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# On Cayley Graphs of Semigroup of Binary Relations

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**Abstract**. Let $Cay(\mathscr{B}_X, G)$ be the Cayley graph associated with the binary relation semigroup $\mathscr{B}_X$ on a finite set $X$ with group elements $G$ of $\mathscr{B}_X$. In this paper, we prove $Cay(\mathscr{B}_X, G)$ is the disjoint union of the induced subgraph $\Gamma(\mathcal{L}_G, G)$ of $Cay(\mathscr{B}_X, G)$, where $\mathcal{L}_G$ varies over the set of all $\mathcal{L}_G$-classes of $\mathscr{B}_X$. Finally, we describe the necessary and sufficient condition for which two elements are adjacent in the Cayley graphs of $\mathscr{B}_X$ relative to Green's equivalence $\mathcal{L}$-Class and $\mathcal{R}$-Class.

## 1 Introduction

A binary relation on set $X$ is a subset of $X \times X$, and $\mathscr{B}_X$ denotes the set of all binary relations on $X$. The product of two relations $\alpha\beta$ on $X$ is defined to be the relation

$\alpha\beta = \{(a, b) \mid (a, c) \in \alpha \text{ and } (c, b) \in \beta \text{ for some } c \in X\}$.

The operation is associative, and hence $\mathscr{B}_X$ is a semigroup. For a finite set $S$ and a non-empty subset $H$ of $S$, the Cayley graph of $S$ with respect to $H$, denoted by $Cay(S, H)$, is defined using left multiplication to be the directed graph with vertex set $S$ and arc set $\{(x, y) \mid hx = y \text{ for some } h \in H\}$. In 1878 Arthur Cayley introduced the Cayley graphs of groups, and the Cayley graphs of semigroups are its generalizations. In 1964, Bosak [2] studied certain graphs over semigroups. Cayley graphs of semigroups are extensively studied since they reflect the structure of semigroup. Numerous studies have been carried out on Cayley graphs of particular semigroups, such as rectangular group, 0-simple semigroup, see for example [3], [5], [6], [8], [13], [14]. Some related studies on structural properties of semigroups can be found in [1], [7], [12].

Green's relations in the semigroup of binary relation $\mathscr{B}_X$ on a set $X$ have been studied primarily in lattice considerations [15]. In [9], binary relations are interpreted as boolean matrices and obtain the characterizations of Green's relations. With the help of these characterizations of $\mathcal{L}$ and $\mathcal{R}$ -classes we obtain the necessary and sufficient conditions for the adjacency of two elements in the Cayley graph of $\mathscr{B}_X$ on a finite set $X$.

$X$ is considered as a finite set throughout this paper, and $\Gamma(\mathcal{L}_G, G)$ denotes the induced subgraph with vertex set $\mathcal{L}_G$ of $Cay(\mathscr{B}_X, G)$.

## 2 Preliminaries

This section describes some basic definitions and results of Semigroup theory and Cayley graphs needed in the sequel.

**Definition 2.1.** [4] Let $S$ be a semigroup. We define $a\mathcal{L}b$ $(a, b \in S)$ if and only if $a$ and $b$ generate the same principal left ideal, that is, if and only if $S^1a = S^1b$. Similarly we define $a\mathcal{R}b$ if and only if $a$ and $b$ generate the same principal right ideal, that is, if and only if $aS^1 = bS^1$. We define $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$.

**Lemma 2.2.** *Let $a$, $b$ be elements of a semigroup $S$. Then $a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$, $yb = a$ and $a\mathcal{R}b$ if and only if there exist $u, v \in S^1$ such that $au = b$, $bv = a$.*

**Notation 2.3.** The $\mathcal{L} - class(\mathcal{R} - class, \mathcal{H} - class)$ containing an element $a$ in a semigroup $S$ will be written as $L_a(R_a, H_a)$.

**Definition 2.4.** [10] Let $G$ be the group of units of a semigroup $S$. Then the equivalence relations $\mathcal{R}_G$, $\mathcal{L}_G$ and $\mathcal{H}_G$ are defined on $S$ as

(i) $x\mathcal{R}_G y$ if and only if $x = yu$ for some $u \in G$

(ii) $x\mathcal{L}_G y$ if and only if $x = uy$ for some $u \in G$

(iii) $x\mathcal{H}_G y$ if and only if $x\mathcal{R}_G y$ and $x\mathcal{L}_G y$.

