with a single point $a \in \mathbb{R}^n$. More precisely, we do not assume that f is nowhere differentiable; instead, we focus on the behavior of the flow around a point where f may fail to be differentiable, and we investigate how this local singularity is reflected in the structure of $H_t f$. Moreover, our analysis is carried out in $\mathbb{R}^n$ under general assumptions on the Hamiltonian.

In this paper, we study a geometric characterization problem for the Hamilton–Jacobi flow. Specifically, we investigate how the validity of a certain geometric equality, which will be denoted by $M_a(t) = Q_a(t)$, depends on the *subdifferential* structure of the initial data. For a given point $a \in \mathbb{R}^n$, we characterize the classes of initial data f for which the geometric equality

$$M_a(t) = Q_a(t) \tag{1.6}$$

holds globally for all $t > 0$ or locally for small $t > 0$. This equality describes the appearance of the graph $q_f(t, \cdot; a)$ in the Hamilton–Jacobi flow $H_t f$, and is closely related to how the subdifferential structure of the initial data f at a influences the behavior of the solution. To formulate this problem, we introduce the set

$$M_a(t) = \{x \in \mathbb{R}^n \mid H_t f(x) = q_f(t, x; a)\}, \quad t > 0, \tag{1.7}$$

which consists of all points where the graph of $q_f(t, \cdot; a)$ appears in $H_t f$. We also introduce the set

$$Q_a(t) = \{a + tDH(p) \mid p \in D^- f(a)\}, \quad t > 0. \tag{1.8}$$

This set can be interpreted as the image of the subdifferential $D^- f(a)$ under the map $p \mapsto a + tDH(p)$, that is, the propagation of the initial slopes at a along the Hamiltonian flow over time $t > 0$. Here, $D^- f(a)$ denotes the (Fréchet) subdifferential of f at $a \in \mathbb{R}^n$ (see Section 2). If $D^- f(a)$ is empty or multi–valued, then f is not differentiable at $a \in \mathbb{R}^n$. Therefore, we assume that $D^- f(a)$ is non-empty to ensure nontrivial behavior.

This shows that the equality $M_a(t) = Q_a(t)$ means that the appearance of the graph $q_f(t, \cdot; a)$ in the solution is completely determined by the subdifferential structure of f at the point a.

In general, we have $M_a(t) \subset Q_a(t)$ (see Proposition 3.1). The simplest possible condition on f to obtain the equality in (1.6) is the following *global subgradient inequality*:

$$f(y) \geq f(a) + p \cdot (y - a), \quad y \in \mathbb{R}^n,\ p \in D^- f(a) \tag{1.9}$$

at $a \in \mathbb{R}^n$. If this inequality holds, then it is not difficult to verify that (1.6) holds for all $t > 0$. The inequality (1.9) is motivated by the concept of *subgradients* in convex analysis (see [9, p.214]).

Our first main result shows that the condition (1.9) is not only sufficient but also necessary for the equality in (1.6) to hold globally for all $t > 0$. In other words, we establish an equivalence between the global subgradient inequality (1.9) and the geometric equality in (1.6).

On the other hand, we also consider the *local subgradient inequality* at $a \in \mathbb{R}^n$:

$$f(y) \geq f(a) + p \cdot (y - a), \quad y \in B_\delta(a),\ p \in D^- f(a) \tag{1.10}$$

for some $\delta > 0$, where $B_r(y)$ denotes the open ball with center $y \in \mathbb{R}^n$ and radius $r > 0$.

Our second main result shows that this local condition implies the equality in (1.6) holds locally for small $t > 0$. Moreover, examples indicate that this local condition does not guarantee equality for all $t > 0$. This shows that the global and local validity of the equality with respect to t are essentially different (see Example 3.6). Apart from the global subgradient inequality (1.9) and the local subgradient inequality (1.10), no other general conditions ensuring (1.6) are known to the author.

Finally, we refer to the assumptions on f. In the first result of Theorem 3.2 giving an equivalent condition to the equality in (1.6) for all $t > 0$, we only assume that $f \in C(\mathbb{R}^n)$ satisfies (1.4). In the second result of Theorem 3.7 giving a sufficient condition for the equality in (1.6) locally, we need to assume that not only $f \in C(\mathbb{R}^n)$ satisfies (1.6) but also f is uniformly continuous on $\mathbb{R}^n$. The author does not know whether or not we can remove the assumption that f is uniformly continuous on $\mathbb{R}^n$ in the second result.

The paper is organized as follows. In Section 2, we give preliminaries to show our main results. In Section 3, we state our main results and prove them. Finally, in Section 4, we summarize the results and discuss their implications.

2 Preliminaries

In this section, we collect several preliminaries that will be used throughout the paper. We recall basic notions and results on subdifferentials, convex functions, and the Lagrangian.

We recall some properties of the (Fréchet) subdifferential. For $g \in C(\mathbb{R}^n)$, the subdifferential $D^-g(\alpha)$ at $\alpha \in \mathbb{R}^n$ is defined by

$$D^-g(\alpha) = \left\{ p \in \mathbb{R}^n \;\middle|\; \liminf_{y \to \alpha} \frac{g(y) - g(\alpha) - p \cdot (y - \alpha)}{|y - \alpha|} \geq 0 \right\}$$

(see [1, p.29]). We collect several known results.

Lemma 2.1. ([1, Lemma 1.8]) *Let $g \in C(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}^n$. Then,*

(i) $D^-g(\alpha)$ is a closed convex subset of $\mathbb{R}^n$;

(ii) if g is differentiable at α, then $D^-g(\alpha) = \{Dg(\alpha)\}$.

