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WEAKLY ENRICHED MAPS AND FIXED POINT APPROXIMATION

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Abstract. This paper investigates non-self weakly inward $((b,\theta))$-enriched contraction mappings defined on closed convex subsets of Banach spaces. By applying an averaging technique, we show that the associated operator becomes a contraction while preserving the weak inwardness property. As a consequence, the mapping admits a unique fixed point, which can be effectively approximated via the Picard iteration. We also derive sufficient conditions guaranteeing the convergence of the iteration within closed balls. Furthermore, we establish stability properties of enriched contractions under addition and convex combinations. These results lead to the existence of fixed point curves for families of weakly inward enriched contractions.

1 Introduction

The Banach Contraction Principle remains one of the cornerstones of fixed point theory, guaranteeing both the existence and uniqueness of fixed points, together with the convergence of the associated iterative sequences. In order to broaden the scope of this principle, numerous generalizations of contraction mappings have been developed. Among these, the class of enriched contractions introduced and investigated by Berinde and Păcurar [1] has attracted considerable attention. This class provides a unifying framework that encompasses classical contractions as well as certain classes of nonexpansive mappings. Furthermore, enriched contractions admit efficient iterative schemes for the approximation of fixed points.

In many applications, however, one encounters non-self mappings. For such operators, inwardness-type conditions play a crucial role in ensuring the existence of fixed points within a prescribed subset. In particular, weak inwardness constitutes a natural and flexible extension of the classical inwardness condition and has been extensively studied in non-self fixed point theory. Motivated by these developments, the present paper focuses on weakly inward (b,θ)-enriched contraction mappings defined on closed convex subsets of Banach spaces.

We first establish existence and uniqueness results for fixed points of non-self weakly inward enriched contractions. Our approach relies on the construction of a suitable averaged operator, which is shown to be a contraction while preserving the weak inwardness property. This technique enables the application of the Picard iteration process to approximate the fixed point.

Subsequently, we derive sufficient conditions under which the averaged operator becomes a Picard operator on closed balls and investigate the convergence behavior of the associated iterative scheme. We also analyze the stability of enriched contractions under addition and convex combinations. As an application of these stability properties, we prove the existence of families of fixed points corresponding to convex combinations of weakly inward enriched contractions, thereby introducing the notion of fixed point curves.

The results obtained in this work extend several existing fixed point theorems for enriched and weakly inward mappings and contribute to the theory of fixed point approximation for a wider class of nonlinear operators.

Before introducing the concept of fixed point curves for enriched contractions, we recall several preliminary definitions and fundamental results that will be used throughout the paper.

2 Preliminaries

Definition 2.1. [1] Let X be a normed linear space. A self map $T : X \to X$ is called a Picard operator if there exist some p in X such that :

(i) $Fix(T) = \{x \in X : T(x) = x\} = \{p\}$.

(ii) $T^n(x_0) \to p$ as $n \to \infty$, for any $x_0 \in X$.

In a complete metric space, every contraction mapping is a Picard operator by the Banach Contraction Principle, which guarantees the existence of a unique fixed point and the convergence of the Picard iteration from any initial point. The converse, however, does not hold in general.

To illustrate this, consider the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{x}{1+x}.$$

This mapping admits a unique fixed point, namely $x = 0$. Moreover, for any initial value $x_0 \in \mathbb{R}$ (for which the iteration is well-defined), the Picard sequence defined by $x_{n+1} = T(x_n)$ converges to 0. Nevertheless, T is not a contraction on $\mathbb{R}$.

Definition 2.2. [1] Let X be a linear normed space. A mapping $T : X \to X$ is said to be an enriched or (b, θ) enriched contraction if there is some $b \in [0, \infty)$ and $\theta \in [0, b + 1)$ such that

$$||b(x - y) + T(x) - T(y)|| \leq \theta||x - y||, \text{ for all } x, y \in X.$$

Every contraction mapping with contraction constant $c \in (0, 1)$ is a $(0, c)$-enriched contraction. However, the converse is not necessarily true; a (b, θ)-enriched contraction need not be a contraction in the classical sense.

For instance, let $X = [0, 1]$ be endowed with the usual norm, and define $T : X \to X$ by $T(x) = 1 - x$. Then T is nonexpansive but not a contraction. Nevertheless, T satisfies the conditions of an enriched contraction.