Evidently $\mathcal{R}_G \subseteq \mathcal{R}$, $\mathcal{L}_G \subseteq \mathcal{L}$ and $\mathcal{H}_G \subseteq \mathcal{H}$, where $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$ are Green's equivalences on $S$.

**Definition 2.5.** A partial transformation semigroup $\mathscr{P}_X$ is the collection of functions from a subset of $X$ into $X$ with composition of transformations as semigroup operation.

**Definition 2.6.** A full transformation semigroup $\mathscr{T}_X$ on a set $X$ is the set of all mappings of a set $X$ onto itself with composition of transformations as semigroup operation.

**Definition 2.7.** The set of all partial one-one mapping on a set $X$ is an inverse semigroup with the composition of mappings as semigroup operation and denote it by $\mathscr{I}_X$, in symmetric inverse semigroup of $X$.

**Notation 2.8.** [9] Let $x \in X, H \subseteq X$ and $\alpha \in \mathscr{B}_X$. Then

(i) $\alpha^{-1} = \{(a, b) \mid (b, a) \in \alpha\}$

(ii) $x\alpha = \{y \in X \mid (x, y) \in \alpha\}$

(iii) $H\alpha = \{b \mid (h, b) \in \alpha$ for some $h \in H\}$

(iv) $\alpha H = H\alpha^{-1}$.

**Lemma 2.9.** *Let $\alpha, \beta \in \mathscr{B}_X$ and $H \subseteq X$. Then*

*(i)* $H(\alpha\beta) = (H\alpha)\beta$ *and* $(\alpha\beta)H = \alpha(\beta H)$

*(ii)* $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$

*(iii)* $\alpha\mathcal{R}\beta$ *if and only if* $\alpha^{-1}\mathcal{L}\beta^{-1}$.

**Definition 2.10.** [15] Let $\alpha \in \mathscr{B}_X$then $V(\alpha) = \{A\alpha \mid A \subseteq X\}$, the set of all possible sums of rows $\alpha$ including the empty sum, which is zero.

**Lemma 2.11.** *Let $\alpha, \beta \in \mathscr{B}_X$. Then $\alpha\mathcal{L}\beta$ if and only if $V(\alpha) = V(\beta)$.*

**Lemma 2.12.** *Let $\alpha, \beta \in \mathscr{B}_X$. Then $\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$ and $\alpha\mathcal{R}\beta$ if and only if $\alpha X = \beta X$.*

**Lemma 2.13.** *In the symmetric inverse semigroup $\mathscr{I}_X$, $\alpha\mathcal{L}\beta$ if and only if $ran(\alpha) = ran(\beta)$ and $\alpha\mathcal{R}\beta$ if and only if $dom(\alpha) = dom(\beta)$.*

**Definition 2.14.** A directed graph or a digraph $D$ is a pair $(V, A)$, where $V$ is a non-empty set whose elements are called the vertices of $D$ and $A$ is a subset $V^{(2)}$ (the set of ordered pairs of distinct elements of $V$), whose elements are called the directed edges or arcs of $D$.

**Definition 2.15.** A sub digraph $H = (U, B)$ of a digraph $D = (V, A)$ is said to be vertex induced subgraph or induced subgraph if $B$ consists of all the arcs of $D$ joining pairs of vertices of $U$. A decomposition of a graph $D = (V, A)$ is a set of subgraphs $H_1, H_2, \ldots, H_K$ that partition the arcs of $D$. A sub digraph $H = (U, B)$ of a digraph $D = (V, A)$ is said to be spanning sub digraph of $D$ if $V(H) = V(D)$.

**Definition 2.16.** [11] Let $S$ be a finite semigroup and let $H$ be a non-empty subset of $S$. The Cayley graph $Cay(S,H)$ of $S$ with respect to $H$ is defined, using left multiplication, to be the directed graph with vertex set $S$ and arc set $\{(x,y) \mid hx = y \text{ for some } h \in H\}$.

