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SOME PROPERTIES OF e-REFLEXIVE RINGS

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Abstract. In this paper, we introduce the concepts of the reflexive property of rings by defining an e-reflexive and strongly e-reflexive ring. We show that e-reflexive rings generalize the concepts of reflexive, symmetric, semiprime, and reversible rings. Although a ring R is e-reflexive does not imply that it is e-reversible for any ring R with $e \in E(R)$, we prove that e-reflexivity is to be an e-reversibility under some additional conditions. We provide examples to prove that e-reflexive rings need not be strongly e-reflexive and vice versa. We also prove that if R is a left APP-ring or left $p.q.$-Baer, then $V_n(R)$ is ε-reflexive. Additionally, we prove that a ring R is e-reflexive if and only if the Dorroh extension D of R by a commutative ring S is e-reflexive. Finally, we discuss various properties related to e-reflexivity.

1 Introduction

Throughout this paper, all rings are assumed to be associative and to possess a multiplicative identity. We denote by $T_2(R)$ the ring of 2×2 upper triangular matrices over R. Moreover, let $E(R)$ denote the set of all idempotents of R, and let $Z(R)$ denote the center of R. Mason introduced the concept of reflexive properties for ideals, which has since been generalized by various authors. Numerous works have been published on this topic, and several related notions have been developed, including idempotent-reflexive right ideals, completely reflexive rings, and weakly reflexive rings (see, for example, [1], [9], [12], [17]). Let I be an ideal of a ring R. In [18], I is defined to be a reflexive ideal if, for all elements $b, d \in R$ the condition $bRd \subseteq I$ implies $dRb \subseteq I$. This notion of reflexive ideals is further applied to the zero ideal of the ring. An ideal $J \subseteq R$ is called semiprime if $dRd \subseteq J$, implies $d \subseteq J$ for every $d \in R$. A ring R is said to be semiprime if its zero ideal is semiprime. It is noteworthy that every semiprime ideal is reflexive, a fact that can be shown by a straightforward computation. Consequently, every ideal in a fully idempotent ring (that is, a ring in which $J^2 = J$ for every ideal J) is reflexive, as shown in [16]. The concept of reflexive rings has been extended to weak reflexivity [18]. A ring R is called weakly reflexive if $drb = 0$, implies $brd \in nil(R)$ for all $d, b \in R$. Furthermore, a ring R is called reversible if $db = 0$ implies $bd = 0$ for all $d, b \in R$, [23]. Rings that contain no nonzero nilpotent elements are called reduced. Commutative rings are symmetric, and symmetric rings are reversible; however, the converses do not hold, as demonstrated in [2, Examples I.5 and II.5] and [10, Examples 5 and 7]. Moreover, although every reversible ring is semicommutative, the converse is not always true, as shown in [11, Lemma 1.4 and Example 1.5]. It is also known that every reversible ring is reflexive. Consequently, there exist reflexive and semicommutative rings that are not symmetric, as noted in [10, Examples 5 and 7].

A strongly reversible ring is defined as a ring in which, for any two polynomials $h(x), k(x) \in R[x]$, the condition $h(x)k(x) = 0$ implies $k(x)h(x) = 0$ [15]. It is important to note that although every reduced ring is strongly reversible, the converse is not necessarily true. The concept of an Armendariz ring was introduced by Rege [19]. A ring R is called Armendariz if for any two polynomials $h(x) = \sum_{i=0}^{n} d_i x^i, k(x) = \sum_{j=0}^{m} b_i x^j \in R[x]$, the condition $h(x)k(x) = 0$ implies $d_i b_j = 0$ for all i and j. A ring is said to be semicommutative if for all $d, b \in R$, the condition $db = 0$ implies that both the left and right annihilators of b and d are zero. Moreover, a ring R is called (strongly) e-symmetric if for all $d, b, c \in R$, the condition $dbc = 0$ implies $dceb = 0$ (equivalently, $dcbe = 0$), as stated in [3]. A ring R is called e-reversible if $db = 0$ implies $bde = 0$ for all $d, b \in R$ [5]. Additionally, R is said to be strongly e-reversible if for all $d, b \in R$, the condition $db = 0$ implies $bed = 0$.

In this paper, we introduce the concept of the reflexive property for rings and, in this direction, define the notions of e-reflexive and strongly e-reflexive rings. Using these properties, we prove that if R is an e-reversible ring, then R is e-reflexive, e-symmetric, and e-semicommutative. Although a ring R is e-reflexive does not imply that it is e-reversible for any ring R with $e \in E(R)$, we prove that e-reflexivity is to be an e-reversibility under some additional conditions. Examples are provided to demonstrate that e-reflexive rings need not be strongly e-reflexive and vice versa. We also show that e-reflexivity does not imply e-symmetry. Moreover, we prove that if R is a left APP-ring or left $p.q.$-Baer, then $V_n(R)$ is ε-reflexive, where $V_n(R)$ denotes the ring of upper triangular matrices over R. Additionally, we prove that a ring R is e-reflexive if and only if the Dorroh extension D of R by a commutative ring S is e-reflexive, and that if R is a von Neumann regular abelian ring, then R is also a strongly e-reflexive ring. Finally, the paper includes several examples and investigates various algebraic properties associated with these concepts.

2 e-reflexive rings

In [3], an idempotent $e \in E(R)$ is called a left minimal idempotent of R if Re is a minimal left ideal. The set of all left minimal idempotents of R is denoted by $ME_l(R)$. An idempotent e is said to be left (respectively, right) semicentral if $de = ede$ (respectively, $ed = ede$) for all $d \in R$. A ring R is called (strongly) left min-abel [4] if either $ME_l(R) = \phi$ or every element $e \in ME_l(R)$ is right (respectively, left) semicentral.