Definition 2.3. [1] Let X be a linear normed space. A mapping $T : X \to X$ is said to be an enriched or $b-$ enriched nonexpansive mapping if there is some $b \in [0, \infty)$ such that

$$||b(x - y) + T(x) - T(y)|| \leq (b + 1)||x - y||, \text{ for all } x, y \in X.$$

The map $T : X \to X$ defined by $T(x) = \dfrac{1}{x}$ for all $x \in X = [\frac{1}{2}, 2]$ is a $3/2-$ enriched nonexpansive mapping but not a nonexpansive mapping. Thus, the class of enriched contractions forms a large and flexible family of operators, which properly includes both contraction and nonexpansive mappings. Moreover, there exist enriched contractions that do not belong to the class of nonexpansive mappings.

Theorem 2.4. *[1] Let X be a Banach space and $T : X \to X$ be a $(b, \theta)-$ enriched contraction. Then*

i) $Fix(T) = \{p\}$.

ii) There exists $\lambda \in (0, 1]$ such that the iterative sequence $\{x_n\}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T(x_n), n = 0, 1, 2 \ldots$$

converges to p, for any $x_0 \in X$.

iii) For $n = 0, 1, 2, 3 \ldots$ and $i = 1, 2, 3, \ldots$,

$$||x_{n+i-1} - p|| \leq \frac{c^i}{1 - c}||x_n - x_{n-1}||, \text{where } c = \frac{\theta}{b + 1}$$

.

This theorem suggests an iterative method, known as Krasnoselskij iterative method for approximating fixed points of enriched contractions.

Definition 2.5. [2] Let K be a closed convex subset of a Banach space X and let $C \subset K$. Then a mapping $T : C \to X$ is said to be *inward mapping (respectively, weakly inward)* on C relative to K if $T(x) \in I_K(x)$ (respectively, $T(x) \in \overline{I_K(x)}$) for $x \in C$ where

$$I_K(x) = \{(1-t)x + ty\colon y \in K \ and \ t \geq 0\}.$$

and $\overline{I_K(x)}$ is a closure of the $I_K(x)$ in X

Proposition 2.6. *[2] Let C be a non empty closed convex subset of a Banach space X. Then $T : C \to X$ is weakly inward if and only if*

$$\lim_{h \to 0^+} \frac{d((1-h)x + hTx, C)}{h} = 0, \quad \forall\, x \in C,$$

where $d(x, C) = \inf\{\|x - y\| : y \in C\}$.

Theorem 2.7. *[2] Let C be a nonempty closed convex subset of a Banach Space X and $T : C \to X$ a weakly inward contarction mapping. Then T has a fixed point in C.*

3 Weakly inward enriched contraction

In the first part of this section, we establish existence and uniqueness results for fixed points of non-self weakly inward enriched contraction mappings. In the second part, we investigate fixed point curves associated with families of weakly inward enriched contraction mappings.

Theorem 3.1. *Let K be a nonempty closed convex subset of a Banach Space X. If $T : K \to X$ is a weakly inward enriched contraction mapping, then T has a unique fixed point in K.*

Proof. Since T is an enriched contraction, there exist constants $b \geq 0$ and $\theta \in [0, b+1)$ such that

$$\| \, b(x-y) + T(x) - T(y) \, \| \leq \theta \|x - y\|, \qquad \forall x, y \in K. \tag{1}$$

If $b = 0$, then (1) reduces to

$$\|T(x) - T(y)\| \leq \theta \|x - y\|, \qquad \theta \in [0, 1),$$

so that T is a contraction. Hence, by the Banach Contraction Principle, T admits a unique fixed point in K.

Assume now that $b > 0$ and set

$$\lambda = \frac{1}{b+1}, \qquad 0 < \lambda < 1.$$

Then (1) can be rewritten as

$$\left\| \left(\frac{1}{\lambda} - 1 \right)(x-y) + T(x) - T(y) \right\| \leq \theta \|x - y\|, \qquad \forall x, y \in K. \tag{2}$$

Define the averaged operator $T_\lambda : K \to X$ by

$$T_\lambda(x) = (1-\lambda)x + \lambda T(x), \qquad x \in K.$$

It follows from (2) that

$$\|T_\lambda(x) - T_\lambda(y)\| \leq \lambda\theta \|x - y\|, \qquad \forall x, y \in K.$$

Since $c = \lambda\theta < 1$, the operator T_λ is a contraction.