**Example 2.17.** Let $X = \{1,2\}$. The binary relations on $X$ are $\alpha_1 = \{(1,1)\}$, $\alpha_2 = \{(1,2)\}$, $\alpha_3 = \{(2,1)\}$, $\alpha_4 = \{(2,2)\}$, $\alpha_5 = \{(1,1),(1,2)\}$, $\alpha_6 = \{(1,1),(2,1)\}$, $\alpha_7 = \{(1,1),(2,2)\}$, $\alpha_8 = \{(1,2),(2,1)\}$, $\alpha_9 = \{(1,2),(2,2)\}$, $\alpha_{10} = \{(2,1),(2,2)\}$, $\alpha_{11} = \{(1,1),(1,2),(2,1)\}$, $\alpha_{12} = \{(1,1),(1,2),(2,2)\}$, $\alpha_{13} = \{(1,2),(2,1),(2,2)\}$, $\alpha_{14} = \{(1,1),(2,1),(2,2)\}$, $\alpha_{15} = \{(1,1),(1,2),(2,1),(2,2)\}$, $\alpha_{16} = \phi$.
Then $\mathscr{B}_X = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_9,\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{14},\alpha_{15},\alpha_{16}\}$.
Also $V(\alpha_1) = \{\{1\}\}, V(\alpha_2) = \{\{2\}\}, V(\alpha_3) = \{\{1\}\}, V(\alpha_4) = \{\{2\}\}, V(\alpha_5) = \{\{1,2\}\}$, $V(\alpha_6) = \{\{1\}\}, V(\alpha_7) = \{\{1,2\}\}, V(\alpha_8) = \{\{1,2\}\}, V(\alpha_9) = \{\{2\}\}, V(\alpha_{10}) = \{\{1,2\}\}$, $V(\alpha_{11}) = \{\{1,2\}\}, V(\alpha_{12}) = \{\{1,2\}\}, V(\alpha_{13}) = \{\{1,2\}\}, V(\alpha_{14}) = \{\{1,2\}\}$, $V(\alpha_{15}) = \{\{1,2\}\}, V(\alpha_{16}) = \phi$
The group of units of $\mathscr{B}_X$ is $G = \{\alpha_7,\alpha_8\}$. The $\mathcal{L}$-classes of $\mathscr{B}_X$ are $\mathcal{L}_1 = \{\alpha_1,\alpha_3,\alpha_6\}$, $\mathcal{L}_2 = \{\alpha_2,\alpha_4,\alpha_9\}$, $\mathcal{L}_3 = \{\alpha_5,\alpha_{10},\alpha_{15}\}$, $\mathcal{L}_4 = \{\alpha_{11},\alpha_{14}\}$, $\mathcal{L}_5 = \{\alpha_{12},\alpha_{13}\}$, $\mathcal{L}_6 = \phi$.
The $\mathcal{R}$-classes of $\mathscr{B}_X$ are $\mathcal{R}_1 = \{\alpha_1,\alpha_2,\alpha_5\}$, $\mathcal{R}_2 = \{\alpha_3,\alpha_4,\alpha_{10}\}$, $\mathcal{R}_3 = \{\alpha_6,\alpha_9,\alpha_{15}\}$, $\mathcal{R}_4 = \{\alpha_{11},\alpha_{12}\}$, $\mathcal{R}_5 = \{\alpha_{13},\alpha_{14}\}$, $\mathcal{R}_6 = \phi$.
The Cayley graphs of $\mathscr{B}_X$ with respect to $G$, $L_1$ and $R_2$ using left multiplication are shown in Figure 1, Figure 2 and Figure 3 respectively.

