

On ϕ - S -semiprime submodules

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Abstract. Let S be a multiplicatively closed subset of R and P be a submodule of an R -module M . Let $S(M)$ be the set of all submodules of M and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. In this paper, we define the concept of ϕ - S -semiprime submodule which is a generalization of S -semiprime submodule. Then we investigate the behavior of this structure under module homomorphisms, localizations, quotient modules, cartesian product of modules and idealizations. We investigate some properties of this class of submodules. Also, some characterizations of ϕ - S -semiprime submodules are given. Finally, we give a characterization of this class of submodules in term of flat modules.

1 Introduction

Throughout this paper R is a commutative ring with non-zero identity and M is a unitary R -module. Many generalizations of prime (resp., primary) ideals are studied in [3, 4, 6, 14]. The class of prime submodules as a generalization of the class of prime ideals has been studied by many authors. For example see [12, 17, 18, 20, 23, 27, 30]. Then many generalizations of prime submodules were studied such as weakly prime (resp. primary) submodules in [9, 22], almost prime (resp. primary) submodules in [15], 2-absorbing submodules in [29], classical prime (resp. primary) submodules in [10], semiprime submodules in [24] and ϕ -semiprime submodules in [11]. Recently, the notion of S -prime submodules, S -primary submodules and S -semiprime submodules which are generalizations of prime submodules have been introduced and studied in [8, 25, 26], respectively. For further study on these concepts see [2, 16, 19, 21, 28].

Let I be an ideal of R and N be a submodule of an R -module M . Then \sqrt{I} denotes the radical of I and $(N :_R M) = \{r \in R \mid rM \subseteq N\}$, which is clearly an ideal of R . Let S be a non-empty subset of R . We call S is a *multiplicatively closed subset* of R , if $0 \notin S$, $1 \in S$ and $ss' \in S$, for all $s, s' \in S$. An R -module M is said to be *S -finite* (resp. *S -principal*) if $sM \subseteq N \subseteq M$ for some finitely generated (resp. principal) submodule N of M and some $s \in S$. Also, M is called *S -Noetherian* if each submodule of M is S -finite. A ring R is called *S -Noetherian* if it is S -Noetherian as an R -module. The concept of S -Noetherian rings is a generalization of the concept of Noetherian rings (see [5]). Let S be a multiplicatively closed subset of R , I be an ideal of R and N be a submodule of the R -module M . We denoted the ring of fractions of R by S (resp. module of fractions of M by S) by R_S (resp. M_S). We use $Sat_S(N, M) := M \cap N_S$ and $Sat_S(I) := R \cap I_S$. Also, we use $(N :_M s) := \{m \in M \mid sm \in N\}$ for some $s \in S$.

Let S be a multiplicatively closed subset of R and P be a submodule of an R -module M with $(P :_R M) \cap S = \emptyset$. We say that P is a *S -semiprime submodule* of an R -module M whenever, if there exists a fixed $s \in S$ and whenever $r^k m \in P$ for some $r \in R$, $m \in M$ and $k \in \mathbb{N}$, then $sr^k m \in P$. We will denote the set of all submodules of M by $S(M)$. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. A proper submodule P of M is called *ϕ -semiprime* if whenever $r \in R$ and $x \in M$ with $r^2 x \in P \setminus \phi(P)$ implies $rx \in P$. We use the functions $\phi_\emptyset(N) = \emptyset$ for semiprime submodules, $\phi_0(N) = 0$ for *weakly semiprime submodules*, $\phi_1(N) = N$ for any submodule,

$\phi_2(N) = (N :_R M)N$ for almost semiprime submodules, $\phi_n(N) = (N :_R M)^{(n-1)}N$ ($n \geq 3$), for n -almost semiprime submodules (see [11]).

In this paper, we introduce the concept of ϕ - S -semiprime submodule as a generalization of S -semiprime submodule. In section 2, we show that under some conditions a ϕ - S -semiprime submodule is an S -semiprime submodule. Then we give several characterizations of ϕ - S -semiprime submodules. Also, we will characterize ϕ -semiprime submodules in terms of ϕ - S -semiprime submodules.

In section 3, we investigate the behavior of ϕ - S -semiprime submodules under module homomorphisms, localizations, quotient modules, cartesian product of modules and idealizations. Also, we give some characterizations of such submodules in (finitely generated faithful) multiplication modules. Furthermore, we give some characterizations of n -almost S -semiprime submodules ($n \geq 2$) in (finitely generated faithful) multiplication modules.

Finally, in section 4, it is proved that if F is a flat R -module and P is an n -almost S -semiprime submodule of M such that $(F \otimes_R N :_R F \otimes_R M) \cap S = \emptyset$, then $F \otimes_R N$ is an n -almost S -semiprime submodule of $F \otimes_R M$. Moreover, if F is a faithfully flat R -module, then P is an n -almost S -semiprime submodule of M if and only if $F \otimes_R N$ is an n -almost S -semiprime submodule of $F \otimes_R M$.

2 ϕ - S -semiprime Submodules

Definition 2.1. Let S be a multiplicatively closed subset of R and P be a submodule of an R -module M with $(P : M) \cap S = \emptyset$. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. We say that P is a ϕ - S -semiprime submodule of an R -module M , whenever if there exists $s \in S$ and whenever $r^k x \in P \setminus \phi(P)$ for some $r \in R$, $x \in M$ and $k \in \mathbb{N}$, then $rsx \in P$. Since $P \setminus \phi(P) = P \setminus (\phi(P) \cap P)$, without loss of generality throughout this paper we will assume that $\phi(P) \subseteq P$. For two functions $\phi_1, \phi_2 : S(M) \rightarrow S(M) \cup \{\emptyset\}$ we write $\phi_1 \leq \phi_2$ if $\phi_1(P) \subseteq \phi_2(P)$ for each $P \in S(M)$.

In the rest of the paper, we use the function $\phi_\emptyset(P) = \emptyset$ for S -semiprime submodules, $\phi_0(P) = 0$ for weakly S -semiprime submodules, $\phi_1(P) = P$ for any submodules, $\phi_2(P) = (P :_R M)P$ for almost S -semiprime submodules, $\phi_n(P) = (P :_R M)^{n-1}P$, ($n \geq 3$) for n -almost S -semiprime submodules and $\phi_\omega(P) = \bigcap_{i=1}^\infty (P : M)^i P$ for ω -semiprime submodules. It is clear that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \dots \leq \phi_{n+1} \leq \phi_n \leq \dots \leq \phi_2 \leq \phi_1$.

Note that every S -semiprime submodule of an R -module M is a ϕ - S -semiprime submodule. But the converse is not true in general. For example consider the \mathbb{Z} -module $M = \mathbb{Z}_{24}$ and the submodule $P = 8\mathbb{Z}_{24}$. Let $S = \mathbb{Z} \setminus 2\mathbb{Z}$ and $\phi = \phi_2$. Since $\phi_2(P) = (P :_{\mathbb{Z}} M)P = P$, P is a ϕ - S -semiprime submodule of M . But P is not an S -semiprime submodule of M . Because $2^2 \times 2 \in P$, but $s \times 2 \times 2 \notin P$ for all $s \in S$. In the following, we show that under some conditions ϕ - S -semiprime submodules are S -semiprime submodules.

Theorem 2.2. Let S be a multiplicatively closed subset of R and M be an R -module. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function and P be a ϕ - S -semiprime submodule of M . If $x + (P :_R M) \not\subseteq \sqrt{(\phi(P) :_R P)}$ for all $x \in R \setminus (P :_R M)$, then P is an S -semiprime submodule of M .