Lemma 2.2. ([1, Lemma 1.7]) *Let $g \in C(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}^n$. Then, $p \in D^-g(\alpha)$ if and only if there exists $\varphi \in C^1(\mathbb{R}^n)$ such that $D\varphi(\alpha) = p$ and $g - \varphi$ has a local minimum at α.*

Based on Lemma 2.2, we give another characterization of the subdifferential. Let $g \in C(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}^n$. We say that a C^1 function ψ is *subtangent* to g at a point α if $\psi(\alpha) = g(\alpha)$ and $\psi(y) \leq g(y)$ in a neighborhood of α. Then we obtain the following characterization:

Lemma 2.3. *Let $g \in C(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}^n$. Then,*

$$D^-g(\alpha) = \left\{ D\psi(\alpha) \;\middle|\; \psi \text{ is a subtangent to } g \text{ at } \alpha \right\}.$$

Next, we recall some properties of convex functions. A function $g : \mathbb{R}^n \to \mathbb{R}$ is convex on $\mathbb{R}^n$ if the inequality

$$g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y), \quad x, y \in \mathbb{R}^n,\ \lambda \in [0, 1]$$

holds (see [9, Section 4]). The function g is said to be strictly convex on $\mathbb{R}^n$ if the strict inequality holds for $x \neq y$ and $\lambda \in (0, 1)$. When g is convex on $\mathbb{R}^n$, a vector $p \in \mathbb{R}^n$ is said to be a *subgradient* of g on $\mathbb{R}^n$ at a point $\alpha \in \mathbb{R}^n$ if

$$g(y) \geq g(\alpha) + p \cdot (y - \alpha), \quad y \in \mathbb{R}^n.$$

This condition states that the graph of g lies above the affine function passing through a point $(\alpha, g(\alpha))$ with gradient p. For given a convex function g on $\mathbb{R}^n$, we denote $\partial g(\alpha)$ the set of all subgradients of g at α (see [9, Section 23]). We summarize some basic properties.

Lemma 2.4. *Let g be a convex function on $\mathbb{R}^n$. Then, $\partial g(\alpha) \neq \emptyset$ and $\partial g(\alpha) = D^- g(\alpha)$ for any $\alpha \in \mathbb{R}^n$. Furthermore, g fulfills the condition (1.4).*

This is a standard result in convex analysis and follows from [9, Theorems 23.1, 23.2, 23.4]. As a result of Lemma 2.1(ii) and Lemma 2.4, we have

Lemma 2.5. *Let g be a convex function on $\mathbb{R}^n$ and differentiable at $\alpha \in \mathbb{R}^n$. Then,*

$$g(y) \geq g(\alpha) + Dg(\alpha) \cdot (y - \alpha), \quad y \in \mathbb{R}^n. \tag{2.1}$$

In particular, the strict inequality holds for $y \neq \alpha$ provided that g is strictly convex on $\mathbb{R}^n$.

Next, we recall the Legendre transform. Let $T : \mathbb{R}^n \to \mathbb{R}$ be convex and superlinear. The Legendre transform T^* of T is defined by

$$T^*(v) = \sup\{p \cdot v - T(p) \,|\, p \in \mathbb{R}^n\}, \quad v \in \mathbb{R}^n$$

(see [4, 3.3.2]). We recall the following fundamental result.

Lemma 2.6. ([3, Theorem A.2.4]) *Let $T : \mathbb{R}^n \to \mathbb{R}$ be convex and superlinear, and T^* be the Legendre transform of T. Then the following properties are equivalent:*

(i) T *is strictly convex on* $\mathbb{R}^n$.

(ii) $D^-T(q_1) \cap D^-T(q_2) = \emptyset$ *for all* $q_1, q_2 \in \mathbb{R}^n$ *with* $q_1 \neq q_2$.

(iii) $T^* \in C^1(\mathbb{R}^n)$.

Given a Hamiltonian H satisfying conditions (1.1)–(1.3), the Lagrangian L is defined as the Legendre transform of H. By Lemma 2.6 and the assumptions on H, we obtain the following properties of L:

Lemma 2.7.

(i) L *is strictly convex on* $\mathbb{R}^n$ *and superlinear.*

(ii) $L \in C^1(\mathbb{R}^n)$.

(iii) A map $DL : \mathbb{R}^n \to \mathbb{R}^n, v \mapsto DL(v)$ *is one-to-one mapping from* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$.

(iv) $(DL)^{-1}(p) = DH(p)$ *for all* $p \in \mathbb{R}^n$.

Example 2.8. ([11, Example 2.15]) For the Hamiltonian

$$H(p) = \frac{1}{2}|p|^2, \quad p \in \mathbb{R}^n,$$

we have

$$L(v) = \sup\{p \cdot v - H(p) \,|\, p \in \mathbb{R}^n\} = \frac{1}{2}|v|^2, \quad v \in \mathbb{R}^n.$$

This quadratic Hamiltonian is a typical example in the theory of Hamilton–Jacobi equations, and will be used in several examples in Section 3.

Next, we introduce some auxiliary notions and recall a characterization of uniformly continuous functions that will be used later. By $UC(\mathbb{R}^n)$, we denote the set of all uniformly continuous functions on $\mathbb{R}^n$. We recall the following characterization.

Lemma 2.9. *Let g be a function on $\mathbb{R}^n$. Then the following are equivalent:*

(i) $g \in UC(\mathbb{R}^n)$.