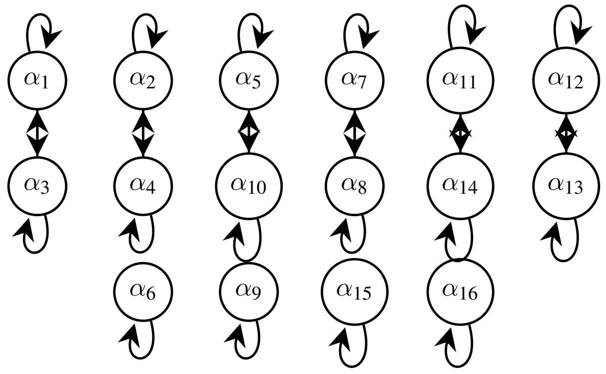

**Figure 1.** $Cay(\mathscr{B}_X, G)$

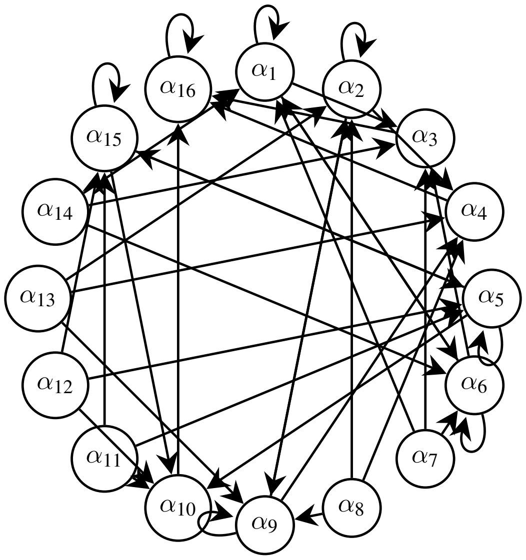

**Figure 2.** $Cay(\mathscr{B}_X, L_1)$

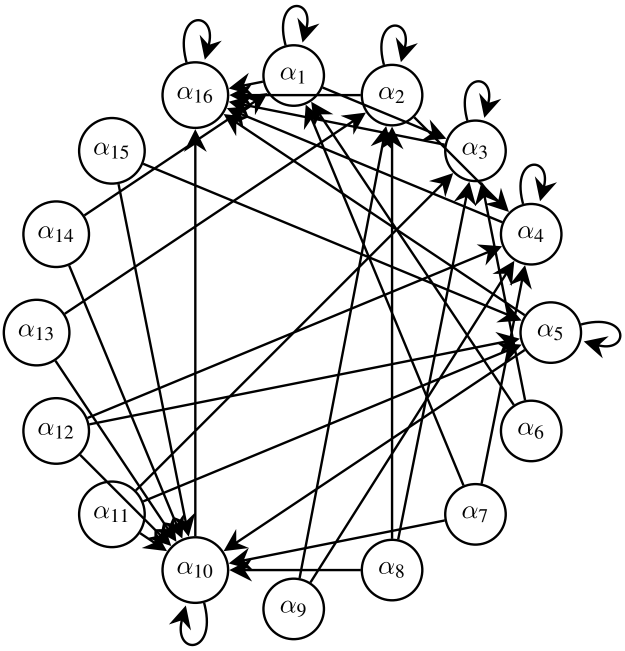

**Figure 3.** $Cay(\mathscr{B}_X, R_2)$

## 3 Results

In this section, we first prove that the Cayley graph $Cay(\mathscr{B}_X, G)$ is the disjoint union of the induced subgraphs $\Gamma(\mathcal{L}_G, G)$, where $\mathcal{L}_G$ varies over the set of all $\mathcal{L}_G$-classes of $\mathscr{B}_X$. We then discuss the necessary and sufficient condition for which two elements are adjacent in the Cayley graphs $Cay(\mathscr{B}_X, \mathcal{L})$, $Cay(\mathscr{B}_X, \mathcal{R})$.

Here, $\Gamma(\mathcal{L}_G, G)$ denotes the induced subgraph of $Cay(\mathscr{B}_X, G)$ with vertex set $\mathcal{L}_G$. Throughout the paper, the Cayley graph $Cay(\mathscr{B}_X, G)$ is defined using left multiplication.

**Proposition 3.1.** *Let $\mathscr{B}_X$ be the binary relation semigroup on a finite set $X$ with group of units $G$. Then for $\alpha, \beta \in \mathscr{B}_X$ with $\alpha \neq \beta$, there exist arcs $(\alpha, \beta)$ and $(\beta, \alpha)$ in the Cayley graph $Cay(\mathscr{B}_X, G)$ if and only if $\alpha \mathcal{L}_G \beta$.*