We next show that T_λ is weakly inward. Because T is weakly inward, for each $x \in K$ there exists a sequence $\{z_n\} \subset I_K(x)$ such that $z_n \to T(x)$. Thus, for each n,

$$z_n = x + \alpha_n(y_n - x), \qquad y_n \in K, \ \alpha_n \geq 0.$$

Define

$$w_n = (1-\lambda)x + \lambda z_n.$$

Then

$$w_n = x + \lambda\alpha_n(y_n - x) \in I_K(x),$$

and

$$w_n \to (1-\lambda)x + \lambda T(x) = T_\lambda(x).$$

Hence $T_\lambda(x) \in \overline{I_K(x)}$, which shows that T_λ is weakly inward.

Therefore, $T_\lambda : K \to X$ is a weakly inward contraction. By the fixed point theorem for weakly inward contractions, T_λ admits a unique fixed point in K.

Finally, observe that

$$T_\lambda(x) = x \iff (1-\lambda)x + \lambda T(x) = x \iff T(x) = x.$$

Thus,

$$\mathrm{Fix}(T) = \mathrm{Fix}(T_\lambda).$$

Consequently, T has a unique fixed point in K.

□

Corollary 3.2. *Let X be a Banach Space. If $T : X \to X$ be a enriched contraction mapping, then for some $\lambda \in [0,1)$ the averaged operator $T_\lambda(x) = (1-\lambda)x + \lambda T(x)$ is a Picard Operator and its iteration converges to the unique fixed point of T.*

In the following result, we establish that the fixed point of a nonself weakly inward enriched contraction can be approximated by means of the Picard iterative scheme applied to the corresponding averaged operator.

Corollary 3.3. *Let X be a Banach Space and B be closed ball with centre at x_0 and radius r. If $T : B \to X$ is a weakly inward $(b,\theta)-$ enriched contraction mapping with $||T(x_0) - x_0|| < (b+1-\theta)r$, then for some $\lambda \in [0,1)$ the averaged operator $T_\lambda(x) = (1-\lambda)x + \lambda T(x)$ is a Picard Operator and its iteration approximates the unique fixed point of T.*

Proof. By Theorem 3.1, the averaged operator

$$T_\lambda(x) = (1-\lambda)x + \lambda T(x), \qquad \lambda = \frac{1}{b+1},$$

is a contraction and admits a unique fixed point in $B[x_0, r]$. We now prove that

$$T_\lambda : B[x_0, r] \to B[x_0, r].$$

First, observe that

$$\|T_\lambda(x_0) - x_0\| = \|\lambda(T(x_0) - x_0)\| < \lambda(b+1-\theta)r = (1-\lambda\theta)r = (1-c)r,$$

where $c = \lambda\theta < 1$.

Let $x \in B[x_0, r]$. Then, using the contraction property of T_λ, we obtain

$$\begin{aligned}\|T_\lambda(x) - x_0\| &\le \|T_\lambda(x) - T_\lambda(x_0)\| + \|T_\lambda(x_0) - x_0\| \\ &< c\|x - x_0\| + (1-c)r \\ &\le cr + (1-c)r = r.\end{aligned}$$

Hence, $T_\lambda\big(B[x_0, r]\big) \subseteq B[x_0, r]$. Consequently, the averaged operator T_λ is a Picard operator on $B[x_0, r]$. □

The following result establishes the stability of enriched contractions under addition and convex combinations.

Let T_1 and T_2 be two mappings defined on a nonempty closed and convex subset K of a Banach space X, where T_1 is a (b_1,θ_1)-enriched contraction and T_2 is a (b_2,θ_2)-enriched contraction. Then the sum T_1+T_2 is a $(b_1+b_2,\theta_1+\theta_2)$-enriched contraction.

Moreover, if T is a (b,θ)-enriched contraction, then for any $\lambda\in[0,1]$, the mapping λT is a (b,θ^*)-enriched contraction, where

$$\theta^*=\lambda\theta+(1-\lambda)b.$$

Consequently, for each $\lambda\in[0,1]$, the convex combination

$$T_\lambda=\lambda T_1+(1-\lambda)T_2$$

is a (b,θ)-enriched contraction, where

$$b=b_1+b_2 \quad\text{and}\quad \theta=\lambda\theta_1+(1-\lambda)\theta_2+(1-\lambda)b_1+\lambda b_2.$$

Next, we derive a necessary condition for the existence of fixed points for a family of weakly inward enriched contractions. Based on this condition, we introduce the notion of a fixed point curve associated with two mappings.