Lemma 2.1. [3, Proposition 2.4]. A ring R is called a left minimal abelian ring if and only if $e = dbe$, then $e = bde$ for each $d, b \in R$ and $e \in ME_l(R)$.

Definition 2.2. A ring R is called e-reflexive if, for all $d, b \in R$, the condition $dRb = 0$ implies $bRde = 0$, where $e \in E(R)$. It is evident that a ring R is reflexive if and only if it is 1-reflexive.

Remark 2.3. Every reflexive is e-reflexive ring for any idempotent element e in the ring; however, the converse does not necessarily hold. For example, consider $R = M_2(\mathbb{R})$, the ring of 2×2 real matrices. Take $A = E_{12}, B = E_{11}, C = E_{21} \in R$. Then $ABC = 0$, while $CBA = E_{22} \neq 0$. Thus, R is not reflexive

This result provides examples of idempotents in reflexive rings

Example 2.4. For any ring R and any idempotent $e \in E(R)$, the following conditions hold:
(1) Every symmetric ring is e-reflexive. Suppose that $dRb = 0$. Since R is a symmetric ring, we have that $dbc = 0$ implies $dcb = 0$ for all $d, b, c \in R$. Taking $c = e$ where e is an idempotent, we obtain $dbe = 0$. By symmetry, this implies $deb = 0$. Hence, $bRde = 0$. Therefore, R is e-reflexive.
(2) Every reversible ring is e-reflexive. This follows from [2, Examples I.5 and II.5] and [10, Examples 5 and 7]. Therefore, the result follows directly from part (1).
(3) Every e-symmetric ring is e-reflexive, but the converse does not hold. To prove the necessary, let R be a ring that is reflexive but not symmetric (for such a ring, see [10, Examples 5 and 7]). Then, $T_2(R)$ is E_{11}-reflexive by Example 2.6, but $T_2(R)$ is not E_{11}-symmetric ring by [7, Proposition 4.1(1)].

A ring R is called e-semicommutative for some $e \in E(R)$ if, for all $d, b \in R, db = 0$ implies $dRbe = 0$.

Proposition 2.5. Every e-reversible ring is both e-reflexive and e-semicommutative ring for any $e \in E(R)$.

Proof. Let R be an e-reversible ring, and suppose $dRb = 0$ for all $d, b \in R$. Then $db = 0$ and $dba = 0$ for all $a \in R$. Since R is e-reversible, it follows that $bade = 0$, which leads to $bRde = 0$. To show that R is e-semicommutative, consider $d, b \in R$ with $db = 0$. By the e-reversibility of R we have $bde = 0$, and thus $bdae = 0$ for all $a \in R$. Again, using the e-reversibility, we find $dabe = 0$. Therefore, $dRbe = 0$. □

The converse of Proposition 2.5 is not valid in general, as shown by the following example.

Example 2.6. Let $R = M_2(\mathbb{Z})$. It is easy to see that R is ϵ-reflexive. Suppose $A = E_{22}$ and $B = (E_{11} + E_{12}) \in R$. Although $AB = 0$, we have $BA\epsilon \neq 0$ for $\epsilon = (E_{21} + E_{22})$. Thus, R is not ϵ-reversible.

In general, the converse of Proposition 2.5 does not hold. However, for Baer rings, we have the following theorem. A ring R is said to be Baer if the left (or right) annihilator of every nonempty subset of R is generated by an idempotent [22].

Theorem 2.7. Let R be a Baer ring and let $e \in E(R)$ be an idempotent. Then the following statements are equivalent;
(1) R is e-reversible;
(2) R is e-reflexive;
(3) R is e-semicommutative.

Proof. $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. This is clear from Proposition 2.5.
$(2) \Rightarrow (1)$. Let $db = 0$ for $d, b \in R$. Then $dbR = 0$, which implies $d \in l_R(bR)$. Since R is a Baer ring, there exists an idempotent $e \in R$ such that $l_R(bR) = eR$. This leads to $eRbR = 0$. Given that R is e-reflexive ring, we have $bReR = 0$ and hence $bd = 0$, as desired.
$(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$. These implications are clear. □

Theorem 2.8. Suppose that R is a ring and $e \in E(R)$ is an idempotent. Then R is an e-reflexive ring if and only if eRe is a reflexive ring and e is left semicentral.

Proof. $(\Rightarrow)$ Let $b \in R$ such that $d = (1-e)be + e$, then $ed = e; de = d; d^2 = d; ede = e; (1-d)de = 0$. Since R is e-reflexive, we have $(1-d)ede = 0$. Hence $(1-d)e = 0$, which implies $e = de = d$. Therefore, $(1-e)be = 0$, and thus $be = ebe$. This shows that e is left semicentral. Now, we show that eRe is reflexive. Let $d, b \in eRe$ such that $drb = 0$ for all $r \in R$. Since eRe is a subring of R and R is e-reflexive, it follows that $drbe = 0$. Because $be = b$, we obtain $brd = 0$. Hence, eRe is a reflexive ring.
$(\Leftarrow)$ Let $d, b \in R$ such that $drb = 0$. Since e is left semicentral, we have $0 = drbe = ederebe$. By the hypothesis that eRe is a reflexive ring, it follows that $brde = eberede = 0$. Therefore, R is e-reflexive. □

Corollary 2.9. Assume that R is e-reflexive and that $\lambda \in E(R)$ is an idempotent. Consequently
(1) $b\lambda e = \lambda be$ for all $b \in R$;
(2) If $\lambda \in eRe$, then for every $b \in R$, $b\lambda = \lambda be$. In particular, this implies that λ is left semicentral in R.

