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Stability Analysis of a Fractional Stochastic Differential System Using the Fixed Point Method

E. Hussen and S. Alargeh

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Corresponding Author: E. Hussen

Abstract In this paper, we propose a new modification of the fixed point method, and apply the Laplace transformation to solve fractional stochastic differential system (FSDS). New conditions are introduced to prove the existence and uniqueness of solutions to FSDS using the contraction mapping principle. In addition, we establish stability conditions for FSDS based on the fixed point method. Due to the inherent complexities in studying FSDS, the proposed conditions provide a more convenient and effective approach compared to existing methods.

1 Introduction

Stochastic behavior is an essential property of many real-world systems, and stability is a top priority in practical applications. The stability analysis of stochastic system is therefore crucial. Both Lyapunov's direct method and the fixed point method have been widely adopted by scholars because of their intuitive concepts, general applicability, clear physical meaning, and rigorous theoretical foundation. These approaches have become the main tools for studying the stability of differential systems. Burton et al. [1, 2, 3, 4] presented the application of fixed point techniques to stability analysis and fractional differential equations. The existence, uniqueness and stability of solutions to stochastic partial differential equations have been investigated extensively [5, 6, 7, 8, 9]. Many recent developments have further advanced the analysis of stochastic integral and differential equations [10, 11, 12, 13, 14, 15, 16].

In 2021, Xiao et al. [17] studied stability of solutions of the following Caputo fractional stochastic differential equations:

$$\begin{aligned} &{}^{C}D_{0^+}^{\alpha}\mathrm{X}(t) = f(t,\mathrm{X}(t)) + g(t,\mathrm{X}(t))\frac{dW(t)}{dt} \\ &\mathrm{X}(0) = \mathrm{X}_0 \end{aligned} \tag{1.1}$$

Where ${}^{C}D_{0^+}^{\alpha}$ denotes the Caputo fractional derivative, $f, g : [0\ ,\ \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions, and $\{W(t)\ ;\ t \in [0\ ,\ \infty)\}$ is a standard scalar Brownian motion on an underlying complete space $(\Omega, \mathcal{F}, \mathrm{P})$ with filtration $\mathrm{F} = \{\mathcal{F}_t\}_{t\geq 0}$.

Xiao et al. [17] applied the contraction mapping principle to derive the existence and uniqueness of the solution of (1.1) by imposing the Lipschitz condition on f and g.

In the same year, Ahmadova et al. [18] investigated the stability of the following Riemann-Liouville fractional stochastic neutral differential equations:

$$\begin{aligned} &{}^{RL}D_{0^+}^{\alpha}\left(\mathrm{X}(t) + g(t,\mathrm{X}(t))\right) = \mathrm{A}\,\mathrm{X}(t) + b(t,\mathrm{X}(t)) + \sigma(t,\mathrm{X}(t))\frac{dW(t)}{dt} \\ &\mathrm{I}_{0^+}^{1-\alpha}\left(\mathrm{X}(t) + g(t,\mathrm{X}(t))\right)\Big|_{t=0} = \rho \end{aligned} \tag{1.2}$$

The coefficients g, b, σ : $[0,\, T[\, \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable and bounded functions, $\mathrm{A} \in \mathbb{R}^{n\times n}$ and the initial condition ρ in integral form is an $\mathcal{F}_0$-measurable. Let $W\,(t)$ denote a standard scalar Brownian motion on a complete probability $(\Omega, \mathcal{F}, \mathrm{P})$ with filtration $\mathrm{F} = \{\mathcal{F}_t\}_{t\geq 0}$.

Ahmadova et al. [18] proved global existence and uniqueness of mild solution of (1.2) under various assumptions using Banach's contraction mapping principle.

Based on these studies, we extend the analysis by considering the stability of solutions to fractional stochastic differential systems (FSDS).

This paper is organized as follows:

In section 2, we recall important definitions and lemmas in fractional calculus. In section 3, we establish existence and uniqueness of solutions for FSDS using contraction mapping. In section 4, we study the stability of FSDS via the fixed point method, and we present some examples to illustrate our results. In section 5, we provide the conclusion and suggestions for future work.

2 Material and Methods

Definition 2.1. [19] The Riemann-Liouville integral of fractional operator order $\alpha > 0$ is defined as

$$\mathrm{I}^{\alpha}_{0^+} f\,(t) = \frac{1}{\Gamma\,(\alpha)} \int\limits_{t_0}^{t} f\,(\tau)\,(t-\tau)^{\alpha-1}\, d\tau \;\; ; \quad t > 0$$

Where $\Gamma\,(\alpha) = \int\limits_{0}^{\infty} \tau^{\alpha-1} e^{-\tau} d\tau$ is Gamma function, $\mathrm{I}^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional order integral.

Definition 2.2. [20] The Riemann-Liouville derivative operator of fractional order $\alpha > 0$ is defined as

$${}^{RL}D^{\alpha}_{0^+} f\,(t) = \frac{1}{\Gamma\,(n-\alpha)} \frac{d^n}{dt^n} \int\limits_{0}^{t} f\,(\tau)\,(t-\tau)^{n-\alpha-1} d\tau \; ; \; n-1 < \alpha < n$$

Where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer of α, ${}^{RL}D^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional order derivative.

Remark 2.3. [20] Alternatively, the Riemann-Liouville derivative operator of fractional order $\alpha > 0$ can be expressed as

$${}^{RL}D^{\alpha}_{0^+} f\,(t) = \frac{d^n}{dt^n} \left(\mathrm{I}^{n-\alpha}_{0^+} f\,(t)\right) \qquad ; n-1 < \alpha \leqslant n$$

Lemma 2.4. *[20] The relationship between the Riemann-Liouville fractional derivative and integral as follows:*

(i) If $f \in C\,[0,\, T]$, then for any point $t \in [0,\, T]$

$${}^{RL}D^{\alpha}_{0^+}\,(\mathrm{I}^{\alpha}_{0^+} f\,(t)) = f\,(t)$$

(ii) If $f \in C\,[0,\, T]$ and $\mathrm{I}^{1-\alpha}_{0^+} f \in C\,[0,\, T]$, then for any point $t \in [0,\, T]$

$$\mathrm{I}^{\alpha}_{0^+}\left({}^{RL}D^{\alpha}_{0^+} f\,(t)\right) = f\,(t) - \frac{\mathrm{I}^{1-\alpha}_{0^+} f\,(t)\Big|_{t=0}}{\Gamma\,(\alpha)} t^{\alpha-1}$$