Theorem 3.4. *Let K be a nonempty closed and convex subset of a Banach space X and $T_1,T_2: K\to X$ be two weakly inward enriched contractions. Then there exist a family of weakly inward contraction maps $\{\psi_t: K\to X : t\in[0,1]\}$ with the property that for each $t\in[0,1]$, ψ_t has a unique fixed point in K.*

Proof. Let K be a nonempty closed and convex subset of a Banach space X. Suppose that $T_1: K\to X$ is a (b_1,θ_1)-enriched contraction and $T_2: K\to X$ is a (b_2,θ_2)-enriched contraction.

Define the mapping $\psi: K\times[0,1]\to X$ by

$$\psi(x,t)=(1-t)T_1(x)+tT_2(x).$$

Set

$$b=b_1+b_2 \quad\text{and}\quad \theta=t\theta_1+(1-t)\theta_2+(1-t)b_1+tb_2, \qquad t\in[0,1].$$

Then, for arbitrary $x,y\in K$ and $t\in[0,1]$, we obtain

$$\|\, b(x-y)+\psi(x,t)-\psi(y,t)\,\|\le\theta\|x-y\|.$$

Hence, ψ is a weakly inward enriched contraction uniformly with respect to $t\in[0,1]$.

For each fixed $t\in[0,1]$, define $\psi_t: K\to X$ by

$$\psi_t(x)=\psi(x,t).$$

Then ψ_t is a weakly inward enriched contraction on K for every $t\in[0,1]$. Consequently, by Theorem 3.1, each ψ_t admits a unique fixed point in K for every $t\in[0,1]$. □

In 1988, Sam B. Nadler and Ushijima [6] introduced the concept of a fixed point curve for linear interpolations of contraction mappings, based on the Banach contraction principle. This concept describes how the fixed points vary continuously when two contraction mappings are connected through a parameterized family. Motivated by this idea, we extend the notion of fixed point curves to the setting of weakly inward enriched contraction mappings. The following definition formalizes this concept for a pair of such mappings on a closed, convex and bounded subset of a Banach space.

Definition 3.5. Let T_1 and T_2 be two weakly inward enriched contractions on a closed, convex subset K of X and $F_W(T_1,T_2)$ denote the set of all unique fixed points of $\psi_t,\ t\in[0,1]$. Then the map $G\colon[0,1]\to F_W(T_1,T_2)$ defined by $G(t)=x_t$ is called the *fixed point curve* for T_1 and T_2.

Lemma 3.6. *Let $T_1,\ T_2\colon K\to X$ be two weakly inward enriched contractions on a closed, convex and subset K of X. Suppose that T_1 and T_2 have no common fixed points. Then the map $G\colon[0,1]\to F_W(T_1,T_2)$ defined by $G(t)=x_t$ is one to one.*

Proof. Suppose, to the contrary, that G is not injective. Then there exist $t_1, t_2 \in [0,1]$ with $t_1 \neq t_2$ such that

$$G(t_1) = G(t_2).$$

By definition of G, this implies

$$x_{t_1} = x_{t_2} = p$$

for some $p \in K$.

Since $x_t = \psi_t(x_t)$ for each $t \in [0,1]$, we have

$$(1-t_1)T_1(p) + t_1T_2(p) = (1-t_2)T_1(p) + t_2T_2(p).$$

Rearranging, we obtain

$$(t_2 - t_1)\big(T_2(p) - T_1(p)\big) = 0.$$

Because $t_1 \neq t_2$, it follows that

$$T_1(p) = T_2(p).$$

Consequently,

$$p = x_{t_1} = \psi_{t_1}(p) = (1-t_1)T_1(p) + t_1T_2(p) = T_1(p) = T_2(p).$$

Thus, p is a common fixed point of both T_1 and T_2, which contradicts the underlying assumption. Therefore, G must be injective. □

Corollary 3.7. *The fixed point curve* $G\colon [0,1] \to F_W(T_1, T_2)$ *is continuous.*

Proof. Define $G : [0,1] \to F_w(T_1, T_2)$ by

$$G(t) = x_t,$$

where x_t is the fixed point of the operator ψ_t or $x_t = \psi_t(x_t)$, with $\psi_t = (1-t)T_1 + tT_2$. Let $t_n \to t$ in $[0,1]$. Then $G(t_n) = \psi_{t_n}(G(t_n))$ and $G(t) = \psi_t(G(t))$.