(ii) For any $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *such that*

$$|g(x) - g(y)| \leq C_\varepsilon |x - y| + \varepsilon, \quad x, y \in \mathbb{R}^n.$$

Proof. We first show (i) implies (ii). Fix $\varepsilon > 0$ and take $\delta_\varepsilon > 0$ so that $|x-y| \le \delta_\varepsilon$ implies $|g(x)-g(y)| \le \varepsilon$. Set $C_\varepsilon = 1/\delta_\varepsilon$. For $x, y \in \mathbb{R}^n$ with $x \neq y$, choose $m \in \mathbb{N} \cup \{0\}$ such that

$$m \le \frac{|x-y|}{\delta_\varepsilon} < m+1,$$

and define $x_k = x + k\delta_\varepsilon \frac{x-y}{|x-y|}$, for each $k \in \{1, 2, \cdots, m\}$. Then

$$|x - x_1| = \delta_\varepsilon, \quad |x_k - x_{k+1}| = \delta_\varepsilon, \quad |x_m - y| < \delta_\varepsilon.$$

Hence,

$$\begin{aligned} |g(x)-g(y)| &\le |g(x)-g(x_1)| + \sum_{k=1}^{m-1} |g(x_k)-g(x_{k+1})| + |g(x_m)-g(y)| \\ &\le (m+1)\varepsilon \le \frac{|x-y|}{\delta_\varepsilon}\varepsilon + \varepsilon = C_\varepsilon|x-y| + \varepsilon. \end{aligned}$$

Next, we show (ii) implies (i). Fix $\varepsilon > 0$ and take $C_\varepsilon > 0$ as in (ii) with $\varepsilon/2$ in place of ε. Set $\delta_\varepsilon = \varepsilon/(2C_\varepsilon)$. Then $|x-y| \le \delta_\varepsilon$ implies

$$|g(x)-g(y)| \le C_\varepsilon|x-y| + \frac{\varepsilon}{2} \le \varepsilon,$$

and hence g is uniformly continuous. □

Thus, $f \in UC(\mathbb{R}^n)$ satisfies the condition (1.4). In particular, if $f \in UC(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that

$$|f(x)-f(y)| \le C|x-y| + 1, \quad x, y \in \mathbb{R}^n.$$

In the following, we fix such a constant C. For a given $t > 0$ and $x \in \mathbb{R}^n$, we set

$$E_t(x) = \Big\{ y \in \mathbb{R}^n \;\Big|\; L\big(\frac{x-y}{t}\big) \le L(0) + \frac{1}{t}(C|x-y|+1) \Big\}.$$

Lemma 2.10. *Let $f \in UC(\mathbb{R}^n)$ and $(t, x) \in (0,\infty) \times \mathbb{R}^n$. Then,*

$$H_t f(x) = \min_{y \in E_t(x)} q_f(t, x; y). \tag{2.2}$$

Furthermore, the following hold:

(i) $E_t(x)$ is a compact set.

(ii) For all $t, t' \in (0, \infty)$ with $t < t'$, $E_t(x) \subset E_{t'}(x)$.

Proof. Fix $(t, x) \in (0,\infty) \times \mathbb{R}^n$. Choose any $y = y(t, x) \in \mathbb{R}^n$ so that $H_t f(x) = q_f(t, x; y)$. Since f is uniformly continuous on $\mathbb{R}^n$, we have

$$tL(0) + C|x-y| + 1 + f(y) \ge tL(0) + f(x) \ge H_t f(x) = q_f(t, x; y) = f(y) + tL\big(\frac{x-y}{t}\big).$$

Thus,

$$L\big(\frac{x-y}{t}\big) \le L(0) + \frac{1}{t}(C|x-y|+1).$$

This implies that $y \in E_t(x)$. Hence, we have (2.2).

We next verify (i). First, suppose that $E_t(x)$ is not bounded on $\mathbb{R}^n$. Then we find a sequence $\{y_j\} \subset E_t(x)$ such that $|y_j| \to \infty$ as $j \to \infty$. Since

$$L\big(\frac{x-y_j}{t}\big) \le L(0) + \frac{1}{t}(C|x-y_j|+1),$$

we have

$$\frac{L\left(\frac{x-y_j}{t}\right)}{\frac{|x-y_j|}{t}} \leq \frac{tL(0)}{|x-y_j|} + C + \frac{1}{|x-y_j|}.$$

Since L is superlinear, we conclude that $\infty \leq C < \infty$ by letting $j \to \infty$. This is a contradiction. Thus $E_t(x)$ is bounded. By continuity of L, it is easy to see that $E_t(x)$ is closed. Hence, $E_t(x)$ is a compact set.

We finally verify (ii). Set $t < t'$ and fix $y \in E_t(x)$ arbitrary. Now we set

$$\omega(r) = rL\left(\frac{x-y}{r}\right) - rL(0), \quad r \in (0, \infty).$$

This function ω is decreasing in $(0, \infty)$. Indeed, by using the inequality (2.1), we have

$$\begin{aligned}\omega'(r) &= L\left(\frac{x-y}{r}\right) - L(0) + \frac{1}{r}DL\left(\frac{x-y}{r}\right) \cdot (y-x) \\ &\leq -\frac{1}{r}DL\left(\frac{x-y}{r}\right) \cdot (x-y) + \frac{1}{r}DL\left(\frac{x-y}{r}\right) \cdot (x-y) = 0.\end{aligned}$$

This implies that the ω is decreasing in $(0, \infty)$. Hence, we have

$$t'L\left(\frac{x-y}{t'}\right) - t'L(0) = \omega(t') \leq \omega(t) \leq C(|x-y|+1).$$

Thus, $y \in E_{t'}(x)$, and we conclude that $E_t(x) \subset E_{t'}(x)$. The proof is complete. □

3 Results

In this section, we state and prove our main results. We also give examples to illustrate these results. First, we have

Proposition 3.1. *Let $a \in \mathbb{R}^n$. Assume that $f \in C(\mathbb{R}^n)$ fulfills (1.4). Then, for all $t > 0$, we have $M_a(t) \subset Q_a(t)$.*

Proof. Fix $t > 0$ and let $x \in M_a(t)$. Since $H_t f(x) = q_f(t, x; a)$, we have

$$f(a) + tL\left(\frac{x-a}{t}\right) \leq f(z) + tL\left(\frac{x-z}{t}\right), \quad z \in \mathbb{R}^n.$$