Definition 2.5. [19] The Laplace transform of a real function $f\,(t)$ is defined as

$$F\,(s) = \mathcal{L}\,\{f\,(t)\} = \int\limits_{0}^{\infty} e^{-st} f\,(t)\, dt$$

Definition 2.6. [19] The convolution of two functions $f\left(t\right)$ and $g\left(t\right)$ is defined as

$$f\left(t\right)*g\left(t\right)=\int\limits_{0}^{t}f\left(t-\tau\right)g\left(\tau\right)d\tau=\int\limits_{0}^{t}f\left(\tau\right)g\left(t-\tau\right)d\tau$$

Definition 2.7. [19] The inverse Laplace transform of a real function $f\left(t\right)$ is defined as

$$f\left(t\right)=\mathcal{L}^{-1}\left\{F(s)\right\}=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}e^{+st}F\left(s\right)ds \qquad ;\quad c=\mathrm{Re}\left(s\right)>c_0$$

Property 2.8. [19] The inverse Laplace transform of the convolution is defined as

$$\mathcal{L}^{-1}\left\{F\left(s\right)\cdot G\left(s\right)\right\}=f\left(t\right)*g\left(t\right)$$

Definition 2.9. [21] We consider the matrix Mittag-Leffler functions with one and two parameters which are defined by

$$\mathrm{E}_{\alpha}\left(t^{\alpha}\mathrm{A}\right)=\sum_{\kappa=0}^{\infty}\frac{\mathrm{A}^{\kappa}t^{\kappa\alpha}}{\Gamma\left(\alpha\kappa+1\right)}$$

$$\mathrm{E}_{\alpha,\beta}\left(t^{\alpha}\mathrm{A}\right)=\sum_{\kappa=0}^{\infty}\frac{\mathrm{A}^{\kappa}t^{\kappa\alpha}}{\Gamma\left(\alpha\kappa+\beta\right)} \qquad ;\ \left(\alpha,\beta>0\,,\ \mathrm{A}\in\mathbb{R}^{n\times n}\right)$$

Where $\mathrm{E}_{\alpha}\left(\cdot\right)$ denotes Mittag-Leffler function with one parameter, $\mathrm{E}_{\alpha,\alpha}\left(\cdot\right)$ denotes Mittag-Leffler function with two parameters, $\mathrm{A}\in\mathbb{R}^{n\times n}$ is a constant matrix.

Lemma 2.10. *[20] The following Laplace transform properties are used in this study:*

(i) $\mathcal{L}\left\{f\left(t\right)\right\}=F\left(s\right)$

(ii) $\mathcal{L}\left\{\mathrm{I}_{0^+}^{\alpha}f\left(t\right)\right\}=s^{-\alpha}F\left(s\right)$

(iii) $\mathcal{L}\left\{{}^{RL}D_{0^+}^{\alpha}f\left(t\right)\right\}=s^{\alpha}F\left(s\right)-\sum\limits_{\kappa=0}^{n-1}s^{\kappa RL}D^{\alpha-\kappa-1}f\left(0\right) \quad ;\ n-1<\alpha<n$

(iv) $\mathcal{L}\left\{t^{\beta-1}\mathrm{E}_{\alpha,\beta}\left(\mp\lambda t^{\alpha}\right)\right\}=\frac{s^{\alpha-\beta}}{s^{\alpha}\pm\lambda}$

Definition 2.11. [20] Let X be a vector space over a real field. A norm on X is a function $\left\|\cdot\right\|:X\to\mathbb{R}^{+}$ such that

(i) $\left\|x\right\|>0 \qquad ;\ \forall x\in X$

(ii) $\left\|x\right\|=0 \ \Leftrightarrow\ x=0$

(iii) $\left\|\lambda x\right\|=\left|\lambda\right|\left\|x\right\| \qquad ;\ \lambda\in\mathbb{R}\ or\ \mathbb{C}$

(iv) $\left\|x+y\right\|\leqslant\left\|x\right\|+\left\|y\right\| \qquad ;\ \forall x,y\in X$

Definition 2.12. [21] A Banach space is a complete normed space, i.e., every Cauchy sequence in X converges in X.

Definition 2.13. A point $x\in\mathrm{X}$ is called a fixed point of a function $f:\mathrm{X}\to\mathrm{X}$, if

$$f\left(x\right)=x \quad ,\ x\in\mathrm{X}$$

Theorem 2.14. *[21] Let X be a Banach space and f be a contraction mapping with Lipschitz constant κ. Then f has a unique fixed point.*

Remark 2.15. Through this paper, we define

$$\Lambda_{\alpha}^{s}=\left\{\lambda\in\mathbb{C}\backslash\left\{0\right\}:\ \left|\arg\,\lambda\right|\ >\frac{\alpha\pi}{2}\right\}$$

For any matrix $\mathrm{A}\in\mathbb{R}^{n\times n}$, the set $spec\left(\mathrm{A}\right)$ is the spectrum of A, i.e.

$$spec\left(\mathrm{A}\right)=\left\{\lambda\in\mathbb{C}:\lambda\,is\,an\,eigenvalue\,of\,the\,matrix\,\mathrm{A}\right\}$$

Lemma 2.16. *[18] Let* $\mathrm{A} \in \mathbb{R}^{n\times n}$ *and suppose that* $spec\,(\mathrm{A}) \subset \Lambda_{\alpha}^{s}$. *Then the following statements are valid.*

(i) There exists $t_0 > 0$ *and a positive constant* $\tilde{\mathrm{M}}$ *which depends on parameters* t_0, α, A *such that*

$$t^{\alpha-1}\left\|\mathrm{E}_{\alpha,\alpha}\left(t^{\alpha}\mathrm{A}\right)\right\| \leqslant \frac{\tilde{\mathrm{M}}}{t^{\alpha+1}} \qquad ;\forall t \geqslant t_0$$

(ii) The quantity

$$t^{1-\alpha}\int\limits_{0}^{t}(t-\tau)^{\alpha-1}\left\|\mathrm{E}_{\alpha,\alpha}\left((t-\tau)^{\alpha}\mathrm{A}\right)\right\|\tau^{\alpha-1}d\tau$$

Is bounded on $[0\,,\,\infty)$, *i.e*

$$\sup_{t>0} t^{1-\alpha}\int\limits_{0}^{t}(t-\tau)^{\alpha-1}\left\|\mathrm{E}_{\alpha,\alpha}\left((t-\tau)^{\alpha}\mathrm{A}\right)\right\|\tau^{\alpha-1}d\tau < \infty$$

We now consider the fractional non homogenous differential system of the form

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}y\,(t) = \mathrm{A}y\,(t) + f\,(t) & ;t>0 \\ \mathrm{I}_{0^+}^{\kappa-\alpha}y\,(t)\big|_{t=0} = b_{\kappa} & ;\kappa = 1,2,...,n \end{cases} \tag{2.1}$$

Where $n-1 < \alpha \leqslant n$ and $\mathrm{A} \in \mathbb{R}^{n\times n}$ constant matrix, $f : [0, T[\, \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded function, $y\,(t) = (y_1\,(t)\,, y_2\,(t)\,, ..., y_n\,(t)) \in \mathbb{R}^n$.