Now,

$$\begin{aligned} \|G(t_n) - G(t)\| &\leq \|\psi_{t_n}(G(t_n)) - \psi_t(G(t))\| \\ &\leq \|\psi_{t_n}(G(t_n)) - \psi_{t_n}(G(t))\| + \|\psi_{t_n}(G(t)) - \psi_t(G(t))\| \\ &\leq k\|G(t_n) - G(t)\| + \|\psi_{t_n}(G(t)) - \psi_t(G(t))\| \end{aligned}$$

Hence,

$$(1-k)\|G(t_n) - G(t)\| \leq \|\psi_{t_n}(G(t)) - \psi_t(G(t))\|.$$

Since ψ_t is continuous in t, the right-hand side tends to 0 as $n \to \infty$.

Therefore,

$$G(t_n) \to G(t).$$

Thus, G is continuous on $[0,1]$.

□

Corollary 3.8. $F_W(T_1, T_2)$ *is a closed set.*

The following result establishes the existence and convergence of a fixed point curve associated with two weakly inward enriched contractions. It characterizes the behavior of fixed points corresponding to a family of operators generated by two given mappings. In particular, by constructing a sequence of weakly inward enriched contractions $\{\psi_m\}$ from the pair T_1 and T_2, we obtain a sequence of unique fixed points $\{x_m\}$ that converges to the fixed point of a suitable limiting operator.

Theorem 3.9. *Let K be a nonempty closed and convex subset of a Banach space X, and let $T_1, T_2 : K \to X$ be two weakly inward enriched contractions with no common fixed points in K.. For each $n \in \mathbb{N}$, define*

$$\psi_n(x) = \psi(x, \lambda_n) = \left(1 - \frac{1}{n}\right) T_1(x) + \frac{1}{n}T_2(x), \qquad x \in K.$$

Then each ψ_n is a weakly inward enriched contraction and hence admits a unique fixed point $x_n \in K$. If $\{\psi_n\}$ converges uniformly on K to a weakly inward enriched contraction ψ with unique fixed point $x \in K$, then $x_n \to x$ in K. In particular, $G(0) = x$.

Proof. For each positive integer n, set $\lambda_n = \frac{1}{n}$ and define

$$\psi_n(x) = \psi(x, \lambda_n) = \left(1 - \frac{1}{n}\right) T_1(x) + \frac{1}{n} T_2(x), \qquad x \in K.$$

Then each ψ_n is a weakly inward enriched contraction. Hence, for every $n \in \mathbb{N}$, there exists a unique fixed point $x_n \in K$ such that

$$\psi_n(x_n) = x_n.$$

Thus, we obtain a sequence of fixed points $\{x_n\}$, where $x_n \in \mathrm{Fix}(\psi_n)$.

Assume that ψ_n converges uniformly to a mapping ψ as $n \to \infty$, where ψ is also a weakly inward enriched contraction with a fixed point $x \in K$, that is,

$$\psi(x) = x.$$

Then

$$\|x_n - x\| = \|\psi_n(x_n) - \psi(x)\| \leq \|\psi_n(x_n) - \psi(x_n)\| + \|\psi(x_n) - \psi(x)\|.$$

Since ψ is an enriched contraction, we have

$$\begin{aligned} \|\psi(x_n) - \psi(x) &\leq \|b(x_n - x) + \psi(x_n) - \psi(x)\| + b\|x_n - x\| \\ &\leq \theta\|x_n - x\| + b\|x_n - x\| \\ &= (\theta + b)\|x_n - x\| \\ &= k\|x_n - x\| \end{aligned}$$

Therefore, $\|x_n - x\| \leq \dfrac{1}{1-k}\|\psi_n(x_n) - \psi(x_n)\|$ for some $k > 0$.