Proof. Let P be a ϕ - S -semiprime submodule of M and s be an element of S such that whenever $r^t m \in P \setminus \phi(P)$ for some $r \in R$, $m \in M$ and $t \in \mathbb{N}$, then $sr^t m \in P$. Now assume that $r^k m \in P$ for some $r \in R$, $m \in M$ and $k \in \mathbb{N}$. Let $r^k m \notin \phi(P)$. Since P is a ϕ - S -semiprime submodule of M , $rs^k m \in P$. Now assume that $r^k m \in \phi(P)$. Let $r^k P \not\subseteq \phi(P)$. Then $r^k n \notin \phi(P)$ for some $n \in P$. It follows that $r^k(m+n) \in P \setminus \phi(P)$. Since P is a ϕ - S -semiprime submodule, $rs(m+n) \in P$ and so $rs^k m \in P$. Hence we can assume that $r^k P \subseteq \phi(P)$. If $(r+r_0)^k m \notin \phi(P)$ for some $r_0 \in (P :_R M)$, then $(r+r_0)^k m \in P \setminus \phi(P)$ and so $(r+r_0)s^k m \in P$. It follows that $rs^k m \in P$. Now we can assume that $(r+(P :_R M))^k m \subseteq \phi(P)$. Since $r+(P :_R M) \not\subseteq \sqrt{(\phi(P) :_R P)}$, there exist $r' \in (P :_R M)$ and $n \in P$ such that $(r+r')^k n \notin \phi(P)$. Then $(r+r')^k(n+m) \in P \setminus \phi(P)$. So $(r+r')s^k(n+m) \in P$. Therefore, $rs^k m \in P$ and hence P is an S -semiprime submodule of M . □

Corollary 2.3. *Let S be a multiplicatively closed subset of R and M be an R -module. Let P be an n -almost S -semiprime submodule of M ($n \geq 2$). If $r \notin \sqrt{((P :_R M)^{(n-1)}P :_R P)}$ for all $r \in R \setminus (P :_R M)$, then P is an S -semiprime submodule of M .*

Proof. Let $r \notin \sqrt{((P :_R M)^{(n-1)}P :_R P)}$ for all $r \in R \setminus (P :_R M)$. Since $(P :_R M) \subseteq \sqrt{((P :_R M)^{(n-1)}P :_R P)}$, $r + (P :_R M) \not\subseteq \sqrt{((P :_R M)^{(n-1)}P :_R P)}$ for all $r \in R \setminus (P :_R M)$. So by Theorem 2.2, P is an S -semiprime submodule of M \square

Corollary 2.4. *Let S be a multiplicatively closed subset of R and M be an R -module. Let P be a weakly S -semiprime submodule of M which is not S -semiprime. Then $r^k P = 0$ for some $r \in R \setminus (P :_R M)$ and $k \in \mathbb{N}$.*

Proof. Let P be a weakly S -semiprime submodule of M which is not S -semiprime. By Theorem 2.2, $r + (P :_R M) \subseteq \sqrt{(0 :_R P)}$ for some $r \in R \setminus (P :_R M)$. Then $r \in \sqrt{(0 :_R P)}$ and hence $r^k P = 0$ for some $r \in R$ and $k \in \mathbb{N}$. \square

Proposition 2.5. *Let S be a multiplicatively closed subset of R and M be an R -module. Let Rx be a non-zero proper submodule of M such that $(0 :_R x) = 0$ and $\sqrt{(Rx :_R M)} = (Rx :_R M)$. If Rx is an n -almost S -semiprime submodule of M ($n \geq 2$), then Rx is an S -semiprime submodule of M .*

Proof. Let Rx be an n -almost S -semiprime submodule of an R -module M . Assume that $r \in \sqrt{((Rx :_R M)^{(n-1)}Rx :_R Rx)}$ for some $r \in R \setminus (Rx :_R M)$. Then $r^k Rx \subseteq (Rx :_R M)^{(n-1)}Rx$ for some $k \in \mathbb{N}$. So $r^k x = r'x$ for some $r' \in (Rx :_R M)$. Since $(0 :_R x) = 0$, $r^k = r' \in (P :_R M)^{(n-1)} \subseteq (Rx :_R M)$. Then $r \in \sqrt{(Rx :_R M)} = (Rx :_R M)$, which is a contradiction. So by Corollary 2.3, Rx is an S -semiprime submodule of M . \square

Let S be a multiplicatively closed subset of R and P be a submodule of an R -module M such that $(P :_R M) \cap S = \emptyset$. It is clear that if P is a ϕ -semiprime submodule of M , then P is a ϕ - S -semiprime submodule of M . But the converse is not true in general. In the following, we investigate some conditions in which the converse is true. We denote the set of all maximal ideal of R by $Max(R)$.

Theorem 2.6. *Let P be a submodule of an R -module M such that $(P :_R M) \subseteq Jac(R)$, where $Jac(R)$ is the Jacobson radical of R . Then the following are equivalent:*

- (i) P is a ϕ -semiprime submodule of M .
- (ii) P is a ϕ - $(R \setminus m)$ -semiprime submodule of M for each $m \in Max(R)$.

Proof. (1) \Rightarrow (2) Let P be a ϕ - S -semiprime submodule of M . Since $(P :_R M) \subseteq Jac(R)$, $(P :_R M) \cap (R \setminus m) = \emptyset$ for each $m \in Max(R)$ and hence P is a ϕ - $(R \setminus m)$ -semiprime submodule of M .

(2) \Rightarrow (1) Assume that P is a ϕ - $(R \setminus m)$ -semiprime submodule of M , for each $m \in Max(R)$. Let $m \in Max(R)$ and $r^k x \in P \setminus \phi(P)$ for some $r \in R$, $x \in M$ and $k \in \mathbb{N}$. Since P is a ϕ - $(R \setminus m)$ -semiprime submodule of M , there exists an $s_m \in R \setminus m$ such that $rs_mx \in P$. Since elements s_m ($m \in Max(R)$) generate the unit ideal, $\sum_{i=1}^k r_i s_{m_i} = 1$ for some $r_i \in R$ and $s_{m_i} \in R \setminus m_i$ ($1 \leq i \leq k$). Then $\sum_{i=1}^k r_i s_{m_i} r x = r x \in P$ and hence P is an ϕ - S -semiprime submodule of M . \square

In the next theorem we give several characterizations of ϕ - S -semiprime submodules.

Theorem 2.7. *Let S be a multiplicatively closed subset of R and M be an R -module. Let P be a proper submodule of M such that $(P :_R M) \cap S = \emptyset$. Then the following are equivalent.*

- (i) P is a ϕ - S -semiprime submodule of M .
- (ii) There exists $s \in S$ such that $(P :_M \langle r^k \rangle) \subseteq (\phi(P) :_M \langle r^k \rangle) \cup (P :_M \langle sr \rangle)$ for all $r \in R$ and $k \in \mathbb{N}$.

(iii) There exists $s \in S$ such that $(P :_M \langle r^k \rangle) \subseteq (\phi(P) :_M \langle r^k \rangle)$ or $(P :_M \langle r^k \rangle) \subseteq (P :_M \langle sr \rangle)$ for all $r \in R$ and $k \in \mathbb{N}$.