Theorem 2.17. *[20] for* $n-1 < \alpha \leqslant n$, *the solution of (2.1) is given by*

$$y\,(t) = \sum_{\kappa=1}^{n} b_{\kappa}t^{\alpha-\kappa}\mathrm{E}_{\alpha,\alpha-\kappa+1}\left(\mathrm{A}t^{\alpha}\right) + \int\limits_{0}^{t}(t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left(\mathrm{A}(t-\tau)^{\alpha}\right)f\,(\tau)\,d\tau \tag{2.2}$$

Where $n-1 < \alpha \leqslant n$ *and* $\mathrm{A} \in \mathbb{R}^{n\times n}$ *constant matrix,* $f : [0, T[\, \times \mathbb{R}^n \to \mathbb{R}^n$ *is bounded function,* $y\,(t) = (y_1\,(t)\,, y_2\,(t)\,, ..., y_n\,(t)) \in \mathbb{R}^n$.

Lemma 2.18. *[18] Suppose that* $\alpha_1\,,\,\alpha_2 > 1$ *and* $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$. *If* $|f\,(t)|^{\alpha_1}$, $|g\,(t)|^{\alpha_2} \in L^1\,(\Omega)$, *then* $|f\,(t)\,g\,(t)| \in L^1\,(\Omega)$ *and*

$$\int\limits_{\Omega}|f\,(t)\,g\,(t)|\,dt \leqslant \left(\int\limits_{\Omega}|f\,(t)|^{\alpha_1}dt\right)^{\frac{1}{\alpha_1}}\left(\int\limits_{\Omega}|g\,(t)|^{\alpha_2}dt\right)^{\frac{1}{\alpha_2}} \tag{2.3}$$

Where $L^1\,(\Omega)$ *represents the Banach space of all Lebesgue measurable functions* $f : \Omega \to \mathbb{R}$ *with* $\int\limits_{\Omega}|f\,(t)|\,dt < \infty$. *Especially, when* $\alpha_1 = \alpha_2 = 2$, *the inequality (2.3) reduces to the Cauchy-Schwartz inequality*

$$\left(\int\limits_{\Omega}|f\,(t)\,g\,(t)|\,dt\right)^2 \leqslant \int\limits_{\Omega}|f\,(t)|^2dt\int\limits_{\Omega}|g\,(t)|^2dt \tag{2.4}$$

Lemma 2.19. *[18] Let* $n \in \mathbb{N}$, $q > 1$ *and* $x_i \in \mathbb{R}_+$, $i = 1,2,...,n$. *Then, the following inequality holds true*

$$\left\|\sum_{i=1}^{n}x_i\right\|^q \leqslant n^{q-1}\sum_{i=1}^{n}\|x_i\|^q$$

In particular, when $q = 2$, we use the following inequality within the estimations in this paper:

$$\left\|\sum_{i=1}^{n}x_i\right\|^2 \leqslant n\sum_{i=1}^{n}\|x_i\|^2 \tag{2.5}$$

3 Results on existence and uniqueness

We consider the following fractional stochastic differential system:

$$\begin{aligned} {}^{RL}D_{0^+}^{\alpha}\mathrm{X}(t) &= \mathrm{A}\,\mathrm{X}(t) + g(t,\mathrm{X}(t))\frac{dW(t)}{dt} \\ \mathrm{I}_{0^+}^{1-\alpha}\mathrm{X}(t)\Big|_{t=0} &= \rho \end{aligned} \tag{3.1}$$

Where $g:\ [0,T[\times\mathbb{R}^n\to\mathbb{R}^n$ is measurable and bounded function, $\mathrm{A}\in\mathbb{R}^{n\times n}$ and the initial condition ρ in integral form is an $\mathcal{F}_0$-measurable. Let $W(t)$ denote a standard scalar Brownian motion on a complete probability $(\Omega,\mathcal{F},\mathrm{P})$ with filtration $\mathrm{F}=\{\mathcal{F}_t\}_{t\geq 0}$.

Remark 3.1. Let H be the Banach space of all $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process ζ satisfying the following norm defined by

$$\|\zeta\|_{\mathrm{H}}^p = \sup_{t\geqslant 0}\mathrm{E}\left\|t^{1-\alpha}\zeta(t)\right\|^p < \infty$$

Definition 3.2. A stochastic process $\{\mathrm{X}(t)\ ;\ t\in[0,\ T]\}$ is called a solution of (3.1) if

(i) $\mathrm{X}(t)$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ with $\int\limits_0^t \|\mathrm{X}(\tau)\|_{\mathrm{H}}^p\,d\tau<\infty$ almost everywhere.

(ii) $\mathrm{X}(t)\in\mathrm{H}$ has continuous path on $t\in[0,\ T]$ and satisfies the following Volterra integral equation of second kind for each $t\in[0,\ T]$:

$$\mathrm{X}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\rho + \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-\tau)^{\alpha-1}\mathrm{A}\mathrm{X}(\tau)\,d\tau + \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-\tau)^{\alpha-1} g(\tau,\mathrm{X}(\tau))\,dW(\tau) \tag{3.2}$$

Theorem 3.3. *for* $0<\alpha<1$*, the solution of (3.1) is given by*

$$\mathrm{X}(t) = t^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(t^{\alpha}\mathrm{A})\rho + \int\limits_0^t (t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}((t-\tau)^{\alpha}\mathrm{A})\,g(\tau,\mathrm{X}(\tau))\,dW(\tau) \tag{3.3}$$

Proof. Applying the Laplace transform to (3.1), we obtain

$$\mathcal{L}\left\{{}^{RL}D_{0^+}^{\alpha}\mathrm{X}(t)\right\} = \mathcal{L}\left\{\mathrm{A}\,\mathrm{X}(t) + g(t,\mathrm{X}(t))\frac{dW(t)}{dt}\right\}$$