*Proof.* Let $\alpha, \beta \in \mathscr{B}_X$ with $\alpha \neq \beta$. Suppose that there exist arcs $(\alpha, \beta)$ and $(\beta, \alpha)$ in the Cayley graph $Cay(\mathscr{B}_X, G)$. Since the Cayley graph is defined using left multiplication, there exist elements $\gamma_1, \gamma_2 \in G$ such that $\beta = \gamma_1\alpha$ and $\alpha = \gamma_2\beta$. Substituting $\beta = \gamma_1\alpha$ into the second equation yields $\alpha = \gamma_2(\gamma_1\alpha) = (\gamma_2\gamma_1)\alpha$. As $\alpha \neq \beta$, this implies that $\gamma_2\gamma_1$ acts as the identity on $\alpha$, and since $G$ is the group of units of $\mathscr{B}_X$, it follows that $\gamma_2\gamma_1 = e$, the identity of $G$. Hence $\gamma_2 = \gamma_1^{-1} \in G$, and therefore $\alpha \mathcal{L}_G \beta$.

Conversely, assume that $\alpha \mathcal{L}_G \beta$ with $\alpha \neq \beta$. By the definition of the $\mathcal{L}_G$-relation, there exists an element $\gamma \in G$ such that $\alpha = \gamma\beta$. Since $G$ is a group, the inverse $\gamma^{-1}$ also belongs to $G$ and satisfies $\beta = \gamma^{-1}\alpha$. Left multiplication by $\gamma$ therefore yields the arc $(\beta, \alpha)$, while left multiplication by $\gamma^{-1}$ yields the arc $(\alpha, \beta)$ in $Cay(\mathscr{B}_X, G)$. □

**Proposition 3.2.** *Let $\mathscr{B}_X$ be the binary relation semigroup on a finite set $X$ with group of units $G$. If $\mathcal{L}_G$ be any $\mathcal{L}_G$-class of $\mathscr{B}_X$, then $\Gamma(\mathcal{L}_G, G)$ is a complete directed graph.*

*Proof.* Let $\alpha, \beta \in \mathcal{L}_G$ with $\alpha \neq \beta$. Then $\alpha \mathcal{L}_G \beta$. By Proposition 3.1, there exist arcs $(\alpha, \beta)$ and $(\beta, \alpha)$ in $Cay(\mathscr{B}_X, G)$. Consequently, there is an arc between $\alpha$ and $\beta$ in the induced subgraph $\Gamma(\mathcal{L}_G, G)$, and therefore $\Gamma(\mathcal{L}_G, G)$ is a complete directed graph. □

**Proposition 3.3.** *Let $\mathscr{B}_X$ be the binary relation semigroup on a finite set $X$ with group of units $G$. Then $Cay(\mathscr{B}_X, G)$ is the disjoint union of $\Gamma(\mathcal{L}_G, G)$, where $\mathcal{L}_G$ varies over the set of all $\mathcal{L}_G$-classes of $\mathscr{B}_X$.*

*Proof.* Let $\mathcal{L}_G$ and $\mathcal{L}'_G$ are two $\mathcal{L}_G$ classes. Since $\mathcal{L}_G$ classes are equivalence classes they are mutually disjoint. Then $\Gamma(\mathcal{L}_G, G)$ and $\Gamma(\mathcal{L}'_G, G)$ are disjoint whenever $\mathcal{L}_G$ and $\mathcal{L}'_G$ are disjoint. Now, $V(Cay(\mathscr{B}_X, G)) = \mathscr{B}_X = \cup\mathscr{L}_G = \cup(V(\Gamma(\mathcal{L}_G, G))) = V(\cup(\Gamma(\mathcal{L}_G, G)))$.

Further let $\alpha, \beta \in \mathscr{B}_X$ with $\alpha \neq \beta$. Suppose that there is an arc from $\alpha$ to $\beta$ in $Cay(\mathscr{B}_X, G)$. Then by Definition 2.16 there exists $\gamma \in G$ such that $\beta = \gamma\alpha$. Then by Definition 2.4 , we have $\alpha\mathcal{L}_G\beta$. By Proposition 3.1, there exist arcs $(\alpha, \beta)$ and $(\beta, \alpha)$ in the induced subgraph $\Gamma(\mathcal{L}_G, G)$. Hence there is an arc from $\alpha$ to $\beta$ in the disjoint union of $\Gamma(\mathcal{L}_G, G)$. Therefore $A(Cay(\mathscr{B}_X, G)) \subseteq A(\cup Cay(\mathcal{L}_G, G))$.