Proof. (1) \Rightarrow (2) Since P is a ϕ - S -semiprime submodule of M , there exists $s \in S$ and whenever $r^k m \in P \setminus \phi(P)$ for some $r \in R, m \in M$ and $k \in \mathbb{N}$, then $rs m \in P$. Let $r \in R$ and $m \in (P :_M \langle r^k \rangle)$. Then $r^k m \in P$. Let $r^k m \notin \phi(P)$. Since P is a ϕ - S -semiprime submodule, $rs m \in P$. So $m \in (P :_M \langle sr \rangle)$. Now let $r^k m \in \phi(P)$. Then $m \in (\phi(P) :_M \langle r^k \rangle)$ and hence $(P :_M \langle r^k \rangle) \subseteq (\phi(P) :_M \langle r^k \rangle) \cup (P :_M \langle sr \rangle)$.

(2) \Rightarrow (3) It is clear.

(3) \Rightarrow (1) Let $r^k m \in P \setminus \phi(P)$ for some $r \in R, m \in M$ and $k \in \mathbb{N}$. Then $m \in (P :_M \langle r^k \rangle)$ and $m \notin (\phi(P) :_M \langle r^k \rangle)$. So by assumption, $m \in (P :_M \langle sr \rangle)$. Then $rs m \in P$ and hence P is a ϕ - S -semiprime submodule of M . □

Corollary 2.8. Let S be a multiplicatively closed subset of R and P be a submodule of an R -module M such that $(P :_R M) \cap S = \emptyset$. Then P is an n -almost S -semiprime submodule of M ($n \geq 2$) if and only if there exists $s \in S$ such that whenever $\langle a^k \rangle N \subseteq P$ and $\langle a^k \rangle N \not\subseteq (P : M)^{(n-1)} P$, then $\langle sa \rangle N \subseteq P$ for all submodules N of $M, a \in R$ and $k \in \mathbb{N}$.

Proof. It follows by Theorem 2.7. □

Theorem 2.9. Let S be a multiplicatively closed subset of R and P be a proper submodule of an R -module M . If P is a ϕ - S -semiprime submodule of M , then there exists $s \in S$ such that $\sqrt{(P :_R x)} \subseteq \sqrt{(\phi(P) :_R x)}$ or $\sqrt{(P :_R x)} \subseteq (P :_R sx)$ for all $x \in M \setminus P$.

Proof. Let $x \in M \setminus P$ and $a \in \sqrt{(P :_R x)}$. Then $a^n x \in P$ for some $n \in \mathbb{N}$. If $a^n x \in \phi(P)$, then $a \in \sqrt{(\phi(P) :_R x)}$. Now let $a^n x \in P \setminus \phi(P)$. Since P is a ϕ - S -semiprime submodule of M , there exists $s \in S$ such that $asx \in P$. It follows that $a \in (P :_R sx)$. So $\sqrt{(P :_R x)} \subseteq \sqrt{(\phi(P) :_R x)} \cup (P :_R sx)$ and hence $\sqrt{(P :_R x)} \subseteq \sqrt{(\phi(P) :_R x)}$ or $\sqrt{(P :_R x)} \subseteq (P :_R sx)$. □

3 ϕ - S -semiprime submodules of some modules

Let S be a multiplicatively closed subset of a ring R . The saturation S^* of S is defined as $S^* = \{x \in R \mid x/1 \text{ is a unit of } R_S\}$. Note that S^* is a multiplicatively closed subset of R containing S .

Definition 3.1. Let S be a multiplicatively closed subset of R, M be an R -module and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. We define $\phi_S : S(M_S) \rightarrow S(M_S) \cup \{\emptyset\}$ by $\phi_S(N_S) = (\phi(Sat_S(N, M)))_S$ if $\phi(Sat_S(N, M)) \neq \emptyset$ and $\phi_S(N_S) = \emptyset$ if $\phi(Sat_S(N, M)) = \emptyset$ for every $N \in S(M)$.

Note that since $\phi(Sat_S(N, M)) \subseteq Sat_S(N, M)$ and $(Sat_S(N, M))_S = N_S, \phi_S(N_S) = (\phi(Sat_S(N, M)))_S \subseteq (Sat_S(N, M))_S = N_S$.

Theorem 3.2. Let S be a multiplicatively closed subset of R and M be an R -module.

- (i) Let $S_1 \subseteq S_2$ be multiplicatively closed subsets of R . If P is a ϕ - S_1 -semiprime submodule and $(P :_R M) \cap S_2 = \emptyset$, then P is a ϕ - S_2 -semiprime submodule.
- (ii) A proper submodule P of M is a ϕ - S -semiprime submodule if and only if it is a ϕ - S^* -semiprime submodule of M .
- (iii) Let $P_S \neq M_S$ and $(\phi(P))_S \subseteq \phi_S(P_S)$ (note that it is true when $\phi = \phi_n$). If P is a ϕ - S -semiprime submodule of M , then P_S is a ϕ_S -semiprime submodule of M_S .

Proof. (1) It is clear by definition.

(2) Let P be a ϕ - S -semiprime submodule. Assume that $(P :_R M) \cap S^* \neq \emptyset$ and let $r \in (P :_R M) \cap S^*$. Since $\frac{r}{1}$ is a unit of $R_S, \frac{r}{1} \frac{a}{s} = \frac{1}{1}$ for some $a \in R$ and $s \in S$. Thus $us = ura \in S \cap (P :_R M)$ for some $u \in S$, which is a contradiction. So $(P :_R M) \cap S^* = \emptyset$ and hence by part (1), P is a ϕ - S^* -semiprime submodule of M . Conversely assume that $r^k x \in P \setminus \phi(P)$ for some $r \in R, x \in M$ and $k \in \mathbb{N}$. Since P is a ϕ - S^* -semiprime submodule, there exists $s \in S^*$

such that $rsx \in P$. Since $\frac{s}{1}$ is a unit of R_S , $us' = usa$ for some $a \in R$ and $u, s' \in S$. Set $s'' := us'$. Then $s''rx = us'rx = usarx \in P$. Hence P is a ϕ - S -semiprime submodule of M .

(3) Let P be a ϕ - S -semiprime submodule of M and $(\frac{r}{s})^k(\frac{x}{s'}) \in P_S \setminus \phi_S(P_S)$ for some $s, s' \in S, r \in R, x \in M$ and $k \in \mathbb{N}$. Since $(\phi(P))_S \subseteq \phi_S(P_S)$ and $\frac{r^k x}{s^k s'} \in P_S \setminus \phi_S(P_S)$, $ur^k x \in P \setminus \phi(P)$ for some $u \in S$. Since P is a ϕ - S -semiprime submodule of M , $turx \in P$ for some $t \in S$. Then $\frac{rx}{ss'} = \frac{turx}{tuss'} \in P_S$. So P_S is a ϕ_S -semiprime submodule of M_S . \square

Corollary 3.3. *Let S be a multiplicatively closed subset of R and M be an S -finite R -module. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function such that ϕ preserves the subset relation. Then if P is a ϕ - S -semiprime submodule of M , then P_S is a ϕ_S -semiprime submodule of M_S .*

Proof. Since $(P :_R M) \cap S = \emptyset$ and M is an S -finite R -module, $P_S \neq N_S$. Also Since $P \subseteq \text{Sat}_S(P, M)$ and ϕ preserves the subset relation, $\phi(P) \subseteq \phi(\text{Sat}_S(P, M))$. Then $(\phi(P))_S \subseteq (\phi(\text{Sat}_S(P, M)))_S = \phi_S(P_S)$. Therefore, by Theorem 3.2, P_S is a ϕ_S -semiprime submodule of M_S . \square