Define the function $\psi : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(z) = -tL\left(\frac{x-z}{t}\right) + tL\left(\frac{x-a}{t}\right) + f(a), \quad z \in \mathbb{R}^n.$$

Then, since ψ is of class C^1, $\psi(a) = f(a)$ and $\psi(z) \leq f(z)$ for any $z \in \mathbb{R}^n$, ψ is a subtangent to f at a. Hence, by Lemma 2.3, we have

$$D\psi(a) \in D^- f(a).$$

Since $D\psi(a) = DL\left(\frac{x-a}{t}\right)$ and Lemma 2.7(iv), we see that $x \in Q_a(t)$. The proof is complete. □

Next, we establish the equivalence between the global subgradient inequality (1.9) and the geometric equality in (1.6).

Theorem 3.2. *Let $a \in \mathbb{R}^n$. Assume that $f \in C(\mathbb{R}^n)$ fulfills (1.4). Then the following are equivalent:*

(M1) f satisfies the global subgradient inequality (1.9) at a.

(M2) The equality $M_a(t) = Q_a(t)$ holds for all $t > 0$.

Proof. First, we show that (M1) implies (M2). Although (M1) implies (M2) is almost trivial, for the sake of completeness, we provide the full proof.

Fix $t > 0$. By Proposition 3.1 we have $M_a(t) \subset Q_a(t)$. Hence, we prove that $M_a(t) \supset Q_a(t)$. Let $x \in Q_a(t)$. By Lemma 2.7(iv), we have

$$DL\Big(\frac{x-a}{t}\Big) \in D^- f(a). \tag{3.1}$$

To show that $x \in M_a(t)$, it suffices to verify that $H_t f(x) = q_f(t, x; a)$. For this purpose, we define $J(y) = q_f(t, x; y) - q_f(t, x; a)$ for $y \in \mathbb{R}^n$ and show that $J(y) \geq 0$ for any $y \in \mathbb{R}^n$. First, by (M1) and (3.1), we have

$$f(y) - f(a) \geq DL(\frac{x-a}{t}) \cdot (y-a), \quad y \in \mathbb{R}^n. \tag{3.2}$$

Next, since L is convex, Lemma 2.5 yields

$$tL(\frac{x-y}{t}) - tL(\frac{x-a}{t}) \geq -DL(\frac{x-a}{t}) \cdot (y-a), \quad y \in \mathbb{R}^n. \tag{3.3}$$

Using (3.2) and (3.3), we estimate $J(y)$ as follows:

$$\begin{aligned} J(y) = q_f(t, x; y) - q_f(t, x; a) &= f(y) - f(a) + tL(\frac{x-y}{t}) - tL(\frac{x-a}{t}) \\ &\geq DL(\frac{x-a}{t}) \cdot (y-a) - DL(\frac{x-a}{t}) \cdot (y-a) = 0. \end{aligned}$$

Therefore, $J(y) \geq 0$ for any $y \in \mathbb{R}^n$, which implies that $H_t f(x) = q_f(t, x; a)$. Hence $x \in M_a(t)$. We conclude that $M_a(t) \supset Q_a(t)$, and thus $M_a(t) = Q_a(t)$.

Next, we show that (M2) implies (M1). We argue by contradiction. Suppose that (M2) holds, but (M1) does not. We will find $t_0 > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$x_0 \in Q_a(t_0) \quad \text{but} \quad x_0 \notin M_a(t_0),$$

which contradicts (M2). First, by the negation of (M1), there exist $p \in D^- f(a)$ and $b \in \mathbb{R}^n \setminus \{a\}$ such that

$$f(b) - f(a) < p \cdot (b-a). \tag{3.4}$$

By this inequality and continuity of DL (Lemma 2.7), we can choose $t_0 > 0$ such that

$$f(b) - f(a) \leq DL\Big(DH(p) - \frac{b-a}{t_0}\Big) \cdot (b-a). \tag{3.5}$$

For this $t_0 > 0$, we set $x_0 = a + t_0 DH(p)$. Then, $x_0 \in Q_a(t_0)$. For this choice of t_0 and x_0, we have

$$q_f(t_0, x_0; b) < q_f(t_0, x_0; a).$$

Indeed, by (3.5) and the definition of $x_0 = a + t_0 DH(p)$, we have

$$\begin{aligned} f(b) - f(a) &\leq DL\Big(DH(p) - \frac{b-a}{t_0}\Big) \cdot (b-a) \\ &= DL(\frac{x_0-b}{t_0}) \cdot (b-a) \end{aligned} \tag{3.6}$$

Next, since L is strictly convex on $\mathbb{R}^n$ and $b \neq a$, it follows from Lemma 2.1 that

$$DL(\frac{x_0-b}{t_0}) \cdot (b-a) < t_0 L(\frac{x_0-a}{t_0}) - t_0 L(\frac{x_0-b}{t_0}). \tag{3.7}$$

By using (3.7) and (3.6), we have

$$f(b) - f(a) \leq DL(\frac{x_0-b}{t_0}) \cdot (b-a) < t_0 L(\frac{x_0-a}{t_0}) - t_0 L(\frac{x_0-b}{t_0})$$

Hence

$$f(b) + t_0 L\left(\frac{x_0 - b}{t_0}\right) < f(a) + t_0 L\left(\frac{x_0 - a}{t_0}\right).$$

This implies that $q_f(t_0, x_0; b) < q_f(t_0, x_0; a)$. Thus, $x_0 \notin M_a(t_0)$, which is a contradiction. Therefore, we have shown that (M2) implies (M1). The proof is complete. □

We now illustrate the result with the following example. In the following examples, we consider the Hamiltonian $H(p) = |p|^2/2$, for $p \in \mathbb{R}^n$, so then the corresponding Lagrangian is given by $L(v) = |v|^2/2$ for $v \in \mathbb{R}^n$ by Example 2.8. We illustrate how the equality $M_a(t) = Q_a(t)$ appears in a concrete example.