This yields

$$s^{\alpha}\mathrm{X}(s) - \rho = \mathrm{A}\,\mathrm{X}(s) + g(\mathrm{X}(s),s)\,dW(s)$$

Hence,

$$\mathrm{X}(s) = (s^{\alpha}\mathrm{I}-\mathrm{A})^{-1}\rho + (s^{\alpha}\mathrm{I}-\mathrm{A})^{-1}g(\mathrm{X}(s),s)\,dW(s)$$

Taking the inverse Laplace transform to both sides, we obtain

$$\mathrm{X}(t) = \mathcal{L}^{-1}\left\{(s^{\alpha}\mathrm{I}-\mathrm{A})^{-1}\rho + (s^{\alpha}\mathrm{I}-\mathrm{A})^{-1}g(\mathrm{X}(s),s)\,dW(s)\right\}$$

Using the probatory

$$\mathcal{L}^{-1}\left\{\frac{p!s^{\alpha-\beta}}{(s^{\alpha}\pm\lambda)^{p+1}}\right\} = t^{\alpha p+\beta-1}\mathrm{E}_{\alpha,\beta}^{(p)}(\mp\lambda t^{\alpha})$$

When $\alpha=\alpha\,,\ \beta=\alpha\,,\ p=0$

$$\mathrm{X}(t) = t^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(t^{\alpha}\mathrm{A})\rho + t^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(t^{\alpha}\mathrm{A}) * g(t,\mathrm{X}(t))\,dW(t)$$

Using the convolution probatory on the second term of last equation, we obtain (3.3). □

Lemma 3.4. *[18] Let $p \geqslant 2$, $t > 0$ and let Φ be an $\mathbb{R}^n$-valued predictable process such that $\mathrm{E}\int\limits_0^t \|\Phi(\tau)\|^p d\tau < \infty$. Then,*

$$\sup_{\tau\in[0\,,\,T]} E\left\|\int\limits_0^\tau \Phi(u)\,dW(u)\right\|^p \leqslant \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \left(\int\limits_0^t \left(\mathrm{E}\|\Phi(\tau)\|^p\right)^{\frac{2}{p}} d\tau\right)^{\frac{p}{2}}$$

Theorem 3.5. *Suppose the following conditions hold:*

(i) The function g satisfies global Lipschitz continuity: there exists $L_\sigma > 0$ such that for all $\mathrm{X}\,,\,Y \in \mathbb{R}^n$, $t \geqslant 0$,

$$\|g(t,\mathrm{X}(t)) - g(t,Y(t))\| \leqslant L_\sigma \|\mathrm{X} - Y\|$$

(ii) $g(\cdot,\,0)$ is essentially bounded i.e.

$$\|g(\tau,\,0)\|_\infty = ess \sup_{\tau\in[0,T]} \|g(\tau,\,0)\| < \infty$$

(iii) The function g satisfies the following condition

$$g(t,\,0) = 0$$

(iv) $\Phi = C_p L_\sigma^p \mathrm{M}^p T^{p(\alpha-1)+\frac{p}{2}} \left(\beta(2\alpha-1, 2\alpha-1)\right)^{\frac{p}{2}} < 1$

Then for any $\rho \in \mathbb{R}^n$, there exists a unique solution of (3.1).

Proof. We define an operator $\Psi:\,\mathrm{H} \to \mathrm{H}$ for $t \in [0\,,\,T]$ by

$$(\Psi\mathrm{X})(t) = t^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(t^\alpha \mathrm{A})\,\rho + \int\limits_0^t (t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}((t-\tau)^\alpha \mathrm{A})\,g(\tau,\mathrm{X}(\tau))\,dW(\tau) \tag{3.4}$$

First, we verify mean continuity of Ψ on $[0\,,\,T]$. Let $\mathrm{X} \in \mathrm{H}$, $t_1 \geqslant 0$ and r be sufficiently small and show that

$$\mathrm{E}\left\|(t_1+r)^{1-\alpha}(\Psi\mathrm{X})(t_1+r) - t_1^{1-\alpha}(\Psi\mathrm{X})(t_1)\right\|^p \leqslant 2^{p-1}\sum_{i=1}^{2}\mathrm{E}\left\|(t_1+r)^{1-\alpha}\mathrm{I}_i(t_1+r) - t_1^{1-\alpha}\mathrm{I}_i(t_1)\right\|^p \tag{3.5}$$

Note that

$$\begin{aligned}
&\mathrm{E}\left\|(t_1+r)^{1-\alpha}\mathrm{I}_1(t_1+r) - t_1^{1-\alpha}\mathrm{I}_1(t_1)\right\|^p \\
&= \mathrm{E}\left\|\left[(t_1+r)^{1-\alpha}(t_1+r)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}((t_1+r)^\alpha \mathrm{A}) - t_1^{1-\alpha}t_1^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(t_1^\alpha \mathrm{A})\right]\,\rho\right\|^p
\end{aligned}$$

$$\mathrm{E}\left\|(t_1+r)^{1-\alpha}\mathrm{I}_1(t_1+r) - t_1^{1-\alpha}\mathrm{I}_1(t_1)\right\|^p = \mathrm{E}\left\|\left[\mathrm{E}_{\alpha,\alpha}((t_1+r)^\alpha \mathrm{A}) - \mathrm{E}_{\alpha,\alpha}(t_1^\alpha \mathrm{A})\right]\,\rho\right\|^p \tag{3.6}$$

The strong continuity of $\mathrm{E}_{\alpha,\alpha}(t^\alpha \mathrm{A})$ implies that the right-hand side of (3.6) goes to zero as $r \to 0$. In view of the Holder's inequality and assumptions (i), (iii), we have