Lemma 3.4. *Let S be a multiplicatively closed subset of R and M be an S -finite R -module. Let P be a submodule of M . Then $(\phi_n)_S(P_S) = (\phi_n(P))_S$.*

Proof. Let M be an S -finite R -module and P be a submodule of M . By Definition 3.1, we have $(\phi_n)_S(P_S) = (\phi_n(\text{Sat}_S(P, M)))_S = (\text{Sat}_S(P, M) :_R M)_S^{n-1}(\text{Sat}_S(P, M))_S (n \geq 2)$. Since M is an S -finite R -module, $(\phi_n)_S(P_S) = ((\text{Sat}_S(P, M))_S :_{R_S} M_S)^{n-1}(\text{Sat}_S(P, M))_S$. Since $P_S = (\text{Sat}_S(P, M))_S$, $(\phi_n)_S(P_S) = (P_S :_{R_S} M_S)^{n-1}P_S = (\phi_n(P))_S$. \square

Corollary 3.5. *Let S be a multiplicatively closed subset of R and M be an S -finite R -module. If P is a ϕ_n - S -semiprime submodule of $M (n \geq 2)$, then P_S is a $(\phi_n)_S$ -semiprime submodule of M_S .*

Proof. It follows by Lemma 3.4 and Theorem 3.2(3). \square

Proposition 3.6. *Let S be a multiplicatively closed subset of R and $f : M \rightarrow M'$ be an epimorphism of R -modules. If P is an n -almost S -semiprime submodule of M containing $\text{Ker}(f)$ ($n \geq 2$), then $f(P)$ is an n -almost S -semiprime submodule of M' .*

Proof. Let $s \in (f(P) :_R M')$ for some $s \in S$. Then $sM' \subseteq f(P)$. Since ϕ is an epimorphism, $f(sM) = sf(M) = sM' \subseteq f(P)$. Then $sM \subseteq P + \text{Ker}(f) = P$. So $s \in (P :_R M)$, which is a contradiction. Hence $(f(P) :_R M') \cap S = \emptyset$. Now let $r^k x \in f(P) \setminus (f(P) :_R M')^{n-1}f(P)$ for some $r \in R, x \in M'$ and $k \in \mathbb{N}$. Since ϕ is an epimorphism, $x = f(m)$ for some $m \in M$ and $r^k x = r^k f(m) = f(r^k m) \in f(P)$. So $f(r^k m) = f(p)$ for some $p \in P$. Since $\text{Ker}(f) \subseteq P$ and $r^k m - p \in \text{Ker}(f)$, $r^k m \in P$. If $r^k m \in (P :_R M)^{n-1}P$, then $r^k x = f(r^k m) \in (P :_R M)^{n-1}f(P) \subseteq (f(P) :_R M')^{n-1}f(P)$, which is a contradiction. So $r^k m \in P \setminus (P :_R M)^{n-1}P$. Since P is an n -almost S -semiprime submodule, $rs'm \in P$ for some $s' \in S$. Then $rs'x = f(rs'm) \in f(P)$. Hence $f(P)$ is an n -almost S -semiprime submodule of M' . \square

Let S be a multiplicatively closed subset of R and M be an R -module. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. For a submodule L of M , we define $\phi_L : S(\frac{M}{L}) \rightarrow S(\frac{M}{L}) \cup \{\emptyset\}$ by $\phi_L(\frac{N}{L}) = \frac{\phi(N)+L}{L}$ if $\phi(N) \neq \emptyset$ and $\phi_L(\frac{N}{L}) = \emptyset$ if $\phi(N) = \emptyset$ for every $N \in S(M)$ with $L \subseteq N$.

Theorem 3.7. *Let S be a multiplicatively closed subset of R and L, P be submodules of an R -module M with $L \subseteq \phi(P)$. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function.*

- (i) *If P is a ϕ - S -semiprime submodule of M , then $\frac{P}{L}$ is a ϕ_L - S -semiprime submodule of $\frac{M}{L}$.*
- (ii) *If $\frac{P}{L}$ is a ϕ_L - S -semiprime submodule of $\frac{M}{L}$ and $L \subseteq P$, then P is a ϕ - S -semiprime submodule of M .*

Proof. (1) Since $(\frac{M/L}{P/L}) \cong \frac{M}{P}$, $(\frac{P}{L} :_R \frac{M}{L}) \cap S = \emptyset$ if and only if $(P :_R M) \cap S = \emptyset$. Now let $r^k x \in \frac{P}{L} \setminus \phi_L(\frac{P}{L})$ for some $r \in R, \bar{x} := x + L \in \frac{M}{L}$ and $k \in \mathbb{N}$. Then $r^k x \in P \setminus \phi(P)$. Since P

is a ϕ - S -semiprime submodule of M , $rsx \in P$ for some $s \in S$. It follows that $rs\bar{x} \in \frac{P}{L}$ and so $\frac{P}{L}$ is a ϕ_L - S -semiprime submodule of $\frac{M}{L}$.

(2) Let $\frac{P}{L}$ be a ϕ_L - S -semiprime submodule of $\frac{M}{L}$. Assume that $r^kx \in P \setminus \phi(P)$ for some $r \in R$, $x \in M$ and $k \in \mathbb{N}$. If $r^k(x + L) \in \phi_L(\frac{P}{L}) = \frac{\phi(P)+L}{L} = \frac{\phi(P)}{L}$, then $r^kx \in \phi(P)$, which is a contradiction. So $r^k(x + L) \in \frac{P}{L} \setminus \phi_L(\frac{P}{L})$. Since $\frac{P}{L}$ is a ϕ_L - S -semiprime submodule of $\frac{M}{L}$, $rs(x + L) \in \frac{P}{L}$ for some $s \in S$. Then $rsx \in P$ and hence P is a ϕ - S -semiprime submodule of M . \square

Let S be a multiplicatively closed subset of R and L be submodules of an R -module M . A submodule P of an R -module M is an S -semiprime submodule if and only if $\frac{P}{L}$ is an S -semiprime submodule of $\frac{M}{L}$ [26, Corollary 3(ii)]. But the converse may not be true in the case of ϕ - S -semiprime submodules. The next example shows that the converse of Theorem 3.7, is not true in general.

Example 3.8. Let $L = P = \langle 20 \rangle$ be submodules of the \mathbb{Z} -module \mathbb{Z} and $S = \{3^n \mid n \in \mathbb{N} \cup \{0\}\}$. Let $\phi = \phi_0$. It is clear that $\frac{P}{L}$ is a ϕ_L - S -semiprime submodules of $\frac{\mathbb{Z}}{L}$. But since $0 \neq 2^2 \times 5 \in P$ and $2 \times s \times 5 \notin P$ for all $s \in S$, P is not ϕ - S -semiprime submodules of M .

Proposition 3.9. Let S be a multiplicatively closed subset of R and P be a proper submodule of an R -module M . Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Then P is a ϕ - S -semiprime submodule of M if and only if $\frac{P}{\phi(P)}$ is a weakly S -semiprime submodules of $\frac{M}{\phi(P)}$.