On the other hand suppose that there is an arc $(\alpha, \beta)$ in the disjoint union of $\Gamma(\mathcal{L}_G, G)$. Then there exists an $\mathcal{L}_G$-class $\mathcal{L}'_G$ such that the arc from $\alpha$ to $\beta$ lies in the induced subgraph $\Gamma(\mathcal{L}'_G, G)$ and thus, $\alpha\mathcal{L}_G\beta$. By Proposition 3.1, there exist arcs $(\alpha, \beta)$ and $(\beta, \alpha)$ in $Cay(\mathscr{B}_X, G)$. It follows that $A(\cup\Gamma(\mathcal{L}_G, G)) \subseteq A(Cay(\mathscr{B}_X, G))$.

Therefore, $A(Cay(\mathscr{B}_X, G)) = A(\cup\Gamma(\mathcal{L}_G, G))$, and hence $Cay(\mathscr{B}_X, G)$ is the disjoint union of $\Gamma(\mathcal{L}_G, G)$, where $\mathcal{L}_G$ ranges over the set of all $\mathcal{L}_G$-classes of $\mathscr{B}_X$. □

Since $\mathscr{P}_X$ and $\mathscr{T}_X$ are subsemigroups of $\mathscr{B}_X$, we have the following corollaries.

**Corollary 3.4.** *Let $\mathscr{P}_X$ be the partial transformation semigroup on a finite set $X$ with group of units $G$. Then $Cay(\mathscr{P}_X, G)$ is the disjoint union of $\Gamma(\mathcal{L}_G, G)$, where $\mathcal{L}_G$ varies over the set of all $\mathcal{L}_G$-classes of $\mathscr{P}_X$.*

**Corollary 3.5.** *Let $\mathscr{T}_X$ be the full transformation semigroup on a finite set $X$ with group elements $G$. Then $Cay(\mathscr{T}_X, G)$ is the disjoint union of $\Gamma(\mathcal{L}_G, G)$, where $\mathcal{L}_G$ varies over the set of all $\mathcal{L}_G$-classes of $\mathscr{T}_X$.*

**Corollary 3.6.** *Let $\mathscr{I}_X$ be the symmetric inverse semigroup on a finite set $X$ with group of units $G$. Then $Cay(\mathscr{I}_X, G)$ is the disjoint union of induced subgraphs $\Gamma(\mathcal{L}_i, G)$, where $\mathcal{L}_i$ runs over all $\mathcal{L}$ classes of $\mathscr{I}_X$.*

*Proof.* We first show that in the symmetric inverse semigroup $\mathscr{I}_X$, $\mathcal{L} = \mathcal{L}_G$. For that, let $\alpha, \beta \in \mathscr{I}_X$. By Lemma 2.13, for any $\alpha, \beta \in \mathscr{I}_X$, one has $\alpha\,\mathcal{L}\,\beta$ if and only if $\text{ran}(\alpha) = \text{ran}(\beta)$. Suppose first that $\alpha\,\mathcal{L}_G\,\beta$. Then there exists $\gamma \in G$ such that $\beta = \gamma\alpha$. Since $\gamma$ is a permutation of $X$, we obtain $\text{ran}(\beta) = \text{ran}(\gamma\alpha) = \gamma(\text{ran}(\alpha)) = \text{ran}(\alpha)$, and hence $\alpha\,\mathcal{L}\,\beta$.

Conversely, assume that $\alpha\,\mathcal{L}\,\beta$. Then $\text{ran}(\alpha) = \text{ran}(\beta)$. For any $x \in \text{dom}(\beta)$, we have $x\beta \in \text{ran}(\beta) = \text{ran}(\alpha)$, so there exists a unique element $y \in \text{dom}(\alpha)$ such that $y\alpha = x\beta$, where uniqueness follows from the injectivity of $\alpha$. Thus the assignment $x \mapsto y$ defines an injective map from $\text{dom}(\beta)$ into $\text{dom}(\alpha)$. Since $X$ is finite, this injective map extends to a bijection of $X$, that is, to a permutation $\gamma \in G$ satisfying $x\gamma = y$ for all $x \in \text{dom}(\beta)$. For such $x$, we obtain $x(\gamma\alpha) = x\beta$. Hence $\gamma\alpha$ and $\beta$ have the same domain and agree on every element of that domain, and therefore $\gamma\alpha = \beta$. This shows that $\alpha\,\mathcal{L}_G\,\beta$. Consequently, $\mathcal{L} = \mathcal{L}_G$ in $\mathscr{I}_X$.