Example 3.3. Let $f(z) = |z|$, $z \in \mathbb{R}^n$. Then, f satisfies the global subgradient inequality (1.9) at $a = 0$, and $D^- f(0) = \overline{B}_1(0)$.

We first compute the flow $H_t f$ and then identify the sets $M_0(t)$ and $Q_0(t)$. For each $t > 0$, it is not difficult to calculate that

$$H_t f(x) = \begin{cases} \frac{|x|^2}{2t}, & \text{if } |x| \le t, \\ |x| - \frac{t}{2}, & \text{if } |x| > t. \end{cases} \tag{3.8}$$

Since $D^- f(0) = \overline{B}_1(0)$ and $DH(p) = p$, the set $Q_0(t)$ is obtained by scaling the unit ball by the factor t, namely,

$$Q_0(t) = \{tp \mid p \in \overline{B}_1(0)\} = \overline{B}_t(0).$$

On the other hand, the expression of $H_t f$ (3.8) shows that the quadratic part is attained precisely on the region $|x| \le t$, which coincides with $\overline{B}_t(0)$. Hence, we have $M_0(t) = \overline{B}_t(0)$.

Therefore, this example shows that the equality in (1.6) holds for all $t > 0$. As shown in Figure 1, the broken line represents the initial value f with $n = 1$, the solid line represents the flow $H_t f(\cdot)$, and the thick solid line shows $q_f(t, \cdot; 0)$.

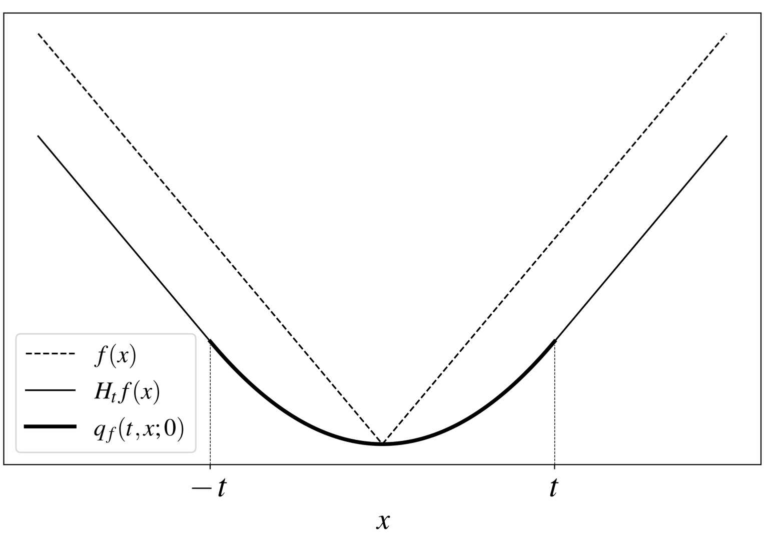

Figure 1. f in Example 3.3 with $n = 1$

Note that if f is a convex function on $\mathbb{R}^n$, then f satisfies the global subgradient inequality (1.9). Thus, we have

Corollary 3.4. *Let $a \in \mathbb{R}^n$. Let f be a convex function on $\mathbb{R}^n$. Then, for any $t > 0$, we have the equality in (1.6) holds.*

Next, we give an example of f such that f satisfies the global subgradient inequality (1.9) but f is not convex on $\mathbb{R}^n$. For the sake of simplicity, we let $n = 1$.

Example 3.5. Let f be the function defined by

$$f(z) = \begin{cases} |z|, & z < 1,\ z > \frac{5}{3}, \\ -2|z - \frac{3}{2}| + 2, & 1 \le z \le \frac{5}{3}, \end{cases}$$

and $a = 0$. Then, this function f is not convex on $\mathbb{R}$. However, it is easy to see that f satisfies the global subgradient inequality at $a = 0$. Therefore, by Theorem 3.2, the equality in (1.6) holds for all $t > 0$, even though f is not convex.

As shown in Figure 2, the broken line represents the initial value f, the solid line represents the flow $H_t f(\cdot)$ with $t = 1$, and the thick solid line shows $q_f(t, \cdot; 0)$ with $t = 1$.

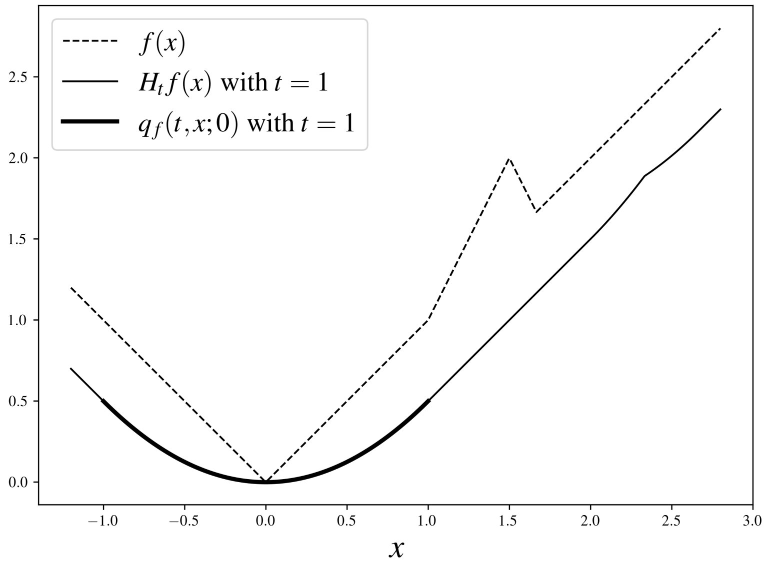

Figure 2. f in Example 3.5

On the other hand, we next consider cases where f does not satisfy the global subgradient inequality (1.9) at $a \in \mathbb{R}^n$ but where the equality in (1.6) still holds for small $t > 0$. We first provide an example:

Example 3.6. Let $n = 1$, $a = 1$ and $f \in C(\mathbb{R})$ be the function defined by

$$f(z) = \begin{cases} |z| - 1, & |z| > 1, \\ 1 - |z|, & |z| \le 1. \end{cases}$$

Then, $D^- f(1) = [-1, 1]$. It is clear that f does not satisfy the global subgradient inequality (1.9) at $a = 1$, although it exhibits a similar behavior locally near $a = 1$.