Proof. Since $\frac{\frac{M}{\phi(P)}}{\frac{P}{\phi(P)}} \cong \frac{M}{P}, (\frac{P}{\phi(P)} :_R \frac{M}{\phi(P)}) \cap S = \emptyset$ if and only if $(P :_R M) \cap S = \emptyset$. Now let P be a ϕ - S -semiprime submodules of M and $0 \neq r^k(x + \phi(P)) \in \frac{P}{\phi(P)}$ for some $r \in R$, $x \in M$ and $k \in \mathbb{N}$. Then $r^kx \in P \setminus \phi(P)$ and so $rsx \in P$ for some $s \in S$. Therefore, $rs(x + \phi(P)) \in \frac{P}{\phi(P)}$ and hence $\frac{P}{\phi(P)}$ is a weakly S -semiprime submodules of $\frac{M}{\phi(P)}$. Conversely, let $\frac{P}{\phi(P)}$ be a weakly S -semiprime submodules of $\frac{M}{\phi(P)}$ and $r^kx \in P \setminus \phi(P)$ for some $r \in R$, $x \in M$ and $k \in \mathbb{N}$. Then $0 \neq r^k(x + \phi(P)) \in \frac{P}{\phi(P)}$ and so $rs'(x + \phi(P)) \in \frac{P}{\phi(P)}$ for some $s' \in S$. It follows that $rs'x \in P$ and hence P is a ϕ - S -semiprime submodule of M . \square

Note that the converse of the Theorem 3.2(3) is not true in general. For example, similar to [25, Example 2.4], consider the \mathbb{Z} -module $\mathbb{Q} \times \mathbb{Q}$. Take the submodule $N = \mathbb{Z} \times \{0\}$ and the multiplicatively closed subset $S = \mathbb{Z} \setminus \{0\}$ of \mathbb{Z} . It is clear that $(N :_{\mathbb{Z}} \mathbb{Q} \times \mathbb{Q}) = 0$. Let s be an arbitrary element of S and choose a prime number p with $\gcd(p, s) = 1$. Then $p^2(\frac{1}{p^2}, 0) = (1, 0) \in N \setminus \phi_m(N)(m \geq 2)$. Since $sp(\frac{1}{p^2}, 0) = (\frac{s}{p}, 0) \notin N$, N is not m -almost S -semiprime submodule. But since $\mathbb{Z}_S = \mathbb{Q}$ is a field, $(\mathbb{Q} \times \mathbb{Q})_S$ is a vector space and hence the proper submodule N_S is an m -almost semiprime submodule of $(\mathbb{Q} \times \mathbb{Q})_S$.

In the following, we investigate some conditions in which the converse of Theorem 3.2 (3), is true. Set $Z(M) := \{r \in R \mid rm = 0, \text{ for some non-zero } m \in M\}$.

Lemma 3.10. Let S be a multiplicatively closed subset of R and M be an S -Noetherian R -module such that $Z(M) \cap S = \emptyset$. Then P_S is a weakly semiprime submodule of M_S if and only if P is a weakly S -semiprime submodule of M .

Proof. Let P_S be a weakly semiprime submodule of M_S . Then $N_S \neq M_S$ and so $(P :_R M) \cap S = \emptyset$. Now let $0 \neq r^kx \in P$, for some $r \in R$, $x \in M$ and $k \in \mathbb{N}$. Since $Z(M) \cap S = \emptyset$, $0 \neq \frac{r^kx}{1} \in P_S$. It follows that $rx \in \text{Sat}_S(P, M)$. Since M is an S -Noetherian R -module, by [21, Lemma 2.15] $\text{Sat}_S(P, M) = (P :_M s)$ for some $s \in S$. Then $rsx \in P$. Conversely, by Theorem 3.2 (3), it is clear. \square

Proposition 3.11. Let S be a multiplicatively closed subset of R and P be a proper submodule of an S -Noetherian R -module M . Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function, $Z(\frac{M}{\phi(P)}) \cap S = \emptyset$ and $\phi_S(P_S) = (\phi(P))_S$. Then P_S is a ϕ_S -semiprime submodule of M_S if and only if P is a ϕ - S -semiprime submodules of M .

Proof. Let P_S be a ϕ_S -semiprime submodule of M_S . Then by [11, Proposition 2.15], $\frac{P_S}{\phi_S(P_S)} = \frac{P_S}{(\phi(P))_S} \cong (\frac{P}{\phi(P)})_S$ is a weakly semiprime submodule of $\frac{M_S}{\phi_S(P_S)} = \frac{M_S}{(\phi(P))_S} \cong (\frac{M}{\phi(P)})_S$ and so by Lemma 3.9, $\frac{P}{\phi(P)}$ is a weakly S -semiprime submodule of $\frac{M}{\phi(P)}$. Therefore, by Proposition 3.7, P is a ϕ - S -semiprime submodules of M . Conversely, by Theorem 3.2(3), it is clear. \square

Corollary 3.12. *Let S be a multiplicatively closed subset of R and P be a proper submodule of an S -Noetherian R -module M . Let $Z(\frac{M}{(\frac{P}{(P:R M)^{(n-1)P}})}) \cap S = \emptyset (n \geq 2)$. Then P_S is an n -almost semiprime submodule of M_S if and only if P is an n -almost S -semiprime submodules of M .*

Proof. Let P_S be an n -almost semiprime submodule of M_S . assume that $\phi = \phi_n (n \geq 2)$. Then by Lemma 3.4 and Proposition 3.10, P is an n -almost S -semiprime submodules of M . Conversely, by Theorem 3.2(3), it is clear. \square

Let M_i be an R_i -module for some $n \in \mathbb{N}$ and $i = 1, 2, \dots, n$. Assume that $M = M_1 \times M_2 \times \dots \times M_n$ and $R = R_1 \times R_2 \times \dots \times R_n$. Then M is clearly an R -module with componentwise addition and multiplication. Also, if S_i is a multiplicatively closed subset of $R_i (1 \leq i \leq n)$, then $S = S_1 \times S_2 \times \dots \times S_n$ is a multiplicatively closed subset of R . If N_i is a submodule of $M_i (1 \leq i \leq n)$, then $N = N_1 \times N_2 \times \dots \times N_n$ is a submodule of M

Let $S = S_1 \times S_2$ be a multiplicatively closed subset of $R = R_1 \times R_2$. Let $P \times M_2$ be a weakly S -semiprime submodule of $M = M_1 \times M_2$ for some submodule P of M_1 and $0 \neq r^k x_1 \in P$ for some $r \in R_1, x_1 \in M_1$ and $k \in \mathbb{N}$. Let $0 \neq x_2 \in M_2$. Then $(0, 0) \neq (r, 1)^k (x_1, x_2) \in P \times M_2$. Since $P \times M_2$ is a weakly S -semiprime submodule of M , $(r, 1)(s_1, s_2)(x_1, x_2) \in P \times M_2$ for some $(s_1, s_2) \in S$. It follows that $s_1 r x_1 \in P$. Then P is a weakly S_1 -semiprime submodule of M_1 . But if P is a weakly S_1 -semiprime submodule of M_1 , then $P \times M_2$ need not be a weakly S -semiprime submodule of M .

Next, we show that if P is a weakly S_1 -semiprime submodule of M_1 and $\{0\} \times M_2 \subseteq \phi(P \times M_2)$, then $P \times M_2$ is a ϕ - S -semiprime submodule of M .