$$
\begin{aligned}
&\mathrm{E}\left\|(t_1+r)^{1-\alpha}\mathrm{I}_2(t_1+r)-t_1^{1-\alpha}\mathrm{I}_2(t_1)\right\|^p = \\
&\mathrm{E}\left\|\begin{array}{l}\displaystyle\int_0^{t_1+r}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1+r-\tau)^{\alpha}\mathrm{A}\right)g\left(\tau,\mathrm{X}(\tau)\right)dW(\tau)\\ \displaystyle-\int_0^{t_1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)g\left(\tau,\mathrm{X}(\tau)\right)dW(\tau)\end{array}\right\|^p \\
&\leqslant 2^{p-1}\mathrm{E}\left\|\int_0^{t_1+r}\begin{bmatrix}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1+r-\tau)^{\alpha}\mathrm{A}\right)\\ -t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\end{bmatrix}g\left(\tau,\mathrm{X}(\tau)\right)dW\right\|^p \\
&\quad+2^{p-1}\mathrm{E}\left\|\int_{t_1}^{t_1+r}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)g\left(\tau,\mathrm{X}(\tau)\right)dW(\tau)\right\|^p \\
&\leqslant 2^{p-1}C_p\mathrm{E}\left(\begin{array}{l}\displaystyle\int_0^{t_1+r}\begin{bmatrix}\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1+r-\tau)^{\alpha}\mathrm{A}\right)\\ -\tau^{\alpha-1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\end{bmatrix}^2\\ \times\left\|\tau^{1-\alpha}g\left(\tau,\mathrm{X}(\tau)\right)\right\|^2d\tau\end{array}\right)^{\frac{p}{2}} \\
&\quad+2^{p-1}C_p\mathrm{E}\left(\begin{array}{l}\displaystyle\int_{t_1}^{t_1+r}\tau^{2\alpha-2}t_1^{1-\alpha}(t_1-\tau)^{2\alpha-2}\left\|\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\right\|^2\\ \times\left\|\tau^{1-\alpha}g\left(\tau,\mathrm{X}(\tau)\right)\right\|^2d\tau\end{array}\right)^{\frac{p}{2}} \\
&\leqslant 2^{p-1}C_pL_\sigma^p\left(\begin{array}{l}\displaystyle\int_0^{t_1+r}\begin{bmatrix}\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1+r-\tau)^{\alpha}\mathrm{A}\right)\\ -\tau^{\alpha-1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\end{bmatrix}^2\\ \times\mathrm{E}\left\|\tau^{1-\alpha}\mathrm{X}(\tau)\right\|^2d\tau\end{array}\right)^{\frac{p}{2}} \\
&\quad+2^{p-1}C_p\mathrm{E}\left(\begin{array}{l}\displaystyle\int_{t_1}^{t_1+r}\tau^{2\alpha-2}t_1^{1-\alpha}(t_1-\tau)^{2\alpha-2}\left\|\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\right\|^2\\ \times\left\|\tau^{1-\alpha}g\left(\tau,\mathrm{X}(\tau)\right)\right\|^2d\tau\end{array}\right)^{\frac{p}{2}} \\
&\leqslant 2^{p-1}C_pL_\sigma^p\left(\begin{array}{l}\displaystyle\int_0^{t_1+r}\begin{bmatrix}\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1+r-\tau)^{\alpha}\mathrm{A}\right)\\ -\tau^{\alpha-1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\end{bmatrix}^2\\ \times\mathrm{E}\left\|\tau^{1-\alpha}\mathrm{X}(\tau)\right\|^2d\tau\end{array}\right)^{\frac{p}{2}} \\
&\quad+2^{p-1}C_pL_\sigma^p\left(\begin{array}{l}\displaystyle\int_{t_1}^{t_1+r}\tau^{2\alpha-2}t_1^{2-2\alpha}(t_1-\tau)^{2\alpha-2}\left\|\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\right\|^2\\ \times\mathrm{E}\left\|\tau^{1-\alpha}\mathrm{X}(\tau)\right\|^2d\tau\end{array}\right)^{\frac{p}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 2^{p-1}C_pL_\sigma^p\left(\int\limits_0^{t_1+r}\begin{bmatrix}\tau^{\alpha-1}(t_1+r)^{1-\alpha}(t_1+r-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1+r-\tau)^{\alpha}\mathrm{A}\right)\\ -\tau^{\alpha-1}t_1^{1-\alpha}(t_1-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\end{bmatrix}^{\frac{2p}{p-2}}d\tau\right)^{\frac{p-2}{2}}\\
&\times\int\limits_0^{t_1+r}\mathrm{E}\left\|\tau^{1-\alpha}\mathrm{X}(\tau)\right\|^p d\tau\\
&+2^{p-1}C_pL_\sigma^p\left(\int\limits_{t_1}^{t_1+r}\left(\tau^{2\alpha-2}t_1^{2-2\alpha}(t_1-\tau)^{2\alpha-2}\left\|\mathrm{E}_{\alpha,\alpha}\left((t_1-\tau)^{\alpha}\mathrm{A}\right)\right\|^2\right)^{\frac{p}{p-2}}d\tau\right)^{\frac{p-2}{2}}\\
&\times\int\limits_0^{t_1+r}\mathrm{E}\left\|\tau^{1-\alpha}\mathrm{X}(\tau)\right\|^p d\tau
\end{aligned}
\tag{3.7}
$$

Therefore, the right-hand side of the above inequality tends to zero as $r \to 0$. Thus, by taking into (3.6) and (3.7) account Ψ is continuous in pth moment on $[0\,,\,T]$.

Next, to prove the global existence and uniqueness of solution, we will show that the operator Ψ has a unique fixed point. Indeed, for any $\mathrm{X}\,,\,Y \in \mathrm{H}$, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathrm{E}\left\|t^{1-\alpha}(\Psi\mathrm{X})(t)-t^{1-\alpha}(\Psi Y)(t)\right\|^p \leqslant\\
&\leqslant \sup_{t\in[0,T]}\mathrm{E}\left\|\int\limits_0^t t^{1-\alpha}(t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t-\tau)^{\alpha}\mathrm{A}\right)\left[g(\tau,\mathrm{X}(\tau))-g(\tau,Y(\tau))\right]dW(\tau)\right\|^p\\
&\leqslant C_p\sup_{t\in[0,T]}\mathrm{E}\left[\left\|\int\limits_0^t \begin{matrix}t^{1-\alpha}(t-\tau)^{\alpha-1}\tau^{\alpha-1}\mathrm{E}_{\alpha,\alpha}\left((t-\tau)^{\alpha}\mathrm{A}\right)\\ \left[\tau^{1-\alpha}g(\tau,\mathrm{X}(\tau))-\tau^{1-\alpha}g(\tau,Y(\tau))\right]d\tau\end{matrix}\right\|^2\right]^{\frac{p}{2}}\\
&\leqslant C_pL_\sigma^p\mathrm{M}^p\left(\sup_{t\in[0,T]}\int\limits_0^t t^{2-2\alpha}(t-\tau)^{2\alpha-2}\tau^{2\alpha-2}d\tau\right)^{\frac{p}{2}}\left\|\mathrm{X}-Y\right\|_{\mathrm{H}}^p\\
&\leqslant C_pL_\sigma^p\mathrm{M}^pT^{p(\alpha-1)+\frac{p}{2}}\left(\beta\left(2\alpha-1,2\alpha-1\right)\right)^{\frac{p}{2}}\left\|\mathrm{X}-Y\right\|_{\mathrm{H}}^p\\
&\left\|\Psi\mathrm{X}-\Psi Y\right\|_{\mathrm{H}}^p\leqslant\Phi\left\|\mathrm{X}-Y\right\|_{\mathrm{H}}^p
\end{aligned}
$$