Moreover, since $\mathscr{I}_X$ is a subsemigroup of the binary relation semigroup $\mathscr{B}_X$, by Proposition 3.3 the Cayley graph $Cay(\mathscr{I}_X, G)$ decomposes in the same way as for $\mathscr{B}_X$, with each induced subgraph corresponding to an $\mathcal{L}_G$-class of $\mathscr{I}_X$. Using $\mathcal{L} = \mathcal{L}_G$, it follows that $Cay(\mathscr{I}_X, G)$ is the disjoint union of the induced subgraphs $\Gamma(L_i, G)$, where $L_i$ runs over all $\mathcal{L}$-classes of $\mathscr{I}_X$. □

**Proposition 3.7.** *Let $\mathscr{B}_X$ be the binary relation semigroup on a finite set $X$ with group of units $G$ and $L$ be an $\mathcal{L}$- class other than $G$. Then for $\alpha$, $\beta \in \mathscr{B}_X$ with $\alpha \neq \beta$, there is an arc from $\alpha$ to $\beta$ in $Cay(\mathscr{B}_X, L)$ if and only if $l\alpha\mathcal{L}\beta$, for every $l \in L$.*

*Proof.* Let $\alpha, \beta \in \mathscr{B}_X$ with $\alpha \neq \beta$. Suppose that there is an arc from $\alpha$ to $\beta$ in $Cay(\mathscr{B}_X, L)$. Then by Definition 2.16 there exists an $l' \in \mathcal{L}$ such that $l'\alpha = \beta$. Also for any $l \in \mathcal{L}$, $V(l) = V(l')$. Then by Definition 2.7 and Lemma 2.11, $V(l\alpha) = \{Al\alpha | A \subseteq X\} = \{Al'\alpha | A \subseteq X\} = V(l'\alpha) = V(\beta)$. Thus $l\alpha\mathcal{L}\beta$ for every $l \in \mathcal{L}$.

Conversely suppose that $l\alpha\mathcal{L}\beta$ for every $l \in \mathcal{L}$. Now let $L_{l\alpha} = \{\gamma \in \mathscr{B}_X | \mathscr{B}_X\gamma = \mathscr{B}_X l\alpha\}$ and consider $H = \{l'\alpha | l' \in \mathcal{L}\}$. Then $L_{l\alpha} = H$. For that, let $l'\alpha \in H$. Then $l' \in \mathcal{L}$. Since $l, l' \in \mathcal{L}$, $V(l) = V(l')$. Consider $V(l\alpha) = \{Al\alpha | A \subseteq X\} = \{Al'\alpha | A \subseteq X\} = V(l'\alpha)$ $\Rightarrow l'\alpha\mathcal{L}l\alpha \Rightarrow l'\alpha \in \mathcal{L}_{l\alpha} \Rightarrow H \subseteq \mathcal{L}_{l\alpha}$.

On the other hand let $\gamma \in \mathcal{L}_{l\alpha}$. Then $\gamma = \alpha_1(l\alpha)$ for some $\alpha_1 \in \mathscr{B}_X$. Then $V(\gamma) = V(\alpha_1 l\alpha)$. Again since $\gamma \in \mathcal{L}_{l\alpha}$, $V(\gamma) = V(l\alpha)$. Therefore $V(\alpha_1 l\alpha) = V(l\alpha) \Rightarrow V(\alpha_1 l) = V(l)$. Hence

$\alpha_1 l \mathcal{L} l$ so that $\alpha_1 l \in \mathcal{L}$. Let $\alpha_1 l = l'$. Then $\gamma = \alpha_1 l \alpha = (\alpha_1 l)\alpha = l'\alpha \in H$. Thus $\mathcal{L}_{l\alpha} \subseteq H$. Hence $H = \mathcal{L}_{l\alpha}$. Since $\beta \in \mathcal{L}_{l\alpha} = H$, it follows that $\beta = l'\alpha$, for some $l' \in \mathcal{L}$. Hence there is an arc from $\alpha$ to $\beta$. □