Proposition 3.13. *Let $S = S_1 \times S_2$ be a multiplicatively closed subset of $R = R_1 \times R_2$ and $M = M_1 \times M_2$ be an $R = R_1 \times R_2$ module. Let $\phi : S(M) \rightarrow S(M) \cup \{0\}$ be a function. If P is a weakly S_1 -semiprime submodule of M_1 such that $\{0\} \times M_2 \subseteq \phi(P \times M_2)$, then $P \times M_2$ is a ϕ - S -semiprime submodule of M .*

Proof. Let $(r_1, r_2)^k (x_1, x_2) \in (P \times M_2) \setminus \phi(P \times M_2)$ for some $(r_1, r_2) \in R, (x_1, x_2) \in M$ and $k \in \mathbb{N}$. Since $(P \times M_2) \setminus \phi(P \times M_2) \subseteq (P \times M_2) \setminus (\{0\} \times M_2) = (P \setminus \{0\}) \times M_2$, $(r_1, r_2)^k (x_1, x_2) \in (P \setminus \{0\}) \times M_2$. Since P is a weakly S_1 -semiprime submodule of M_1 , $r_1 s_1 m_1 \in P$ for some $s_1 \in S_1$. Thus $(r_1, r_2)(s_1, 1)(x_1, x_2) \in P \times M_2$ and hence $P \times M_2$ is a ϕ - S -semiprime submodule of M . \square

Proposition 3.14. *With the same notation of Proposition 3.13, let $\phi : S(M) \rightarrow S(M) \cup \{0\}$ be a function such that $\phi_\omega \leq \phi$. If P is a weakly S_1 -semiprime submodule of M_1 , then $P \times M_2$ is a ϕ - S -semiprime submodule of M .*

Proof. Since $\{0\} \times M_2 \subseteq (P \times M_2 :_R M)^i (P \times M_2) = ((P :_R M_2)^i P) \times M_2$ for all $i \geq 1$, $\{0\} \times M_2 \subseteq \bigcap_{i=1}^\infty (P \times M_2 :_R M)^i (P \times M_2) = \phi_\omega(P \times M_2) \subseteq \phi(P \times M_2)$. So by Proposition 3.13, $P \times M_2$ is a ϕ - S -semiprime submodule of M . \square

Proposition 3.15. *Let $S = S_1 \times S_2 \times \dots \times S_n$ be a multiplicatively closed subset of $R = R_1 \times R_2 \times \dots \times R_n$ and $M = M_1 \times M_2 \times \dots \times M_n$ be an $R = R_1 \times R_2 \times \dots \times R_n$ module, where M_i be an R_i -module and S_i be a multiplicatively closed subset of $R_i (1 \leq i \leq n, n \geq 2)$. Let $\phi : S(M) \rightarrow S(M) \cup \{0\}$ be a function and $P = P_1 \times P_2 \times \dots \times P_n$ be a ϕ - S -semiprime submodule of M for some submodule P_i of $M_i (1 \leq i \leq n)$. Assume that $\phi(P) = \psi_1(P_1) \times \psi_2(P_2) \times \dots \times \psi_n(P_n)$ for some function $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{0\} (1 \leq i \leq n)$. Then P_j is a ψ_j - S_j -semiprime submodule of M_j with $P_j \neq M_j$ and $(P_j :_R M_j) \cap S_j = \emptyset$, for each $j (1 \leq j \leq n)$.*

Proof. Let P be a ϕ - S -semiprime submodule of M . Let $P_j \neq M_j$ and $(P_j :_R M_j) \cap S_j = \emptyset$ for some $j (1 \leq j \leq n)$. Let $r^k x \in P_j \setminus \psi_j(P_j)$ for some $r \in R_j, x \in M_j$ and $k \in \mathbb{N}$. Then $(0, \dots, 0, r_j, 0, \dots, 0)^k (0, \dots, 0, x_j, 0, \dots, 0) \in P \setminus \phi(P)$. Since P is a ϕ - S -semiprime submodule of M , $(0, \dots, 0, r_j, 0, \dots, 0)_{s_j} (0, \dots, 0, x_j, 0, \dots, 0) \in P$ for some $s_j \in S$. Then $r_j s_j x_j \in P_j$ and hence P_j is a ψ_j - S_j -semiprime submodule of M_j . \square

Corollary 3.16. *Let $S = S_1 \times S_2 \times \cdots \times S_n$ be a multiplicatively closed subset of $R = R_1 \times R_2 \times \cdots \times R_n$ and $M = M_1 \times M_2 \times \cdots \times M_n$ be an $R = R_1 \times R_2 \times \cdots \times R_n$ module, where M_i be an R_i -module and S_i be a multiplicatively closed subset of R_i ($1 \leq i \leq n, n \geq 2$). Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function and $P = P_1 \times P_2 \times \cdots \times P_n$ be a ϕ - S -semiprime submodule of M for some submodule P_i of M_i ($1 \leq i \leq n$). Assume that $\phi(P) = \psi_1(P_1) \times \psi_2(P_2) \times \cdots \times \psi_n(P_n)$ for some function $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{\emptyset\}$ ($1 \leq i \leq n$). Then there exists at least one j ($1 \leq j \leq n$) such that P_j is a ψ_j - S_j -semiprime submodule of M_j .*

Proof. Since $(P :_R M) \cap S = ((P_1 :_{R_1} M_1) \times \cdots \times (P_n :_{R_n} M_n)) \cap S = ((P_1 :_{R_1} M_1) \cap S) \times \cdots \times ((P_n :_{R_n} M_n) \cap S)$ and P is a proper submodule of M with $(P :_R M) \cap S = \emptyset$, we have $(P_j :_{R_j} M_j) \cap S = \emptyset$ for some j ($1 \leq j \leq n$). So by Proposition 3.15, P_j is a ψ_j - S_j -semiprime submodule of M_j . □

Corollary 3.17. *Let $S = S_1 \times S_2 \times \cdots \times S_n$ be a multiplicatively closed subset of $R = R_1 \times R_2 \times \cdots \times R_n$ and $M = M_1 \times M_2 \times \cdots \times M_n$ be an $R = R_1 \times R_2 \times \cdots \times R_n$ module, where M_i be an R_i -module and S_i be a multiplicatively closed subset of R_i ($1 \leq i \leq n, n \geq 2$). Let $P = P_1 \times P_2 \times \cdots \times P_n$ be an m -almost S -semiprime submodule of M for some submodule P_i of M_i ($1 \leq i \leq n, m \geq 2$). Then P_j is an m -almost S_j -semiprime submodule of M_j for some j ($1 \leq j \leq n$).*

Proof. Since $\phi_m(P) = (P :_R M)^{(m-1)}P = (P_1 :_{R_1} M_1)^{(m-1)}P_1 \times \cdots \times (P_n :_{R_n} M_n)^{(m-1)}P_n = \phi_m(P_1) \times \cdots \times \phi_m(P_n)$, the proof follows from Proposition 3.15 and Corollary 3.16. □

Suppose that M be an R -module. The idealization $R(+M) = \{(a, m) \mid a \in R, m \in M\}$ of M is a commutative ring whose addition is componentwise and whose multiplication is defined as $(a, m)(b, n) = (ab, an + bm)$ for each $a, b \in R$ and $n, m \in M$. Let I be an ideal of R and N be a submodule of M . Then $I(+N)$ is an ideal of $R(+M)$ if and only if $IM \subseteq N$. In this case $I(+N)$ is called a homogeneous ideal of $R(+M)$. Also, if S is a multiplicatively closed subset of R and N is a submodule of M , then $S(+N) = \{(s, n) \mid s \in S, n \in N\}$ is a multiplicatively closed subset of $R(+M)$, see [7, Theorem 3.8].