Proof. Since R is an e-reflexive, it follows from Theorem 2.8 that eRe is a reflexive and that e is left semicentral in R. Consequently, we have:
(1) $e\lambda e \in E(eRe)$, which implies that $e\lambda e$ is central in eRe due to the fact that eRe is an Abelian ring. Thus, $b\lambda e = ebee\lambda e = e\lambda eebe = \lambda be$.
(2) Since $\lambda \in E(eRe), \lambda = \lambda e = e\lambda$, by (1), we have $b\lambda = \lambda be$. Consequently, $b\lambda = b\lambda\lambda = \lambda be\lambda = \lambda b\lambda$, therefore, λ is left semicentral in R. □

We now present a characterization of left minimal abelian rings.

Proposition 2.10. The following statements are equivalent for a ring R;
(1) R is a left minimal abelian ring;
(2) R is e-symmetric for every $e \in ME_l(R)$;
(3) R is e-reflexive for every $e \in ME_l(R)$.

Proof. (1) $\Rightarrow$ (2). This follows from [3, Theorem 2.5].
(2) $\Rightarrow$ (3) This implication follows from Example 2.4.
(3) $\Rightarrow$ (1) Let $e \in ME_l(R)$. Since R is e-reflexive, t follows from Theorem 2.8 that e is left semicentral. Therefore, R is a left minimal abelian ring. □

Theorem 2.11. A ring R is left minimal abelian if and only if R is e-reflexive for every $e \in ME_l(R)$.

Proof. ($\Rightarrow$) Let $d, b \in R$ with $dRb = 0$. Suppose that $bRde \neq 0$. Then $Re = RbRde$, and hence $e = xbRde$ for some $x \in R$. Since R is a left min-abel ring, by Lemma 2.1, we obtain $e = dxbRe = Rdxbe$. Consequently, $e = xbRde = xbeRde = xbdxbReRde = xbdexbReRde = xbdRdxbexbReRde = 0$, which is a contradiction. Therefore, $bRde = 0$, and hence R is e-reflexive. ($\Leftarrow$) This follows directly from Proposition 2.10. □

3 Strongly e-reflexive rings

In this section, we investigate properties of rings by introducing the notion of strong e-reflexivity. Let R be a ring and let $e \in E(R)$ be an idempotent.

Definition 3.1. Let R be a ring and let $e \in E(R)$. The ring R is called strongly e-reflexive if, for any $d, b \in R$, the condition $dRb = 0$ implies $bRed = 0$. It is evident that a ring R is reflexive if and only if it is strongly 1-reflexive.

The following example shows that an e-reflexive ring is not necessarily strongly e-reflexive.

Example 3.2. (1) Let R be a reflexive ring. Then, by Proposition 2.5 (2(ii)) in [1], $T_2(R)$ is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$-reflexive. On the other hand, take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in T_2(R)$, then $ABD = 0$. But $DB \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} A \neq 0$. Therefore, $T_2(R)$ is not a strongly e-reflexive.
(2) Every strongly e-symmetric ring is also strongly e-reflexive, where e is an idempotent element in the set of idempotents $E(R)$. Let R be a strongly e-symmetric ring. We want to show that R is strongly e-reflexive. Take any elements $d, b \in R$ such that $dRb = 0$. This means that $dab = 0$ for every $a \in R$. Now, consider $c = e$ (since $e \in E(R)$). We then have $dbe = 0$. Since $dRb = 0$, we can apply the strongly e-symmetric property to the elements d, b, and e. By the definition of strong e-symmetry, from $dbe = 0$, it follows that $(dceb = 0) \Rightarrow dcbe = 0$. In our case, setting $c = e$ gives $d(e)b = 0$ which implies $de \cdot be = 0$, and thus $deb = 0$. Next, we need to show that $bRed = 0$. Indeed, consider $bRed = b(e)d$. Since $b(e)d = 0$, we conclude that $bRed = 0$. Therefore, if $dRb = 0$, then $bRed = 0$, proving that R is strongly e-reflexive. Conversely, consider a ring R (see [10, Examples 5 and 7]) that is strongly 1_R-reversible but not strongly 1_R-symmetric. □

Theorem 3.3. A ring R is strongly e-reflexive if and only if $e \in Z(R)$ and eRe is reflexive.

Proof. ($\Rightarrow$) Assume that R is strongly e-reflexive. For any $d \in R$, set $\phi = e + ed(1-e)$ and $\psi = e + (1-e)de$. Then we have $e\phi = \phi, \phi e = e, \psi e = \psi$ and $e\psi = e$. Since $\phi(1-\phi)e = 0 = (1-e)e\psi$, it follows that $\phi e(1-\phi) = 0 = (1-e)\psi e$, and hence $e(1-\phi) = 0 = (1-e)\psi$. Thus, $\phi = e\phi = e = e\psi = \psi$, which implies $ed(1-e) = 0 = (1-e)de$ for each $d \in R$. Therefore, $e \in Z(R)$. Moreover, since R is e-reflexive. Theorem 2.8 implies that eRe is a reflexive ring. ($\Leftarrow$) Conversely, assume that $e \in Z(R)$ and that eRe is a reflexive ring, Then, by Theorem 2.8, R is e-reflexive. Since e is central, R is strongly e-reflexive. This completes the proof. □

The following result provides a condition under which the class of strongly e-reflexive rings coincides with that of e-reflexive rings.