On the other hand, the equality in (1.6) holds for small time. More precisely,

$$M_1(t) = Q_1(t), \quad t \in (0, \frac{1}{2}]. \tag{3.9}$$

However, this equality does not persist for a larger time. For example,

$$Q_1(2) \setminus M_1(2) \neq \emptyset. \tag{3.10}$$

This example shows that there is a clear difference between the local and global behavior of the equality in (1.6). Even though f does not satisfy the global subgradient inequality at $a = 1$, the equality $M_1(t) = Q_1(t)$ holds for small $t \in (0, \frac{1}{2}]$ as in (3.9), but it fails for larger times, as shown in (3.10).

As shown in Figure 3, the broken line represents the initial value f, the solid line represents the flow $H_t f(\cdot)$ with $t = \frac{1}{2}$, and the thick solid line shows $q_f(t, \cdot; 1)$ with $t = \frac{1}{2}$.

Proof of (3.9) and (3.10). Fix $t \in (0, \frac{1}{2}]$. We prove that $M_1(t) \supset Q_1(t)$ by verifying that $q_f(t,x;1)$ attains the minimum for $x \in Q_1(t)$. Fix $x \in Q_1(t) = [1-t, 1+t]$. We set $F(z) = q_f(t,x;z)$ for $z \in \mathbb{R}$ and we compute that $\min_{z\in\mathbb{R}} F(z) = \frac{1}{2t}(x-1)^2$. By using $1-t \le x \le 1+t$ and $t \le \frac{1}{2}$, we have

$$\begin{aligned}\min_{z\in\mathbb{R}} F(z) &= \min\Big\{\min_{z\le -1} F(z),\ \min_{-1\le z\le 0} F(z),\ \min_{0\le z\le 1} F(z),\ \min_{z\ge 1} F(z)\Big\}\\ &= \min\Big\{\frac{1}{2t}(x+1)^2,\ 1+\frac{x^2}{2t},\ \frac{1}{2t}(x-1)^2,\ \frac{1}{2t}(x-1)^2\Big\} = \frac{1}{2t}(x-1)^2.\end{aligned}$$

Thus, we have $H_t f(x) = \frac{1}{2t}(x-1)^2 = q_f(t,x;1)$. This implies that the equality in (1.6) for $t \in (0, \frac{1}{2}]$ holds.

Next, we prove that $Q_1(2)\backslash M_1(2) \neq \emptyset$. Letting $x = -1$, then $x \in Q_1(2)(= [-1,3])$ but $x \notin M_1(2)$. Indeed,

$$H_2 f(-1) = \min_{z\in\mathbb{R}} q_f(2,-1;z) = 0 < 1 = q_f(2,-1;1).$$

This implies that $x \notin M_1(2)$. Thus, $Q_1(2)\backslash M_1(2) \neq \emptyset$. □

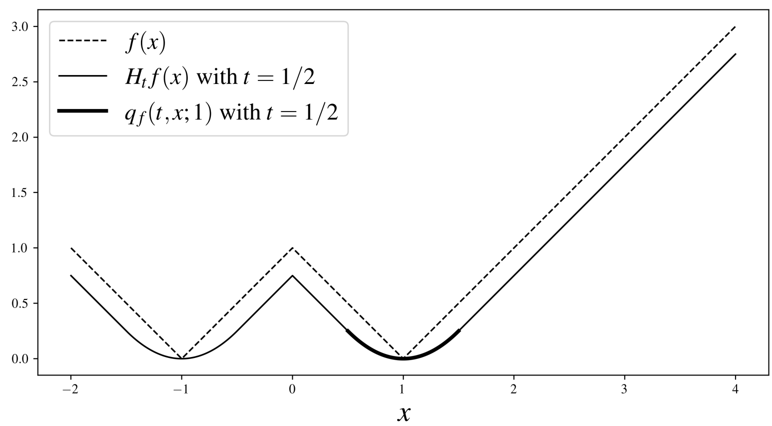

Figure 3. f in Example 3.6 with $t = \frac{1}{2}$

By Example 3.6, we observe a clear difference between the local and global behavior of the equality in (1.6). Even though f does not satisfy the global subgradient inequality at $a \in \mathbb{R}^n$, the equality $M_a(t) = Q_a(t)$ still holds for small $t > 0$, while it fails for larger times.

This example suggests that the validity of (1.6) for small times depends only on the local behavior of f near the point a. Motivated by this observation, we introduce a local version of the subgradient inequality at $a \in \mathbb{R}^n$, that is,

$$f(y) \ge f(a) + p\cdot(y-a), \quad y \in B_\delta(a),\ p \in D^- f(a) \tag{3.11}$$

for some $\delta > 0$. In other words, this means that f satisfies a subgradient inequality in the neighborhood of $a \in \mathbb{R}^n$. In the following result, we further suppose that $f \in UC(\mathbb{R}^n)$.