Theorem 3.18. *Let M be an R -module, S be a multiplicatively closed subset of R and $I(+P)$ be a homogeneous ideal of $R(+M)$. If $I(+P)$ is an n -almost $S(+M)$ -semiprime ideal of $R(+M)$ ($n \geq 2$), then I is an n -almost S -semiprime ideal of R ($n \geq 2$). Furthermore, if $(N :_R M) \cap S = \emptyset$, then P is an n -almost S -semiprime submodule of M .*

Proof. Since $(I(+P) \cap (S(+M))) = \emptyset, I \cap S = \emptyset$. Now let $r^k a \in I \setminus I^n$ for some $r, a \in R$ and $k \in \mathbb{N}$. Then $(r, 0)^k(a, 0) \in I(+P)$. If $(r, 0)^k(a, 0) \in (I(+P))^n \subseteq (I(+M))^n \subseteq (I^n(+M))$, then $r^k a \in I^n$, which is a contradiction. So $(r, 0)^k(a, 0) \in I(+P) \setminus (I(+P))^n$. Since $I(+P)$ is an n -almost $S(+M)$ -semiprime ideal of $R(+M)$, there exists an $(s, x) \in S(+M)$ for some $s \in S$ and $x \in M$ such that $(s, x)(r, 0)^k(a, 0) \in I(+P)$. So $sra \in I$ and hence I is an n -almost S -semiprime ideal of R .

Now let $(P :_R M) \cap S = \emptyset$ and $r^k x \in P \setminus (P :_R M)^{(n-1)}P$ for some $r \in R, x \in M$ and $k \in \mathbb{N}$. Then $(r, 0)^k(0, x) \in I(+P)$. If $(r, 0)^k(0, x) = (0, r^k x) \in (I(+P))^n \subseteq I^n(+M)$, then $r^k x \in I^{(n-1)}P \subseteq (P :_R M)^{(n-1)}P$, which is a contradiction. So $(r, 0)^k(0, x) \in I(+P) \setminus (I(+P))^n$. Therefore, there exists $(s, m) \in S(+M)$ for some $s \in S$ and $m \in M$ such that $(s, m)(r, 0)^k(0, x) \in I(+P)$. Then $sr x \in P$ and hence P is an n -almost S -semiprime submodule of M . □

Let M be an R -module. Recall that M is a multiplication R -module if for every submodule N of M , there exists an ideal I of R such that $N = IM$. It is easy to see that an R -module M is multiplication if and only if $N = (N :_R M)M$, for every submodule N of M . In the following, we give some characterizations of n -almost S -semiprime submodules in (finitely generated faithful) multiplication modules.

Lemma 3.19. *Let I be an ideal of a commutative ring R and P be a submodule of a finitely generated faithful multiplication R -module M . Then $(IP :_R M) = I(P :_R M)$.*

Proof. See [15, Lemma 3.4]. □

Theorem 3.20. *Let S be a multiplicatively closed subset of R and M be a finitely generated faithful multiplication R -module. Let P be a proper submodule of M and $n \geq 2$. Then the following conditions are equivalent:*

- (i) P is an n -almost S -semiprime submodule of M .
- (ii) $(P :_R M)$ is an n -almost S -semiprime ideal of R .
- (iii) $P = QM$ for some n -almost S -semiprime ideal Q of R .

Proof. (1) \Rightarrow (2) Let $r^k b \in (P :_R M) \setminus (P :_R M)^n$ for some $r, b \in R$ and $k \in \mathbb{N}$. Then $r^k b M \subseteq P$ and $r^k b M \not\subseteq (P :_R M)^{(n-1)} P$. So by Corollary 2.8, there exists $s \in S$ such that $s r b M \subseteq P$. Then $s r b \in (P : M)$ and hence $(P :_R M)$ is an n -almost S -semiprime ideal of R .

(2) \Rightarrow (1) Let $r^k x \in P \setminus (P :_R M)^{(n-1)} P$ for some $r \in R, x \in M$ and $k \in \mathbb{N}$. Then $\langle r \rangle^k (\langle x \rangle :_R M) \subseteq (\langle r^k x \rangle :_R M) \subseteq (P :_R M)$. If $\langle r \rangle^k (\langle x \rangle :_R M) \subseteq (P :_R M)^n$, then by Lemma 3.19, $\langle r^k \rangle (\langle x \rangle :_R M) \subseteq ((P :_R M)^{(n-1)} P :_R M)$. So $\langle r \rangle^k \langle x \rangle = \langle r \rangle^k (\langle x \rangle :_R M) M \subseteq (P :_R M)^{n-1} P$, which is a contradiction. Thus $\langle r \rangle (\langle r^k x \rangle :_R M) \subseteq (P :_R M) \setminus (P :_R M)^n$. Since $(P :_R M)$ is an n -almost S -semiprime ideal of R , there exists $s \in S$ such that $s \langle r \rangle (\langle x \rangle :_R M) \subseteq (P :_R M)$. So $s \langle r \rangle \langle x \rangle = s \langle r \rangle (\langle x \rangle :_R M) M \subseteq (P :_R M) P = P$. Therefore, $r s x \in P$ and hence P is an n -almost S -semiprime submodule of M .

(2) \Leftrightarrow (3) Set $Q = (P :_R M)$. The proof follows by [13, Theorem 3.1]. □

Lemma 3.21. *Let P be a submodule of a faithful multiplication R -module M and I be a finitely generated faithful multiplication ideal of R . Then*

- (i) $P = (IP :_M I)$.
- (ii) If $N \subseteq IM$, then $(JP :_M I) = J(P :_M I)$ for any ideal J of R .

Proof. It follows from [1]. □

Theorem 3.22. *Let N be a submodule of a faithful multiplication R -module M and I be a finitely generated faithful multiplication ideal of R . Then P is an n -almost S -semiprime submodule of IM ($n \geq 2$) if and only if $(P :_M I)$ is an n -almost S -semiprime submodule of M .*

Proof. Let P be an n -almost S -semiprime ideal of IM . Since $(P :_R IM) = ((P :_M I) :_R M)$, $((P :_M I) :_R M) \cap S = \emptyset$. Now let $r^k x \in (P :_M I) \setminus ((P :_M I) :_R M)^{(n-1)} (P :_M I)$ for some $r \in R, x \in M$ and $k \in \mathbb{N}$. Then $r^k (Ix) \subseteq P \setminus (P :_R IM)^{(n-1)} P$. Because if $r^k (Ix) \subseteq (P :_R IM)^{(n-1)} P = ((P :_M I) :_R M)^{(n-1)} P$, then by Lemma 3.21, $r^k x \in (((P :_M I) :_R M)^{(n-1)} P :_M I) = ((P :_M I) :_R M)^{(n-1)} (P :_M I)$, which is a contradiction. Since P is an n -almost S -semiprime submodule of IM , by Corollary 2.8, there exists $s \in S$ such that $r s (Ix) \subseteq P$. So $r s x \in (P :_M I)$ and hence $(P :_M I)$ is an n -almost S -semiprime submodule of M .