Proposition 3.4. Let R be a ring with $e \in E(R)$. Then R is strongly e-reflexive and e is a left (right) semicentral idempotent if and only if R is right (left) e-reflexive.

Proof. Suppose R is a strongly e-reflexive and e is left semicentral idempotent. Assume, $drb = 0$ and elements d, b belong to R, so, $bred = 0 = brde$. As a result, R is right e-reflexive. According to Theorem 3.3 and [8, Theorem 3.1], the converse is true. □

Proposition 3.5. Let R be a ring with $\lambda \in E(R)$. Then R is reflexive if and only if R is strongly λ-reflexive and $(1-\lambda)R(1-\lambda)$ is reflexive.

Proof. ($\Rightarrow$) Obviously.
($\Leftarrow$) If R is a strongly λ-reflexive, According to Theorem 3.3, λ is a central idempotent and $\lambda R\lambda$ is reflexive. Consequently, λ is a central idempotent, can get $\lambda R\lambda \ = R/(1-\lambda)R(1-\lambda)$ and $(1-\lambda)R(1-\lambda) \ = R/\lambda R\lambda$. By hypothesis $\lambda R\lambda$ and $(1-\lambda)R(1-\lambda)$ are reflexive, so, $R/(1-\lambda)R(1-\lambda)$ and $R/\lambda R\lambda$ are reflexive. Therefore, $R/((1-\lambda)R(1-\lambda) \cap \lambda R\lambda))$ is a reflexive. But $(1-\lambda)R(1-\lambda) \cap \lambda R\lambda = 0$. For that reason, R is reflexive ring. □

Lemma 3.6. The following statements are equivalent for a ring R with $e \in E(R)$;
(1) R is a reflexive ring;
(2) R is both an e-reflexive and $(1-e)$-reflexive.

Proof. (1) $\Rightarrow$ (2). Obviously. (2) $\Rightarrow$ (1). Suppose two elements d, b in R in which $drb = 0$ for any $r \in R$. Since R is $(1-e)$-reflexive ring, therefore $brd(1-e) = 0$. This implies that $brd = 0$ as R is an e-reflexive ring. Hence, R is a reflexive. □

Corollary 3.7. Let R be a ring and let $e \in E(R)$. Then the following statements are equivalent.
(1) R is a reflexive ring;
(2) R is a strongly e-reflexive ring and $(1-e)R(1-e)$ is a reflexive ring.

Proof. Assume that R is reflexive ring. This implies that R is both an e-reflexive and an abelian, leading to the conclusion that $e \in Z(R)$. According to Theorem 3.3, R is a strongly e-reflexive. As $(1-e)R(1-e)$ is a subring of R, it follows that $(1-e)R(1-e)$ is also a reflexive. Conversely, let us assume R is a strongly e-reflexive and that $(1-e)R(1-e)$ is a reflexive. According to Theorem 3.3, this implies that R is strongly $(1-e)$-reflexive since $(1-e) \in Z(R)$. Hence, R is e-reflexive and $(1-e)$-reflexive. As a result, by Lemma 3.6, R must be a reflexive. □

In [7], a ring R is called a strongly left minimal abelian ring if $ME_l(R) \subseteq Z(R)$.

Corollary 3.8. For a ring R, the following statements are equivalent;
(1) R is a strongly left minimal abelian ring;
(2) R is strongly e-symmetric for any $e \in ME_l(R)$;
(3) R is strongly e-reflexive for any $e \in ME_l(R)$.

Proof. The proof is an immediate consequence of Proposition 2.10. □

Proposition 3.9. *Let* $(R_i)_{i\in J}$ *be rings and let* $(e_i)_{i\in J} \in E(\prod_{i\in J} R_i)$*, where* J *is a finite index set. Then the product ring* $\prod_{i\in J} R_i$ *is strongly* $(e_i)_{i\in J}$*-reflexive if and only if each* R_i *is strongly* $(e_i)_{i\in J}$*-reflexive.*

Proof. Suppose $R = \prod_{i\in J} R_i$ be the direct product of the rings $(R_i)_{i\in J}$ with each R_i being strongly (e_i)-reflexive for all $i \in J$. Denote the projection from R to R_i as $\prod_i$. Let $\mu, \theta \in R$. Let $\mu_i = \prod_i \mu$, $\theta_i = \prod_i \theta$ and $\varphi_i = \prod_i \varphi$ for any $\varphi \in R$. Then $\mu_i, \theta_i \in R_i$. Consider $\mu = (0, 0, \ldots, \mu_i, \ldots, 0, 0), \theta = (0, 0, \ldots, \theta_i, \ldots, 0, 0) \in R = \prod_{i\in J} R_i$ are such that $\mu\varphi\theta = 0$. As R is a strongly $(e_i)_{i\in J}$-reflexive, $\theta\varphi e\mu = 0$ for $e = (e_i)_{i\in J} \in E(\prod_{i\in J} R_i)$. As a result, $\theta_i\varphi_i e_i\mu_i = 0$, Therefore, R_i is a strongly (e_i)-reflexive. In particular, suppose $\mu = (\mu_i)_{i\in J}, \theta = (\theta_i)_{i\in J} \in R$ such that $\mu\theta = 0$. Then we have, $\mu_i\theta_i = 0$ for each $i \in J$. Since R_i is a strongly (e_i)-reflexive, $\theta_i\varphi e_i\mu_i = 0$ for all $i \in J$, $e_i \in E(R_i)$. For this reason, $\theta\varphi e\mu = 0$. Thus, R is strongly e-reflexive. □

A ring R is called von Neumann regular if, for every $d \in R$, there exists $\phi \in R$ such that $d = d\phi d$. The following result provides a condition under which an abelian ring is strongly e-reflexive.