Theorem 3.7. *Let $a \in \mathbb{R}^n$. Assume that $f \in UC(\mathbb{R}^n)$ fulfills the local subgradient inequality (3.11) at a for some $\delta > 0$. Then there exists $t_0 > 0$ such that the equality in (1.6) holds for all $t \in (0, t_0]$.*

In order to prove Theorem 3.7, we preliminarily give some lemmas.

Lemma 3.8. *Let $a \in \mathbb{R}^n$. Assume that f fulfills the local subgradient inequality (3.11) at a for some $\delta > 0$. Then, $D^- f(a)$ is bounded.*

Proof. Let f satisfy the local subgradient inequality (3.11) for some $\delta > 0$. Supposing that $D^- f(a)$ is not bounded, then we choose a sequence $\{p_j\} \subset D^- f(a)$ such that $|p_j| \to \infty$ as

$j \to \infty$. We choose a sequence $\{y_j\}$ with $y_j = a + \delta p_j/2|p_j|$, which belongs to $B_\delta(a)$. By (3.11), we have

$$f(y_j) \geq f(a) + p_j \cdot (y_j - a), \quad j \in \mathbb{N}. \tag{3.12}$$

Since $\{y_j\}$ is a bounded sequence, we find the subsequence $\{y_{j_k}\}$ and $\gamma \in \mathbb{R}^n$ such that

$$\lim_{k\to\infty} y_{j_k} = \gamma.$$

Thus, by using (3.12), we conclude that

$$f(\gamma) = \lim_{k\to\infty} f(y_{j_k}) \geq f(a) + \lim_{k\to\infty} p_{j_k} \cdot (y_{j_k} - a) = f(a) + \frac{\delta}{2} \lim_{k\to\infty} |p_{j_k}| = \infty,$$

which is a contradiction. Hence $D^- f(a)$ is bounded. □

Lemma 3.9. *Let $\delta > 0$, $m > 0$ and $a \in \mathbb{R}^n$. Then, there exists $\tau > 0$ such that*

$$if \ \ z \in \overline{B}_{m\tau}(a) \quad then \quad E_\tau(z) \subset B_\delta(a). \tag{3.13}$$

Proof. Suppose that the conclusion of the lemma is false. Then, we choose a sequence $\{z_j\} \subset \overline{B}_{\frac{m}{j}}(a)$ and $\{w_j\} \subset \mathbb{R}^n$ such that

$$w_j \in E_{\frac{1}{j}}(z_j) \quad and \quad |w_j - a| \geq \delta \quad for\ all\ j \in \mathbb{N}.$$

Then, we have $|z_j - w_j| \to 0$ as $j \to \infty$. Indeed, supposing that $|z_j - w_j| \not\to 0$ as $j \to \infty$, then we can find a $\eta > 0$ such that $|z_j - w_j| \geq \eta$ for sufficiently large number $j \in \mathbb{N}$. Since

$$\frac{L((z_j - w_j)j)}{|z_j - w_j|j} \leq \frac{L(0)}{|z_j - w_j|j} + C + \frac{1}{|z_j - w_j|},$$

we conclude that $\infty \leq C + \frac{1}{\eta} < \infty$, which is a contradiction. Thus $|z_j - w_j| \to 0$ as $j \to \infty$. From the above, we have

$$\delta \leq |w_j - a| \leq |w_j - z_j| + |z_j - a| \to 0 \quad \text{as} \quad j \to \infty,$$

which is a contradiction, since $\delta > 0$. Hence there exists $\tau > 0$ which satisfies (3.13). □

Proof of Theorem 3.7. First of all, we find a $t_0 \in (0, \infty)$ in the theorem. Set

$$m = \sup_{p \in D^- f(a)} |DH(p)|.$$

It is well–defined by Lemma 3.8. By using this lemma again, we can find a $t_1 \in (0, \infty)$ such that

$$\sup_{p \in D^- f(a)} L\big(DH(p)\big) \leq L(0) + \frac{1}{t_1}. \tag{3.14}$$

Also, by Lemma 3.9, we can find $t_2 \in (0, \infty)$ such that if $z \in \overline{B}_{mt_2}(a)$ then $E_{t_2}(z) \subset B_\delta(a)$. Finally, we set $t_0 = \min\{t_1, t_2\}$.

Fix $t \in (0, t_0]$. By Proposition 3.1, it is sufficient to show that $M_a(t) \supset Q_a(t)$. Fix $x \in Q_a(t)$. Then, by $DL(\frac{x-a}{t}) \in D^- f(a)$ and (3.11), we have

$$f(y) \geq f(a) + DL\big(\frac{x-a}{t}\big) \cdot (y - a), \quad y \in B_\delta(a). \tag{3.15}$$

We first verify that $a \in E_t(x)$. We set $q = DL(\frac{x-a}{t})$. By $q \in D^- f(a)$ and (3.14), we have

$$L\big(\frac{x-a}{t}\big) = L((DL)^{-1}(q)) = L(DH(q)) \leq \sup_{p \in D^- f(a)} L(DH(p))$$
$$\leq L(0) + \frac{1}{t} \leq L(0) + \frac{1}{t}(C|x - a| + 1).$$

This implies $a \in E_t(x)$.