Conversely, assume that $(P :_M I)$ is an n -almost S -semiprime submodule of M and let $r^k x \in P \setminus (P :_R IM)^{(n-1)} P$ for some $r \in R, x \in IM$ and $k \in \mathbb{N}$. Then $r^k (\langle x \rangle :_M I) \subseteq (\langle r^k x \rangle :_M I) \subseteq (P :_M I)$. If $r^k (\langle x \rangle :_M I) \subseteq ((P :_M I) :_R M)^{(n-1)} (P :_M I) = (P :_R IM)^{(n-1)} (P :_M I)$, then by Lemma 3.21, $r^k \langle x \rangle = r^k (I \langle x \rangle :_M I) = r^k (\langle x \rangle :_M I) I \subseteq (P :_R IM)^{(n-1)} (P :_M I) I = (P :_R IM)^{(n-1)} P$, which is a contradiction. So $r^k (\langle x \rangle :_M I) \subseteq (P :_M I) \setminus ((P :_M I) :_R M)^{(n-1)} (P :_M I)$. Since $(P :_M I)$ is an n -almost S -semiprime submodule of M , there exists an $s \in S$ such that $r s (\langle x \rangle :_M I) \subseteq (P :_M I)$. So $r s \langle x \rangle = r s (I \langle x \rangle :_M I) = r s (\langle x \rangle :_M I) I \subseteq (P :_M I) I = P$. Therefore, $r s x \in P$ and hence P is an n -almost S -semiprime submodule of IM . □

4 n -almost S -semiprime submodules and flat modules

An R -module M is said to be flat if for every exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ of R -modules, the induced sequence of R -modules $0 \rightarrow N' \otimes_R M \rightarrow N \otimes_R M \rightarrow N'' \otimes_R M \rightarrow 0$ is exact. In the following, we give a characterization of n -almost S -semiprime submodule ($n \geq 2$) by flat module.

Lemma 4.1. *Let M be an R -module and P be a submodule of M . If F is a flat R -module, then we have $F \otimes_R (P :_M r) = (F \otimes_R P :_{F \otimes_R M} r)$ for all $r \in R$.*

Proof. See, [9, Lemma 3.2]. □

Theorem 4.2. *Let S be a multiplicatively closed subset of R and M be an R -module. Let P be a submodule of M with $(P :_R M) \cap S = \emptyset$. Then*

- (i) *If F is a flat R -module and P is an n -almost S -semiprime submodule of M such that $(F \otimes_R P :_R F \otimes_R M) \cap S = \emptyset$, then $F \otimes_R P$ is an n -almost S -semiprime submodule of $F \otimes_R M$ ($n \geq 2$).*
- (ii) *Let F be a faithfully flat R -module. Then P is an n -almost S -semiprime submodule of M if and only if $F \otimes_R P$ is an n -almost S -semiprime submodule of $F \otimes_R M$.*

Proof. (1) Since P is an n -almost S -semiprime submodule of M , by Theorem 2.7 there exists $s \in S$ such that $(P :_M r^k) \subseteq ((P :_M)^{(n-1)}P :_M r^k)$ or $(P :_M r^k) \subseteq (P :_M sr)$ for all $r \in R$ and $k \in \mathbb{N}$. Then $F \otimes_R (P :_M r^k) \subseteq F \otimes_R ((P :_R M)^{(n-1)}P :_M r^k)$ or $F \otimes_R (P :_M r^k) \subseteq F \otimes_R (P :_M sr)$ for all $r \in R$ and $k \in \mathbb{N}$. So by Lemma 4.1, $(F \otimes_R P :_{F \otimes_R M} r^k) \subseteq (F \otimes_R (P :_R M)^{(n-1)}P :_{F \otimes_R M} r^k) = ((P :_R M)^{(n-1)}(F \otimes_R P) :_{F \otimes_R M} r^k) \subseteq ((F \otimes_R P :_R F \otimes_R M)^{(n-1)}(F \otimes_R P) :_{F \otimes_R M} r^k)$ or $(F \otimes_R P :_{F \otimes_R M} r^k) \subseteq (F \otimes_R P :_{F \otimes_R M} sr)$ for all $r \in R$. Therefore, $F \otimes_R P$ is an n -almost S -semiprime submodule of $F \otimes_R M$.

(2) At first, we will show that $(P :_R M) = (F \otimes_R P :_R F \otimes_R M)$. It is clear that $(P :_R M) \subseteq (F \otimes_R P :_R F \otimes_R M)$. Now let $r \in (F \otimes_R P :_R F \otimes_R M)$ for some $r \in R$. Then $r(F \otimes_R M) = F \otimes_R rM \subseteq F \otimes_R P$. Since $0 \rightarrow F \otimes_R rM \rightarrow F \otimes_R P$ is an exact sequence and F is a faithfully flat R -module, $0 \rightarrow rM \rightarrow P$ is an exact sequence. Then $r \in (P :_R M)$. So $(F \otimes_R P :_R F \otimes_R M) \subseteq (P :_R M)$ and hence $(P :_R M) = (F \otimes_R P :_R F \otimes_R M)$. Now let P be an n -almost S -semiprime submodule of M . Since $(P :_R M) \cap S = \emptyset$, then $(F \otimes_R P :_R F \otimes_R M) \cap S = \emptyset$ and hence by part (1), $F \otimes_R P$ is an n -almost S -semiprime submodule of $F \otimes_R M$.

Conversely, assume that $F \otimes_R P$ is an n -almost S -semiprime submodule of $F \otimes_R M$. Since $(F \otimes_R P :_R F \otimes_R M) \cap S = \emptyset$, $(P :_R M) \cap S = \emptyset$. By Theorem 2.7, there exists an $s \in S$ such that $(F \otimes_R P :_{F \otimes_R M} r^k) \subseteq ((F \otimes_R P :_R F \otimes_R M)^{(n-1)}(F \otimes_R P) :_{F \otimes_R M} r^k)$ or $(F \otimes_R P :_{F \otimes_R M} r^k) \subseteq (F \otimes_R P :_{F \otimes_R M} sr)$ for all $r \in R$ and $k \in \mathbb{N}$. If $(F \otimes_R P :_{F \otimes_R M} r^k) \subseteq (F \otimes_R P :_{F \otimes_R M} sr)$, then by Lemma 4.1, $F \otimes_R (P :_M r^k) \subseteq F \otimes_R (P :_M sr)$. So $0 \rightarrow F \otimes_R (P :_M r^k) \rightarrow F \otimes_R (P :_M sr)$ is an exact sequence. Since F is a faithfully flat R -module, $0 \rightarrow (P :_M r^k) \rightarrow (P :_M sr)$ is an exact sequence. Then $(P :_M r^k) \subseteq (P :_M sr)$. Now let $(F \otimes_R P :_{F \otimes_R M} r^k) \subseteq ((F \otimes_R P :_R F \otimes_R M)^{(n-1)}(F \otimes_R P) :_{F \otimes_R M} r^k) = ((P :_R M)^{(n-1)}(F \otimes_R P) :_{F \otimes_R M} r^k)$. By Lemma 4.1, $F \otimes_R (P :_M r^k) \subseteq F \otimes_R ((P :_R M)^{(n-1)}P :_M r^k)$. So $0 \rightarrow F \otimes_R (P :_M r^k) \rightarrow F \otimes_R ((P :_R M)^{(n-1)}P :_M r^k)$ is an exact sequence. Since F is a faithfully flat R -module, $0 \rightarrow (P :_M r^k) \rightarrow ((P :_R M)^{(n-1)}P :_M r^k)$ is an exact sequence. Then $(P :_M r^k) \subseteq ((P :_R M)^{(n-1)}P :_M r^k)$ and hence P is an n -almost S -semiprime submodule of M . □

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