Since $\psi_n \to \psi$ uniformly on K, it follows that

$$x_n \to x \quad \text{as } n \to \infty.$$

Define $G : [0, 1] \to K$ by $G(\lambda_n) = x_n$. By the above convergence and the continuity of G, we obtain

$$G(0) = x.$$

□

Theorem 3.9 shows that if the sequence $\{\psi_m\}$ converges uniformly to a weakly inward enriched contraction ψ, then the associated sequence of fixed points converges to the unique fixed point of ψ. Consequently, this result provides a rigorous theoretical foundation for the construction and approximation of the fixed point curve generated by two weakly inward enriched contractions.

Example 3.10. Let $X = \mathbb{R}$ and $K = [0, 1]$. Define the mappings $T_1, T_2 : K \to X$ by

$$T_1(x) = \frac{1}{2}x + \frac{1}{4} \quad \text{and} \quad T_2(x) = \frac{1}{2}x - \frac{1}{4}.$$

Both T_1 and T_2 are weakly inward (b, θ)-enriched contractions on K.

For $\lambda \in [0, 1]$, consider the convex combination

$$\psi_\lambda(x) = (1 - \lambda)T_1(x) + \lambda T_2(x) = \frac{1}{2}x + \left(\frac{1}{4} - \frac{1}{2}\lambda\right).$$

Let $\lambda_n = \frac{1}{n}$ and set $\psi_n = \psi_{\lambda_n}$. Then each ψ_n is a contraction on K and hence admits a unique fixed point $x_n \in K$, determined by

$$x_n = \frac{1}{2} - \frac{1}{n}.$$

Since $\psi_n \to \psi = T_1$ uniformly on K as $n \to \infty$, and $x_n \to \frac{1}{2}$, it follows that the fixed points of ψ_n converge to the fixed point of T_1. Consequently, the associated fixed point curve G satisfies $G(0) = \frac{1}{2}$.

Theorem 3.11. *Let X be a uniformly convex Banach space and let K be a non-empty, closed, convex and bounded subset of X . If $T\colon K \to X$ is a weakly inward $b-$ enriched nonexpansive mapping with $b \neq 0$, then T has a fixed point in K.*

Proof. Let $T : K \to X$ be a b-enriched nonexpansive mapping. Then, for all $x, y \in K$, we have

$$\| b(x-y) + T(x) - T(y) \| \leq (b+1)\|x-y\|.$$

Set

$$\lambda = \frac{1}{b+1},$$

and define the averaged operator $T_\lambda : K \to X$ by

$$T_\lambda(x) = (1-\lambda)x + \lambda T(x), \qquad x \in K.$$

For arbitrary $x, y \in K$, we compute

$$\begin{aligned} \|T_\lambda(x) - T_\lambda(y)\| &= \|(1-\lambda)(x-y) + \lambda(T(x) - T(y))\| \\ &= \frac{1}{b+1}\| b(x-y) + (T(x) - T(y)) \| \\ &\leq \frac{1}{b+1}(b+1)\|x-y\| \\ &= \|x-y\|. \end{aligned}$$

Thus, T_λ is nonexpansive.

Since K is convex and T is weakly inward, it follows from the convex combination defining T_λ that T_λ is also weakly inward. Therefore, by Theorem 2.7, the mapping T_λ admits a fixed point in K.

Finally, observe that $T_\lambda(x) = x \iff (1-\lambda)x + \lambda T(x) = x \iff T(x) = x$.

Hence, $\mathrm{Fix}(T_\lambda) = \mathrm{Fix}(T)$, and consequently T has a fixed point in K.

□

Using Theorem 3.11, the following corollary establishes the existence of an approximate fixed point for the averaged operator T_λ.

Corollary 3.12. *Let K be a non-empty closed convex and bounded subset of a uniformly convex Banach space X and let $T\colon K \to K$ be a $b-$ enriched non expansive mapping. Then there exist a sequence $\{x_n\}$ in K such that*

$$\lim_{n\to\infty} \|x_n - T_\lambda(x_n)\| = 0.$$

Proof. Since $\lambda = \dfrac{1}{b+1}$, the averaged operator T_λ is nonexpansive, that is,

$$\|T_\lambda(x) - T_\lambda(y)\| \leq \|x-y\|, \qquad x, y \in K.$$

For a fixed $y \in K$, define $T_n : K \to K$ by

$$T_n(x) = \frac{1}{n}y + \left(1 - \frac{1}{n}\right) T_\lambda(x), \qquad n \in \mathbb{N}.$$