Where

$$\Phi = C_pL_\sigma^p\mathrm{M}^pT^{p(\alpha-1)+\frac{p}{2}}\left(\beta\left(2\alpha-1,2\alpha-1\right)\right)^{\frac{p}{2}} < 1$$

Therefore, by assumption (iv), Ψ is a contraction mapping and hence there exists a unique fixed point, which is a solution of (3.1) on $[0\,,\,T]$. □

4 Results on Stability Solution

Remark 4.1. The trivial solution of Fractional Stochastic Differential System is asymptotically stable if and only if the spectrum $spec\,(\mathrm{A})$ of the matrix $\mathrm{A} \in \mathbb{R}^{n\times n}$ satisfies the condition

$$spec\,(\mathrm{A}) \subset \Lambda_\alpha^s$$

Definition 4.2. Let $p \geqslant 2$ be an integer. The solution of system (3.1) is said to be stable in pth moment if for arbitrarily given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{t\geqslant 0}\mathrm{E}\left(\left\|\mathrm{X}(t)\right\|^p\right) < \varepsilon\,,\ \ when\ \left\|\rho\right\| < \delta$$

Remark 4.3. We assume that $\mathcal{D}$ is a space of all $\mathbb{R}^n$-valued F_t-adapted process φ satisfying the following norm defined by

$$\|\varphi\|^p = \sup_{t\geqslant 0} E\|\varphi(t)\|^p$$

Where φ is continuous in t such that $E\|\varphi(t)\|^p \underset{t\to\infty}{\to} 0$. It is clear that $\mathcal{D}$ is a Banach space with respect to the norm $\|.\|^p$.

Theorem 4.4. *Suppose that following conditions hold:*

- *(i) Let* $\mathrm{A} \in \mathbb{R}^{n\times n}$ *be constant matrix and suppose that* $spec(\mathrm{A}) \subset \Lambda_\alpha^s$.
- *(ii) The function* g *satisfies global Lipschitz continuity: there exists* $L_\sigma > 0$ *such that for all* $\mathrm{X}, Y \in \mathbb{R}^n$, $t \geqslant 0$,
$$\|g(t,\mathrm{X}(t)) - g(t,Y(t))\| \leqslant L_\sigma \|\mathrm{X} - Y\|$$
- *(iii)* $g(\cdot, 0)$ *is essentially bounded i.e.*
$$\|g(\tau, 0)\|_\infty = ess \sup_{\tau\in[0,T]} \|g(\tau, 0)\| < \infty$$
- *(iv) The function* g *satisfies the following condition*
$$g(t, 0) = 0$$

Then, the solution of (3.1) is stable in the pth moment.

Proof. First, we show that $\Psi(\mathcal{D}) \subset \mathcal{D}$. Let $\mathrm{X} \in \mathcal{D}$, we have

$$\begin{aligned} \mathrm{E}\|(\Psi\mathrm{X})(t)\|^p &\leqslant 2^{p-1}E\left\|t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\mathrm{A})\rho\right\| \\ &+2^{p-1}\mathrm{E}\left\|\int_0^t (t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}((t-\tau)^\alpha\mathrm{A})\,g(\tau,\mathrm{X}(\tau))\,dW(\tau)\right\|^p \end{aligned} \tag{4.1}$$

Now we estimate the terms on the right-hand side of (4.1). By assumption (i) and according to the lemma 2.16, first we have

$$2^{p-1}E\left\|t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\mathrm{A})\rho\right\| \underset{t\to\infty}{\to} 0$$

By assumption (i) we have

$$\begin{aligned} &\mathrm{E}\left\|\int_0^t (t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}((t-\tau)^\alpha\mathrm{A})\,g(\tau,\mathrm{X}(\tau))\,dW(\tau)\right\|^p \\ &\leqslant C_p\mathrm{E}\left(\int_0^t (t-\tau)^{2\alpha-2}\left\|\mathrm{E}_{\alpha,\alpha}((t-\tau)^\alpha\mathrm{A})\right\|^2\|g(\tau,\mathrm{X}(\tau))\|^2 d\tau\right)^{\frac{p}{2}} \\ &\leqslant C_pL_\sigma^p\left(\int_0^t (t-\tau)^{2\alpha-2}\left\|\mathrm{E}_{\alpha,\alpha}((t-\tau)^\alpha\mathrm{A})\right\|^2\mathrm{E}\|\mathrm{X}(\tau)\|^2 d\tau\right)^{\frac{p}{2}} \\ &\leqslant C_pL_\sigma^p\left(\int_0^t\left[(t-\tau)^{2\alpha-2}\left\|\mathrm{E}_{\alpha,\alpha}((t-\tau)^\alpha\mathrm{A})\right\|^2\right]^{\frac{p}{p-2}} d\tau\right)^{\frac{p-2}{2}}\int_0^t \mathrm{E}\|\mathrm{X}(\tau)\|^p d\tau \\ &\leqslant C_pL_\sigma^p\left(\int_0^t\left[\tau^{2\alpha-2}\left\|\mathrm{E}_{\alpha,\alpha}(\tau^\alpha\mathrm{A})\right\|^2\right]^{\frac{p}{p-2}} d\tau\right)^{\frac{p-2}{2}}\int_0^t \mathrm{E}\|\mathrm{X}(\tau)\|^p d\tau \underset{t\to\infty}{\to} 0 \end{aligned}$$

Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $\delta < \varepsilon$ satisfies $2^{p-1}C_pL_\sigma^p\mathrm{M}^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}\varepsilon < \varepsilon$, if $\mathrm{X}(t) = \mathrm{X}(t,p)$ is solution of (3.1) with $\|\rho\|^p < \delta$, then $(\Psi\mathrm{X})(t) = \mathrm{X}(t)$ satisfies $\mathrm{E}\|\mathrm{X}(t)\|^p < \varepsilon$ for every $t \geqslant 0$. If there exists $\hat{t}$ such that $\mathrm{E}\|\mathrm{X}(\hat{t})\|^p = \varepsilon$ and $\mathrm{E}\|\mathrm{X}(\tau)\|^p < \varepsilon$ for $\tau \in [0,\hat{t}]$. Then,

$$\mathrm{E}\|\mathrm{X}(\hat{t})\|^p < 2^{p-1}C_pL_\sigma^p\mathrm{M}^p\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{p}{2}}\varepsilon < \varepsilon$$