Proposition 3.10. *Let* R *be a von Neumann regular abelian ring. Then, for every* $e \in E(R)$*, the ring* R *is strongly* e*-reflexive.*

Proof. Since an abelian von Neumann regular ring R is reduced, it follows that R is reflexive. Consequently, R is strongly e-reflexive. □

As defined in [3], an idempotent e in a ring R is left (right) semicentral if $de = ede$ (or $ed = ede$) for all $d \in R$.

Proposition 3.11. *Let J be a reduced ideal of a ring R, and let $\overline{e} \in E(R/J)$. If R/J is strongly $\overline{e}$-reflexive, then R is strongly e-reflexive.*

Proof. Suppose $d, b \in R$ and assume that $drb = 0$ for all $r \in R$. Then, in the quotient ring R/J, we have $\overline{d}\,\overline{r}\,\overline{b} = \overline{0}$. Since R/J is strongly $\overline{e}$-reflexive, it follows that $\overline{b}\,\overline{r}\,\overline{e}\,\overline{d} = \overline{0}$. This implies that $bred \in J$. Therefore, we can compute $(bred)^2 = bredbred = brdbred = 0$, since J is reduced. Hence, $bred = 0$. and consequently, R is strongly e-reflexive. □

4 Some extensions of reflexive rings

In this section, we examine various extensions of e-reflexive rings and characterize them from different perspectives. According to [25], a ring R is called quasi-Armendariz if whenever polynomials $h(x) = \sum_{i=0}^{n} d_i x^i$ and $k(x) = \sum_{j=0}^{m} b_i x^j$ are polynomials in $R[x]$ satisfy $h(x)R[x]k(x) = 0$, it follows that $d_i R b_j = 0$ for each i, j. Additionally, as noted in [14], let R be an algebra over a commutative ring S. For an algebra R over a commutative ring S, the Dorroh extension of R by S is defined as the Abelian group $D = R \bigoplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$, where $r_i \in R$ and $s_i \in S$.

Proposition 4.1. Let R be a ring and let $e \in E(R)$. Then R is e-reflexive if and only if the Dorroh extension D of R by a commutative ring S is e-reflexive.

Proof. Since every $s \in S$ can be expressed as $s = s1 \in R$, it follows that $R = \{r + s : (r, s) \in D\}$. Let R be an e-reflexive. If $(r_1, s_1)D(r_2, s_2) = 0$, then for any $(r, s) \in D$, we have $(r_1, s_1)(r, s)(r_2, s_2) = 0$. This leads to the equation $r_1 r r_2 + s_1 r r_2 + s r_1 r_2 + s_2 r_1 r + s_1 s r_2 + s_1 s_2 r + s s_2 r_1 = 0$, along with $s_1 s s_2 = 0$. Therefore, the condition $(r_1, s_1)(r, s)(r_2, s_2) = 0$ is equivalent to $(r_1 + s_1)(r + s)(r_2 + s_2) = 0$ with $s_1 s s_2 = 0$. This implies $(r_1 + s_1)R(r_2 + s_2) = 0$ and $s_2 S s_1 = 0$. Given that R is e-reflexive and S is commutative, we find that $(r_2 + s_2)R(r_1 + s_1)(e, e) = 0$ and $s_2 S s_1 e = 0$. Thus, $(r_2 + s_2)(r + s)(r_1 + s_1)(e, e) = 0$, with $s_2 s s_1 e = 0$, for $e \in E(R)$. This leads to $r_2 r r_1 e + r_2 r s_1 e + r_2 r_1 s e + r r_1 s_2 e + r_2 s s_1 e + r s_2 s_1 e + r_1 s_2 s e = 0$ and $s_2 s s_1 e = 0$. Therefore, $(r_2, s_2)(r, s)(r_1, s_1)(e, e) = 0$, which implies $(r_2, s_2)D(r_1, s_1)(e, e) = 0$. Thus, D is e-reflexive.
Conversely, Conversely, suppose that D is e-reflexive. For any $d, b \in R$ with $dRb = 0$. Then for any $(r, s) \in D$, we have $d(r + s)b = 0$. This results in $(d, 0)(r, s)(b, 0) = 0$ for any $(r, s) \in D$. As D is e-reflexive, we obtain $(b, 0)(r, s)(d, 0)(e, e) = 0$, leading to $b(r + s)de = 0$, thus, R is e-reflexive. □

Corollary 4.2. Let $e \in E(R)$. If S is a multiplicatively closed subset of R consisting of central regular elements, then the localization $S^{-1}R$ is an e-reflexive ring.

Proof. This follows from Proposition 4.1. □

Lemma 4.3. [21, Theorem 10.19] A ring R is semiprime if and only if the polynomial ring $R[x]$ is semiprime.