For $x_1, x_2 \in K$, we compute

$$\begin{aligned}\|T_n(x_1) - T_n(x_2)\| &= \left\| \left(1 - \frac{1}{n}\right) \left(T_\lambda(x_1) - T_\lambda(x_2)\right) \right\| \\ &= \left(1 - \frac{1}{n}\right) \|T_\lambda(x_1) - T_\lambda(x_2)\| \\ &\leq \left(1 - \frac{1}{n}\right) \|x_1 - x_2\|.\end{aligned}$$

Hence T_n is a contraction with Lipschitz constant $L_n = 1 - \frac{1}{n} < 1$.
By the Banach Contraction Principle, there exists a unique $x_n \in K$ such that $T_n(x_n) = x_n$.
From the fixed point equation,

$$x_n = \frac{1}{n} y + \left(1 - \frac{1}{n}\right) T_\lambda(x_n).$$

Rearranging,

$$x_n - T_\lambda(x_n) = \frac{1}{n}\big(y - T_\lambda(x_n)\big).$$

Taking norms, $\|x_n - T_\lambda(x_n)\| = \frac{1}{n}\|y - T_\lambda(x_n)\|$.
Since K is bounded (or $\{x_n\}$ remains bounded), $\|y - T_\lambda(x_n)\|$ is bounded independently of n. Therefore,

$$\|x_n - T_\lambda(x_n)\| \leq \frac{C}{n} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus, $\lim_{n \to \infty} \|x_n - T_\lambda(x_n)\| = 0$.

□

4 Conclusion

In this paper, we studied non-self weakly inward (b, θ)-enriched contraction mappings defined on closed convex subsets of Banach spaces. By means of an averaging technique, we proved that the associated averaged operator becomes a contraction while preserving the weak inwardness property. This method allowed us to establish the existence and uniqueness of fixed points for such mappings and to approximate them via the Picard iteration process.

Moreover, we derived sufficient conditions under which the averaged operator acts as a Picard operator on closed balls, thereby providing a constructive framework for fixed point approximation. We also investigated the stability of enriched contractions under addition, scalar multiplication, and convex combinations. These stability properties enabled the construction of parameterized families of weakly inward enriched contractions and led to the introduction of the notion of fixed point curves. In particular, we established the convergence of sequences of fixed points associated with uniformly convergent families of operators, thus furnishing a rigorous foundation for the approximation and continuity of fixed point curves.

Additionally, existence results were obtained for weakly inward enriched nonexpansive mappings in uniformly convex Banach spaces.

The results presented herein extend several classical fixed point theorems and broaden the applicability of fixed point approximation techniques to a wider class of nonlinear operators.

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References

[1] Berinde, V., Păcurar, M.: Approximating Fixed Points of Enriched Contractions in Banach Spaces. Journal of Fixed Point Theory and Applications, Vol-22 (2020) https://doi.org/10.48550/arXiv.1909.02382

[2] R P Aggarwal, Donal O'Regan and Sahu: Fixed point theory of a Lipschitizian-type mappings with applications, Springer, Vol 6, New York, 2009

[3] Riyas P, K T Ravindran, : Fixed Point Curve for Weakly Inward Contractions and Approximate Fixed Point Theory. Analysis in Theory and Applications, Vol-29, No-4 (2013): 342-347

[4] Riyas P, K T Ravindran, : Unique Fixed Point Theorem for Weakly Inward Contractions and Fixed Point Curve in 2-Banach Spaces. Mathematical Journal of Interdisciplinary Sciences, Vol-3, No-2 (2015): 17-182

[5] Riyas P, Shiju George, Harikrishnan P.K., : Fixed Point Curve for Non-expansive mappings in 2-Banach Spaces. Gulf Journal of Mathematics, Vol-20, No-1 (2025): 330-337

[6] Sam B Nadler and K Ushijima, : Fixed Point Curves for Linear interpolations of contraction maps, Appl. Anal, Vol-29(1988)

[7] T Vidali. : Fixed Point Curves generated by nonexpansive mappings, Fixed Point Theory, 6(2)(2005), 333-339

[8] W A Kirk,: Approximate Fixed Points of nonexpansive maps, Fixed Point Theory, 10(2)(2009), 275-288

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