Therefore, the solution of (3.1) is stable in the pth moment. □

Example 4.5. Consider the following fractional stochastic differential system:

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}X(t) = \mathrm{A}\,X(t) + g(\mathrm{X},t)\,\dfrac{dW(t)}{dt} \\ \mathrm{X}(0) = \mathrm{X}_0 > 0 \end{cases} \tag{4.2}$$

Where $\alpha = \frac{1}{2}$, $g(\mathrm{X},t) = \begin{pmatrix} 0 & \sin X_1(t) \end{pmatrix}^T \in \mathbb{R}^2$, $\mathrm{A} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$. First, we find the eigenvalues of A:

$$|\mathrm{A} - \lambda\mathrm{I}| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = -1 \ and\ \lambda_2 = -2$$

$$|\arg\lambda_i| = \pi > \frac{\alpha\pi}{2} = \frac{\pi}{4}$$

By remark 4.1 we can conclude that, the trivial solution of FSDS (4.2) is asymptotically stable. Using Python (Matplotlib), the solution trajectories for $\alpha = \frac{1}{2}$ are shown in the following Figure 1.

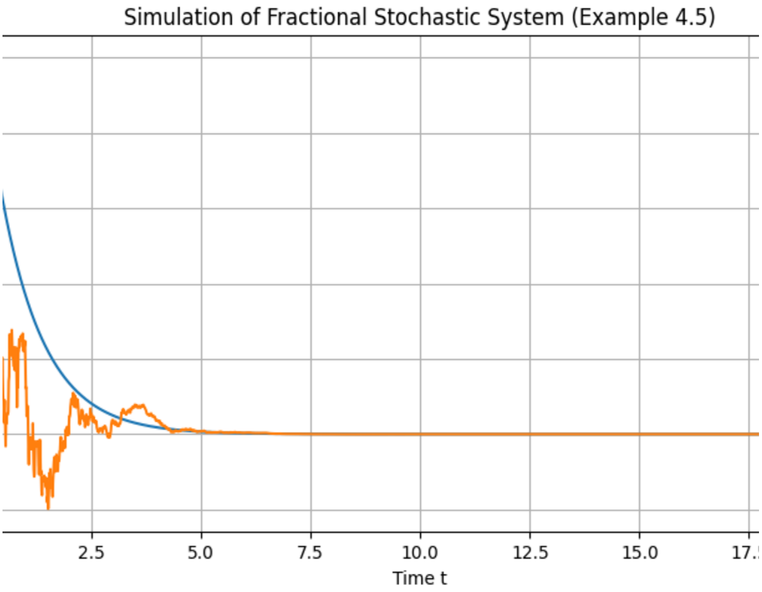

Figure 1. Asymptotically stable when $\alpha = \frac{1}{2}$.

Example 4.6. Consider the following fractional stochastic differential system:

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}X(t) = \mathrm{A}\,X(t) + g(\mathrm{X}(t),t)\,\dfrac{dW(t)}{dt} \\ \mathrm{X}(0) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T \end{cases} \tag{4.3}$$

with $\alpha = \frac{1}{2}$, $t \in [0,1]$, $A = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \in \mathbb{R}^{3\times 3}$ and

$$g\left(X(t), t\right) = \frac{1}{e^t + 39} \begin{pmatrix} \frac{|x_1(t)|}{1+|x_1(t)|} + \frac{1}{1+t} \sin(t) \\ \frac{|x_2(t)|}{1+|x_2(t)|} + \frac{1}{1+t} \cos(t) \\ 0 \end{pmatrix} \in \mathbb{R}^2$$

First, we find the eigenvalues of A:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 2 & 0 \\ 0 & -3-\lambda & 0 \\ 1 & 0 & -5-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = -1 \ \& \ \lambda_2 = -3 \ \& \ \lambda_3 = -5$$

$$|\arg \lambda_i| = \pi > \frac{\alpha\pi}{2} = \frac{\pi}{4} \quad ; \ i = 1,2,3$$

$$g\left(X(t), t\right) - g\left(Y(t), t\right) = \frac{1}{e^t + 39} \begin{pmatrix} \frac{|x_1(t)|}{1+|x_1(t)|} + \frac{1}{1+t} \sin(t) - \frac{|y_1(t)|}{1+|y_1(t)|} - \frac{1}{1+t} \sin(t) \\ \frac{|x_2(t)|}{1+|x_2(t)|} + \frac{1}{1+t} \cos(t) - \frac{|y_2(t)|}{1+|y_2(t)|} - \frac{1}{1+t} \cos(t) \\ 0 \end{pmatrix}$$

$$g\left(X(t), t\right) - g\left(Y(t), t\right) = \frac{1}{e^t + 39} \begin{pmatrix} \frac{|x_1(t)|}{1+|x_1(t)|} - \frac{|y_1(t)|}{1+|y_1(t)|} \\ \frac{|x_2(t)|}{1+|x_2(t)|} - \frac{|y_2(t)|}{1+|y_2(t)|} \\ 0 \end{pmatrix}$$

then,

$$\begin{aligned} &\|g\left(X(t), t\right) - g\left(Y(t), t\right)\|^2 = \left(\frac{1}{e^t + 39}\right)^2 \\ &\times \left[\left(\frac{|x_1(t)|}{1+|x_1(t)|} - \frac{|y_1(t)|}{1+|y_1(t)|}\right)^2 + \left(\frac{|x_2(t)|}{1+|x_2(t)|} - \frac{|y_2(t)|}{1+|y_2(t)|}\right)^2\right] \\ &= \left(\frac{1}{e^t + 39}\right)^2 \\ &\quad \times \left(\frac{|x_1(t)| - |y_1(t)|}{(1+|x_1(t)|)(1+|y_1(t)|)}\right)^2 + \left(\frac{1}{e^t + 39}\right)^2 \left(\frac{|x_2(t)| - |y_2(t)|}{(1+|x_2(t)|)(1+|y_2(t)|)}\right)^2 \end{aligned}$$

But we know that $\left|\, |x(t)| - |y(t)| \,\right| \leqslant |x(t) - y(t)|$, then,

$$\|g\left(X(t), t\right) - g\left(Y(t), t\right)\|^2 \leqslant \frac{1}{1600} (|x_1 - y_1|)^2 + \frac{1}{1600} (|x_2 - y_2|)^2 = \frac{1}{1600} \|X - Y\|^2$$

Therefore,

$$\|g\left(X(t), t\right) - g\left(Y(t), t\right)\| \leqslant \frac{1}{40} \|X - Y\|$$

Hence, the Lipschitz condition is hold with $L = \frac{1}{40}$. Thus, according to Theorem 3.5 the solution of FSDS (4.3) is stable in the pth moment. Using Python (Matplotlib), the solution trajectories for $\alpha = \frac{1}{2}$ are shown in Figs 2,3 and 4.