Next, set $J(y) = q_f(t, x; y) - q_f(t, x; a)$ for $y \in E_t(x)$. We prove that $J(y) \geq 0$ for any $y \in E_t(x)$. Then the equality $H_t f(x) = q_f(t, x; a)$ holds by Lemma 2.10. Since

$$|x - a| = |tDH(q)| \leq t_0 |DH(q)| \leq t_0 m \leq t_2 m,$$

we have $x \in \overline{B}_{mt_2}(a)$. By using Lemma 3.9, we have $E_{t_2}(x) \subset B_\delta(a)$. Thus, by Lemma 2.10-(ii) and $t \leq t_2$, we have $E_t(x) \subset B_\delta(a)$. Next, fix $y \in E_t(x)$ arbitrarily. Thus, by using Lemma 2.5 and (3.15), we have

$$\begin{aligned} J(y) &= f(y) - f(a) + tL\Big(\frac{x-y}{t}\Big) - tL\Big(\frac{x-a}{t}\Big) \\ &\geq f(y) - f(a) - DL\Big(\frac{x-a}{t}\Big) \cdot (y - a) \\ &\geq DL\Big(\frac{x-a}{t}\Big) \cdot (y-a) - DL\Big(\frac{x-a}{t}\Big) \cdot (y-a) = 0. \end{aligned}$$

Therefore, $x \in M_a(t)$. Hence, we conclude that $M_a(t) \supset Q_a(t)$. The proof is complete. □

Recall that a function $g : \mathbb{R}^n \to \mathbb{R}$ is locally convex at $\alpha \in \mathbb{R}^n$ if there exists a $\delta > 0$ such that g is convex on $B_\delta(\alpha)$. If a uniformly continuous function f is locally convex at $a \in \mathbb{R}^n$, then we see that f satisfies the local subgradient inequality (1.10) by Lemma 2.4. Thus we have

Corollary 3.10. *Let $a \in \mathbb{R}^n$. Let $f \in UC(\mathbb{R}^n)$ be a locally convex at a. Then, there exists $t_0 > 0$ such that the equality in (1.6) for all $t \in (0, t_0]$.*

From this corollary, local convexity of f at $a \in \mathbb{R}^n$ ensures the equality in (1.6) for a local time.

Finally, we show that, in Theorem 3.7, the converse assertion does not hold in general, that is, even if the local equality in (1.6) for $t \in (0, t_0]$, $f \in UC(\mathbb{R}^n)$ does not always satisfy the local subgradient inequality at $a \in \mathbb{R}^n$. Indeed, we have

Example 3.11. Let $n = 1$ and $f \in UC(\mathbb{R})$ be the function defined by

$$f(z) = \begin{cases} -z, & z < 0, \\ \arctan z, & z \geq 0. \end{cases}$$

Then, $D^- f(0) = [-1, 1]$. Since $f(y) = \arctan y < y$ for $y \in (0, \infty)$ and $1 \in D^- f(0)$, f does not satisfy the local subgradient inequality at $a = 0$ for any $\delta > 0$.

On the other hand, it is easy to see that the local equality

$$M_0(t) = Q_0(t), \quad t \in (0, 1]$$

holds. Thus, the converse assertion does not always hold true in Theorem 3.7. This example shows that the local validity of (1.6) does not necessarily imply the local subgradient inequality.

4 Conclusion

In this paper, we studied a geometric characterization problem for the Hamilton–Jacobi flow associated with the Hopf–Lax formula. More precisely, we investigated the relationship between the geometric equality

$$M_a(t) = Q_a(t)$$

and the subdifferential structure of the initial data f at a point $a \in \mathbb{R}^n$.

We proved that the global subgradient inequality is equivalent to the validity of the equality $M_a(t) = Q_a(t)$ for all $t > 0$. This provides a complete characterization of the global behavior of the flow in terms of the subdifferential.

We also analyzed the local counterpart of this problem. We showed that the local subgradient inequality implies the validity of $M_a(t) = Q_a(t)$ for small time, and examples demonstrate that this property does not extend to global time and that the converse implication does not hold in general. These results reveal a clear distinction between the local and global behavior of the equality.

References

[1] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Systems and Control: Foundations and Applications*, Brikhäuser Boston, Inc. Boston, MA, (1997).

[2] M. Bardi and S. Faggian, *Hopf-type Estimates and formulas for nonconvex nonconcave Hamilton–Jacobi equations*, SIAM Journal on Mathematical Analysis, **29**, 1067–1086, (2006).

[3] P. Cannarsa and C. Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control, Progress in Nonlinear Differential Equations and their Applications*, **58**, Birkhäuser Boston Inc. Boston, MA, (2004).

[4] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, **19**, American Mathematical Society, (2010).

[5] Y. Fujita, N. Hamamuki, A. Siconolfi and N. Yamaguchi, *A class of nowhere differentiable functions satisfying some concavity-type estimate*, Acta Mathematica Hungarica, **160**, 343–359, (2020).

[6] Y. Fujita, N. Hamamuki and N. Yamaguchi, *All the generalized characteristics for the solution to a Hamilton-Jacobi equation with the initial data of the Takagi function*, SN Partial Differential Equations and Applications, **1**, 1–20, (2020).

[7] Y. Fujita, N. Hamamuki and N. Yamaguchi, *A self-affine property of evolutional type appearing in a Hamilton-Jacobi flow starting from the Takagi function*, Michigan Math J., **71**, 105–120, (2022).

[8] Y. Fujita, A. Siconolfi and N. Yamaguchi, *Hamilton-Jacobi flows with nowhere differentiable initial data*, Mathematische Annalen, **385**, 1061–1084, (2023).

[9] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series Princeton University Press, Princeton, N. J., (1997). Reprint of the 1970 original. Princeton Landmarks in Mathematics. Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1997.

[10] T. D. Van, *Hopf-Lax-Oleinik-type formulas for viscosity solutions to some Hamilton–Jacobi equations*, Vietnam J. Math., **32**, 241-275, (2004).

[11] H. V. Tran, *Hamilton-Jacobi equations theory and applications*, Graduate Studies in Mathematics, **213**, American Mathematical Society, (2021).

Author information

Kotaro Matsuoka, HOKURIKU COMPUTER SERVICE CO., LTD., 47-4 Fuchumachi-Shimahongo, Toyama-shi, Toyama 939-2708, Japan.
E-mail: `hcskmatsuo@gmail.com`

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