**Proposition 3.8.** *Let $\mathscr{B}_X$ be the binary relation semigroup on a finite set $X$ with group of units $G$ and $R$ be an $\mathcal{R}$- class other than $G$. Then for $\alpha \in G$ and $\beta \in \mathscr{B}_X$, there is an arc from $\alpha$ to $\beta$ in $Cay(\mathscr{B}_X, R)$ if and only if $r\alpha\mathcal{R}\beta$ for every $r \in R$.*

*Proof.* Let $\alpha \in G$ and $\beta \in \mathscr{B}_X$. Suppose that there is an arc from $\alpha$ to $\beta$ in $Cay(\mathscr{B}_X, R)$. Then by Definition 2.16, there exists an $r' \in \mathcal{R}$ such that $r'\alpha = \beta$. Also, for any $r \in \mathcal{R}$, $rX = r'X$. Then $Xr^{-1} = Xr'^{-1}$. Consider, $r'\alpha X = X(r'\alpha)^{-1} = X(\alpha^{-1}r'^{-1}) = (X\alpha^{-1})r'^{-1} = (X\alpha^{-1})r^{-1} = X(\alpha^{-1}r^{-1}) = X(r\alpha)^{-1} = r\alpha X$. Thus $r'\alpha\mathcal{R}r\alpha$. Hence $r\alpha\mathcal{R}\beta$ for every $r \in \mathcal{R}$.

Conversely, suppose that $r\alpha\,\mathcal{R}\,\beta$ for every $r \in R$. Now let $\mathcal{R}_{r\alpha} = \{\gamma \in \mathscr{B}_X \mid \gamma\mathscr{B}_X = r\alpha\mathscr{B}_X\}$ and consider $H = \{r'\alpha \mid r' \in \mathcal{R}\}$. We claim that $\mathcal{R}_{r\alpha} = H$. First, let $r'\alpha \in H$. Then $r' \in \mathcal{R}$. Since $r$ and $r'$ belong to the same $\mathcal{R}$-class, we have $Xr^{-1} = Xr'^{-1}$. Hence $r'\alpha X = r\alpha X$, which implies $r'\alpha\,\mathcal{R}\,r\alpha$, and therefore $r'\alpha \in \mathcal{R}_{r\alpha}$. Thus $H \subseteq \mathcal{R}_{r\alpha}$. On the other hand, let $\gamma \in \mathcal{R}_{r\alpha}$. Then $\gamma = (r\alpha)\alpha_1$ for some $\alpha_1 \in \mathscr{B}_X$. Since $\alpha \in G$ is invertible, we may insert the identity $\alpha^{-1}\alpha = 1_X$ to obtain $\gamma = (r\alpha)\alpha_1(\alpha^{-1}\alpha) = r(\alpha\alpha_1\alpha^{-1})\alpha$. Define $\alpha' = \alpha\alpha_1\alpha^{-1} \in \mathscr{B}_X$. Then $\gamma = r\alpha'\alpha$. Let $r\alpha' = r'$. Since $\alpha' \in \mathscr{B}_X$, there exists $\alpha'' \in \mathscr{B}_X$ such that $r'\alpha'' = r$, and hence $r\,\mathcal{R}\,r'$, which implies $r' \in \mathcal{R}$. Therefore, $\gamma = (r\alpha')\alpha = r'\alpha \in H$. Thus $\mathcal{R}_{r\alpha} \subseteq H$, and consequently $\mathcal{R}_{r\alpha} = H$. Since $\beta \in \mathcal{R}_{r\alpha} = H$, there exists $r'' \in \mathcal{R}$ such that $\beta = r''\alpha$. Hence there is an arc from $\alpha$ to $\beta$ in $Cay(\mathscr{B}_X, R)$. □

**Remark 3.9.** The conditions "$l\alpha\,\mathcal{L}\,\beta$ for every $l \in L$" in Proposition 3.7 and "$r\alpha\,\mathcal{R}\,\beta$ for every $r \in R$" in Proposition 3.8 are essential. This is because the existence of an arc from $\alpha$ to $\beta$ in the respective Cayley graph depends on the relation holding uniformly for all elements of the class $L$ or $R$; a single element satisfying the relation is not sufficient to guarantee that $\beta$ lies in the image of all translates by $L$ or $R$.

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