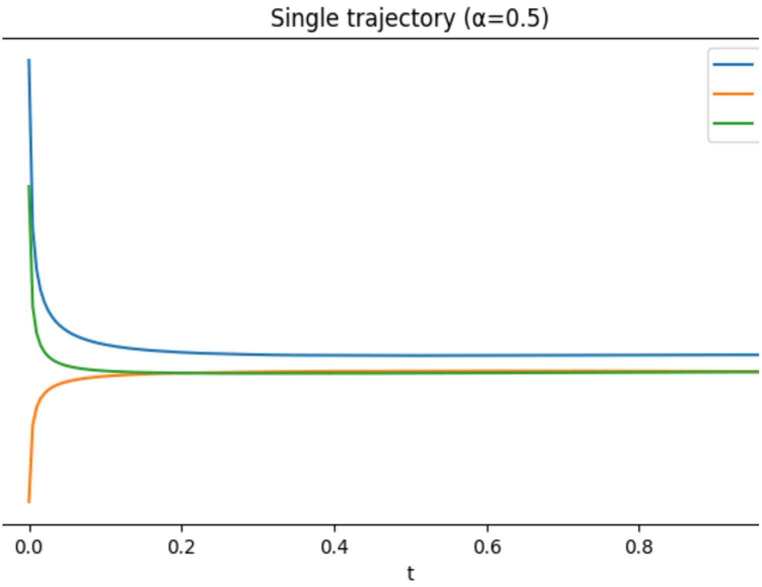

Figure 2. Single trajectory of the FSDS components $(X_1(t), X_2(t), X_3(t))$.

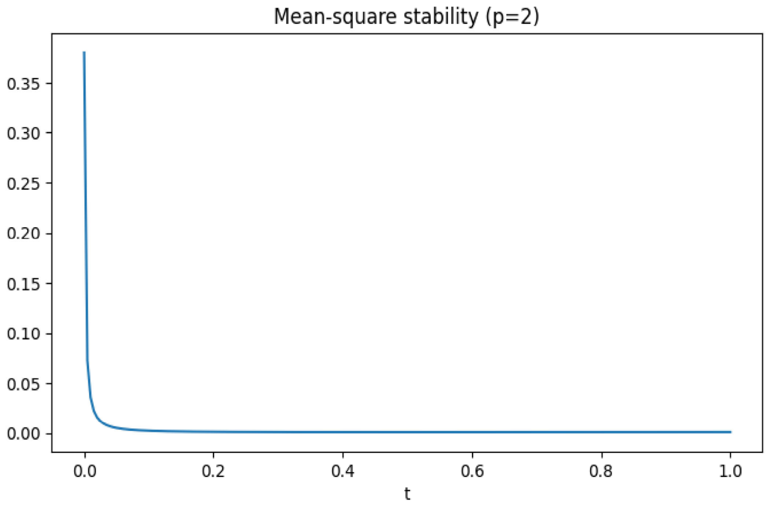

Figure 3. Mean-Square stability curve showing decay over time.

5 Conclusion

In this paper, we studied the stability of fractional stochastic differential systems (FSDS). The fixed point method was employed to establish the existence and uniqueness of solutions, while the contraction mapping principle was used to prove stability conditions. A numerical example was provided to demonstrate the applicability of the theoretical results. The main contributions of this work are:

- A new modification of the fixed point method applied to FSDS.
- Clear conditions for the existence, uniqueness, and stability of solutions.
- An illustrative example verifying the theoretical framework.

Limitations and Future Work:

- The analysis is restricted to systems with certain Lipschitz and boundedness assumptions; relaxing these conditions could be an interesting direction.
- Extending the approach to multi-dimensional systems with more general noise structures (e.g., Levy noise of fractional Brownian motion) would be valuable.
- Further exploration of numerical schemes tailored for FSDS stability analysis is recommended.

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Author information

E. Hussen, PhD Student, Department of Mathematics, College of Science, Homs University, Syria.
E-mail: `e.hussain@homs-univ.edu.sy`

S. Alargeh, Department of Mathematics, College of Science, Homs University, Syria.
E-mail: `salarje@homs-univ.edu.sy`

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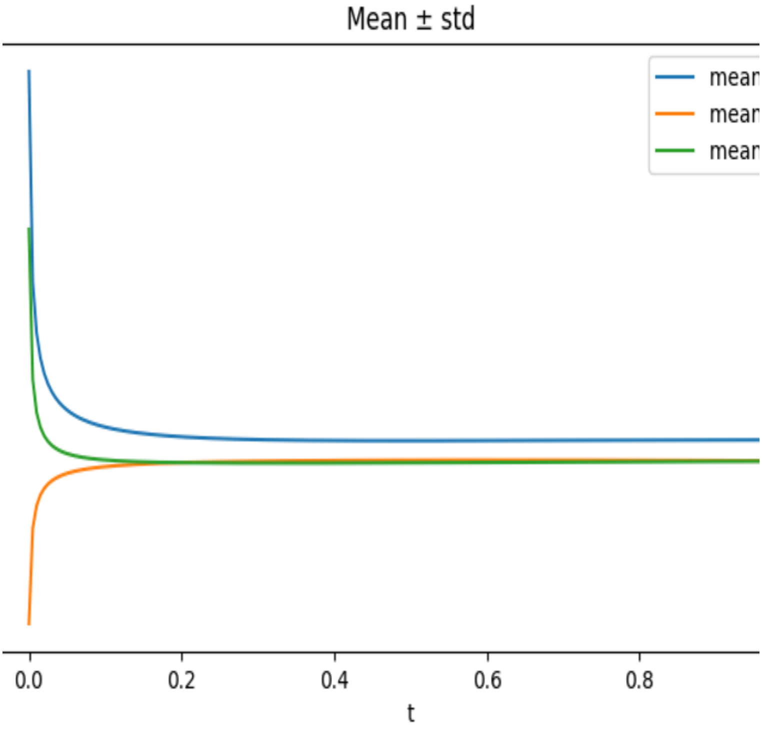

Figure 4. Component-wise mean ± standard deviation across 200 trajectories