Proposition 4.4. *Assume that R is a quasi-Armendariz ring. Then R is an e-reflexive ring with $e \in E(R)$ if and only if $R[x]$ is an e-reflexive ring with $e \in E(R[x])$.*

Proof. It suffices to show that $R[x]$ is e-reflexive. Let R be an e-reflexive and $h(x)R[x]k(x) = 0$, where $h(x) = \sum_{i=0}^{n} d_i x^i$ and $k(x) = \sum_{j=0}^{m} b_i x^j$ are polynomials in $R[x]$. As R is a quasi-Armendariz, it follows that $d_i R b_j = 0$ for all i and j. Given that R is e-reflexive, we get $b_j R d_i e = 0$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. As a result, we conclude that $k(x)R[x]h(x)e = 0$, Thus, $R[x]$ is an e-reflexive. □

Corollary 4.5. Assume that R is an Armendariz ring. Then R is an e-reflexive ring with $e \in E(R)$ if and only if $R[x]$ is an e-reflexive ring with $e \in E(R[x])$.

Proof. If R is an Armendariz ring, then it is quasi-Armendariz. Therefore, the result follows from Proposition 4.4. □

Corollary 4.6. Assume that R is a semiprime ring. Then R is an e-reflexive ring with $e \in E(R)$ if and only if $R[x]$ is an e-reflexive ring with $e \in E(R[x])$.

Proof. If R is a semiprime ring, then R is a quasi-Armendariz as shown in [25, Corollary 3.8]. Thus, the conclusion follows from Lemma 4.3 and Proposition 4.4. □

Let R be a ring. It was proved in [1], that R is a reflexive ring if and only if $M_n(R)$ is a reflexive for all $n \geq 1$. We will now examine the following ring:

$$V_n(R) = \left\{ \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ 0 & b_1 & b_2 & \cdots & b_{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & b_2 \\ 0 & 0 & 0 & \cdots & b_1 \end{pmatrix} \mid b_1, b_2, b_3, \ldots, b_n \in R \right\}.$$

The purpose of this note is to show that if R is a left APP-ring, then $V_n(R)$ is e-reflexive. Furthermore, we will show if $V_n(R)$ is an e-reflexive with $e \in E(R)$, then R is also e-reflexive. An ideal I of R is said to be right s-unital if, for every $d \in I$ there exists an element $x \in I$ such that $dx = d$. According to Tominaga [27, Theorem 1], I is right s-unital if and only if, for any finite set of elements $d_1, d_2, \ldots, d_n \in I$, there exists an element $x \in I$ such that $d_i = xd_i$ (resp. $d_i = d_i x$) for each $i = 1, 2, \ldots, n$. As stated in [28], a ring R is called a left APP-ring if the left annihilator $l_R(Rd)$ is right s-unital as an ideal of R for any element $d \in R$. Right APP-rings can be defined in a similar manner. Recall a ring R is a left $p.q.$-Baer ring if the left annihilator of a principal left ideal of R is generated by an idempotent (see, for example, [29], [30] and [31]). Clearly every left $p.q.$-Baer ring is a left APP-ring (thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings). In [32], we prove that if R is a left APP-ring, then R is reflexive if and only if the skew generalized power series ring over R is reflexive. The following results follows from [25] and [33], respectively.

Lemma 4.7. *Every left APP-ring is quasi-Armendariz, but not conversely.*

Lemma 4.8. *Let R be a left APP-ring and $d_1, \ldots, d_n, b_1, \ldots, b_m \in R$. If $d_i R b_j = 0$ for all i and j, then there exists $e \in R$ such that $d_i = d_i e$ and $eRb_j = 0$ for all i and j.*

Theorem 4.9. *For a reduced ring R. If R is a left APP-ring with $e \in E(R)$, then $V_n(R)$ is ε-reflexive, where* $\varepsilon = \begin{pmatrix} e & er_1 & \cdots & er_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ *is an idempotent matrix in $V_n(R)$.*

Proof. Assume that R is left APP and $A_i,\ B_j \in V_n(R)$ such that $A_i V_n(R) B_j = 0$. Let

$$A_i = \begin{pmatrix} d_1^i & d_2^i & d_3^i & \cdots \\ 0 & d_1^i & d_2^i & \cdots \\ 0 & 0 & d_1^i & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, B_j = \begin{pmatrix} b_1^j & b_2^j & b_3^j & \cdots \\ 0 & b_1^j & b_2^j & \cdots \\ 0 & 0 & b_1^j & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Set $f_p = (d_p^i)_{i=1}^{\ell}, g_p = (b_p^j)_{j=1}^{m}$ for any p with $p \geq 1$. Then from $A_i V_n(R) B_j = 0$ it follows that for any $\lambda_p = (c_p^k)_{k=1}^{h}$ with $p \geq 1$, we have

$$\begin{pmatrix} f_1 & f_2 & f_3 & \cdots \\ 0 & f_1 & f_2 & \cdots \\ 0 & 0 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots \\ 0 & \lambda_1 & \lambda_2 & \cdots \\ 0 & 0 & \lambda_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g_1 & g_2 & g_3 & \cdots \\ 0 & g_1 & g_2 & \cdots \\ 0 & 0 & g_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

Note that $d_i c_k b_j = 0$ for all i, j and k. Since $f\lambda g = 0$, we have the following equations:

$$d_1c_1b_1 = 0 \tag{1}$$

$$d_1c_1b_2 + d_1c_2b_1 + d_2c_1b_1 = 0 \tag{2}$$

$$d_1c_1b_3 + d_1c_2b_2 + d_1c_3b_1 + d_2c_1b_2 + d_2c_2b_1 + d_3c_1b_1 = 0 \tag{3}$$

$$\vdots$$

$$d_1c_1b_m + d_1c_2b_m + \cdots + d_1c_{m+1}b_1 + \cdots + d_mc_1b_2 + d_mc_2b_1 + d_{m+1}c_1b_1 = 0 \tag{4}$$

$$\vdots$$

$$d_1c_1b_{n-1} + d_1c_2b_{n-2} + \cdots + d_{n-2}c_2b_1 + d_{n-1}c_1b_1 = 0 \tag{5}$$

$$d_1c_1b_n + d_1c_2b_{n-1} + \cdots + d_{n-1}c_1b_2 + d_{n-1}c_2b_1 + d_nc_1b_1 = 0 \tag{6}$$

where $1 \leq m \leq n$. It is important to note that R is reflexive, $dRbRb = 0$ if and only if $dRb = 0$ for $d, b \in R$. We will using these facts freely in the following calculations. From Equation (1), we find that $d_1Rb_1 = 0$. Consequently, by Lemma 4.8, there exists an element $e \in R$ with $d^i = d^ie$ and $eRb^j = 0$ for all indices i, j. This implies that $f = fe$ and $eRg = 0$. Thus, for $j = 2$, we have $g_j \in r_R(aR)$ where $a \in R$ is any arbitrary element. By assumption, $r_R(aR)$ is s-unital, and again applying Lemma 4.8, we find an element $e \in r_R(aR)$ with $g_j = eg_j$, for $j = 2$. As $aRe = 0$, we conclude that $f_1Reg_1 = 0$, which leads to $f_1Rg_1 = 0$. Multiplying Equation (2) by Rb_1 on the right, we obtain $d_2Rb_1Rb_1 = 0$ resulting in $d_2Rb_1 = 0$. Then Equation (2) implies $d_1c_1b_2 = 0$. Substitute et for c_1 in $d_1c_1b_2 = 0$ to yield $d_1(et)b_2 = 0$, for t is an arbitrary element of R, then we have $d_1Rb_2 = 0$. Thus by Lemma 4.8 again, there exist $y \in R$ such that $d^i = d^iy$ and $yRb^j = 0$ for all i and j. Hence $f = fy$ and $yRg = 0, yRg_2 = 0$. Thus $f_1Rg_2 = 0$ and so $f_2Rg_1 = 0$. Now Equation (3) becomes

$$d_1c_1b_3 + d_2c_1b_2 + d_3c_1b_1 = 0$$

Multiply this equality on the right by Rb_1 and Rb_2 in turn, to obtain $d_3Rb_1 = 0, d_2Rb_2 = 0$ and $d_1Rb_3 = 0$. By Lemma 4.8, there exist $z \in R$ such that $d^i = d^iz$ and $zRb^j = 0$ leading to $f = fz$ and $zRg = 0$. Consequently, we find that $f_3Rg_1 = 0$. Applying Lemma 4.8 once more, we identify $x \in R$ such that $d^i = d^ix$ and $xRb^j = 0$. Since $b^j \in r_R(xR)$ is s-unital, we have $f = fx$ and $xRg = 0$. This results in $f_2Rg_2 = 0$ and $f_1Rg_3 = 0$. In summarizing, we have that

$$d_iRb_j = 0 \qquad for \qquad i + j = 2, 3, 4.$$

Assuming inductively that $d_iRb_j = 0$ for $i + j = 2, 3, \ldots m$ for $m - 1 \leq n$, we can rewrite Equation (4) as

$$d_1c_1b_{m-1} + d_2c_1b_m + d_2c_1b_{m-1} + \cdots + d_mc_1b_2 + d_{m-1}c_1b_1 = 0 \tag{7}$$

Next, multiplying Equation (7) on the right by $Rb_1, Rb_2, \ldots,$ and Rb_m sequentially gives us $d_{m-1}Rb_1 = 0, d_mRb_2 = 0, \ldots,$ and $d_2Rb_m = 0$, which implies $d_1Rb_{m-1} = 0$. These findings indicate that $d_iRb_j = 0$ for all i and j with $i + j = m - 1$. Therefore, we conclude that $d_iRb_j = 0$ for all i and j with $1 \leq i + k \leq n$. Given that R is e-reflexive, we have $b_jRd_ie = 0$. Thus, there exists an arbitrary element $h \in R$ such that $d^i = d^ih$ and $hRb^j = 0$ for all i and j. leading to $b^j \in r_R(hR)$. By hypothesis, $r_R(hR)$ is left s-unital, and applying Lemma 4.8, again we find $f_p = f_ph$ and $hRg_p = 0$. This means $g_p \in r_R(hR)$ for $p = 1, 2, \ldots$ is left s-unital. According to the induction hypothesis, we have $g_1Rf_1\epsilon = 0, g_1Rf_2\epsilon = 0, g_2Rf_1\epsilon = 0, \ldots, g_1Rf_n\epsilon = 0, \ldots, g_nRf_1\epsilon = 0$. This leads to $g\lambda f\epsilon = 0$, which establishes that $V_n(R)$ is ϵ-reflexive. □

Proposition 4.10. *If $V_n(R)$ is ε-reflexive where ε is an idempotent matrix, then R is e-reflexive with $e \in E(R)$.*

Proof. Assume that d, b are in R with $dRb = 0$. Let $A = dE_{11}, B = bE_{11} \in V_n(R)$, so $ACB = 0$ for any $C = cE_{11} \in V_n(R)$ thus, we conclude that $BCA\epsilon = 0$, as $V_n(R)$ is ε-reflexive. Hence, $bcde = 0$. Therefore, R is e-reflexive. □

Corollary 4.11. *Let R be a ring with $e \in E(R)$. If R is a left $p.q.$-Baer ring, then $V_n(R)$ is ε-reflexive.*

Proof. If R is a left $p.q.$-Baer ring, then according to [30], R is a left APP-ring. Therefore, the conclusion follows from Theorem 4.9. □

Corollary 4.12. *Let R be a ring with $e \in E(R)$. If R is a semiprime ring, then $V_n(R)$ is ε-reflexive.*

Proof. If R is semiprime, then it is a left $p.q.$-Baer ring, according to [30]. Consequently, the result follows from Corollary 4.11. □

References

[1] K. K. Tai., Yang. L., *Reflexive Property of Rings,* Comm. Algebra, **40**(4)(2012), 1576–1594.

[2] D. Anderson, Camillo. V., *Semigroups and rings whose zero products commute.* Comm. Algebra, **27**(6) (1999), 2847–2852.

[3] M. Fanyun, Junchao. W., *e-Symmetric rings.* Comm. Contemp. Math, (2018), 1750039, 1–8.

[4] J. C. Wei., *Certain rings whose simple singular modules are nil-injective,* Turkish J. Math. **32** (4) (2008), 393–408.

[5] K. C. Avanish., Rohit K. Verma., *On e-Reversible Rings,* Palestine Journal of Mathematics, **11**(1) (2022), 217–225

[6] S. C. Uday., *On some classes of reflexive rings,* Asian-European Journal of Mathematics, **8**(1) (2015), 1550003.

[7] M. Fanyun,., Junchao W., *Some properties of e-symmetric rings.* Turkish J. Math, **42** (2018), 2389–2399.

[8] K. K. Tai., Yang Lee., *Reflexive property on idempotent,* Bull. Korean Math. Soc. **50**(6) (2013), 1957–1972.

[9] E. Ali., *Nilpotent elements and nil-reflexive property of generalized power series.* Adv. Pure Math., **12** (2022), 676–692.

[10] G. Marks., *Reversible and symmetric rings.* J. Pure and Appl. Algebra **174** (2002), 311–318.

[11] K. Kim., Lee Y., *Extensions of reversible rings.* J. Pure Appl. Algebra, **185**(2003), 207–223.

[12] E. Ali., A. Elshokry., *On a generalization of Quasi-Armendariz Rings,* Palestine Journal of Mathematics, **13**(1) (2024), 142–150.

[13] Kelarev V., *Ring constructions and applications.* (World Scientific Publishing Co. Pte. Ltd., Singapore. 2002.

[14] Dorroh J. L., *Concerning adjunctions to algebras,* Bull. Amer. Math. Soc. **38**(2) (1932), 85–88.

[15] G. Yang, Zhong. Liu., *On strongly reversible rings.* Taiwanese J. Math, **12**(1) (2008), 129–136.

[16] R. C. Courter, *Rings all of whose factor rings are semiprime.* Canad. Math. Bull. **12**(4) (1969), 417–426.

[17] G. Mason, *Reflexive ideals,* Comm. Algebra, **9**(17)(1981), 1709-1724.

[18] Zhong. Liu, X. Zhu Q. Gu., *Reflexive rings and their extensions,* Math. Slovaca, **63**(3) (2013), 417–430.

[19] M. B. Rege., S. Chhawchharia, *Armendariz rings,* Proc. Japan Acad. Ser. A Math. Sci., **73**(1) (1997), 14–17.

[20] P. Pollingher, A. Zaks., *On Baer and quasi-Baer rings,* Duke Math. J, **37** (1970), 127–138.

[21] T. Y. Lam., *A First Course in Noncommutative Rings,* Springer-Verlag, NewYork, 1991.

[22] I. Kaplansky., *Rings of Operators*. Mathematics Lecture Note Series. W. A. Benjamin, New York, 1968.

[23] P. M. Cohn., *Reversible rings,* Bull. London Math. Soc., **31**(6) (1999), 641-648.

[24] W. E. Clark., *Twisted matrix units semigroup algebras,* Duke Math. J, **34** (1967), 417–424.

[25] Hirano, Y., *On annihilator ideals of a polynomial ring over a noncommutative ring*, J. Pure Appl. Algebra **168** (2002), 45–52.

[26] Muhittin B., Tai K. K, *Quasi-Armendariz Property for skew polynomial rinds*, Commun. Kore. Math. Soc. **26**(4) (2011), 557–573.

[27] H. Tominaga., *On s-unital rings,* Math. J. Okayama Univ, **18** (1976), 117-134.

[28] Zhan. Renyu., *A generalization of PP-rings and p.q.-Baer rings*, Glasgow Math. J. **48** (2006), 217–229.

[29] Birkenmeier, G.F., Kim, J.Y., Park, J.K., *On polynomial extensions of principally quasi-Baer rings,* Kyungpook Mathematical J. **40** (2000), 247–254.

[30] Birkenmeier, G.F., Kim, J.Y. and Park, J.K., *Principally quasi-Baer rings,* Comm. Algebra **29** (2001), 639–660.

[31] Zhong. Liu., *A note on principally quasi-Baer rings*, Comm. Algebra **30**(2002), pp. 3885–3890.

[32] E. Ali., A. Elshokry., *A note on (S, ω)-quasi-Armendariz rings,* Palestine Journal of Mathematics, **12**(4) (2023), 452–464.

[33] Zhong. Liu., Zhang WenHui, *A note on Quasi-Armendariz ring,* Math. J. Okayama Univ. **52** (2010), 89–95.

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