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TRANSLATION-INVARIANT STRUCTURES IN BIPOLAR FUZZY SBG-ALGEBRAS

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Abstract. In this paper, we present a comprehensive study of bipolar-valued fuzzy sets in the context of Sheffer stroke BG-algebras (SBG-algebras). We introduce the notions of bipolar fuzzy SBG-subalgebras and SBG-ideals, and provide their formal definitions and characterizations. Necessary and sufficient conditions for these fuzzy algebraic structures are established via the use of negative and positive cuts, and explicit algorithms are developed for their verification. The interplay between bipolar-valued fuzzy sets and their crisp analogues is illustrated with constructive examples, highlighting the transition from fuzzy to classical settings. We further prove that the intersection of arbitrary families of bipolar fuzzy SBG-ideals remains a bipolar fuzzy SBG-ideal, and demonstrate that the complement of the negative membership function, in conjunction with the positive membership function, generates new fuzzy SBG-ideals and subalgebras. These findings extend the algebraic apparatus of fuzzy logic, providing both theoretical insights and practical tools for the analysis of bipolar uncertainty in algebraic structures.

1 Introduction

In mathematical modeling, simplifying structures by minimizing the number of axioms or operations is essential. Tarski [24] and Sheffer [22] demonstrated this by using minimal operations to define Abelian groups and Boolean functions, respectively. Building on this, McCune et al. [10] axiomatized Boolean algebras using only the Sheffer stroke.

Logical algebraic structures have wide applications in areas like artificial intelligence, computer science, quantum logics, and probability theory. Sheffer stroke basic algebras, introduced by Oner and Senturk [16], streamline basic algebras with a single operation, offering efficiency in technological applications. This idea was extended to Sheffer stroke BG-algebras in [18].

The evolution of algebraic structures continued with Imai and Iséki's introduction of BCK- and BCI-algebras [2], followed by Neggers and Kim's B-algebras [15], and Kim and Kim's BG-algebras [7], a generalization of B-algebras.

Fuzzy set theory, introduced by Zadeh in 1965 [26], has seen various extensions, including intuitionistic, interval-valued, vague, and bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets, introduced by Lee in 2000 [8], extend fuzzy sets by incorporating a membership degree range of [-1, 0]. Subsequent research explored bipolar fuzzy structures in different algebraic contexts, such as BCH-algebras [4, 5], BCI-algebras [9, 6], KU-algebras [11], and BG-algebras [21]. More recent work has expanded these concepts into bipolar fuzzy translations in different algebraic structures such as [19, 3, 20].

Bipolar-valued fuzzy sets have significant applications across various fields, enhancing the development of effective algorithms for complex problem-solving. In the context of BG-algebras, Ahn and Lee [1] studied fuzzy subalgebras, while Muthuraj et al. [13, 14] focused on fuzzy ideals and multi-fuzzy subalgebras. Building on these, Oner, Senturk, and Rezaei examined bipolar-valued fuzzy translations in Sheffer stroke MTL-algebras [17].

This study complements recent advances in Diophantine fuzzy approaches to BE and BCK/BCI-algebras [23, 12], by introducing bipolar-valued analogues in the context of SBG-algebras.

In this paper, we introduce a novel construction of bipolar fuzzy sets, SBG-ideals, and (α, β)-translations within SBG-algebras, exploring their fundamental properties and providing significant results. Our work offers a foundation for further research in fields that utilize bipolar fuzzy concepts. Section 3 examines bipolar fuzzy sets in SBG-algebras, including their inclusion, bounds, and (α, β)-translations. We prove that these translations maintain the properties of SBG-subalgebras and SBG-ideals and discuss the preservation of these structures under homomorphisms.

2 Preliminaries

In this section, we provide definitions, lemma, and proposition relevant to the concepts of Sheffer stroke BG-algebras, their ideals, subalgebras, and bipolar fuzzy sets, which will be used throughout the paper.

Definition 2.1. [22] Let $H = \langle H; | \rangle$ be a groupoid. The operation $|$ is called a Sheffer stroke operation if it meets the following conditions:

$(S1)$ $x|y = y|x$,

$(S2)$ $(x|x)|(x|y) = x$,

$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,

$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

To enhance the readability of this manuscript about Sheffer stroke BG-algebras, we will consistently use the following notation:

$$x|(y|y) = x^y.$$

Definition 2.2. [18] A Sheffer stroke BG-algebra (abbreviated as SBG-algebra) is a structure $\langle A; | \rangle$ of type (2) where 0 is a fixed element in A and the following conditions hold for all $x, y, z \in A$:
(SBG_1) $x^x | x^x = 0$,
(SBG_2) $0^y | (x^y | x^y) = x|x$.

Proposition 2.3. *[18] Consider an SBG-algebra $\langle A; | \rangle$. The binary relation*

$$x \leq y \quad \text{if and only if} \quad y^x = 0|0$$

defines a partial order on A.

Definition 2.4. [18] A nonempty subset G of a Sheffer stroke BG-algebra H is referred to as an SBG-subalgebra of H if for all $x, y \in G$, the element $x^y | x^y$ is in G.

Definition 2.5. [18] A nonempty subset G of a Sheffer stroke BG-algebra H is called an SBG-ideal of H if it satisfies the following conditions for all $x, y \in G$:

(i) $0 \in G$,

(ii) If $x^y | x^y \in G$ and $y \in G$, then $x \in G$.

Definition 2.6. [27] Let X be a nonempty set. A bipolar fuzzy set B in X is an object of the form $B = \{(x, f^-(x), f^+(x)) \mid x \in X\}$, where $f^+ : X \to [0, 1]$ and $f^- : X \to [-1, 0]$ are functions. The positive membership degree $f^+(x)$ indicates the extent to which an element x satisfies the property associated with the bipolar fuzzy set B, while the negative membership degree $f^-(x)$ indicates the extent to which x satisfies an implicit counter-property associated with B.

If $f^+(x) \neq 0$ and $f^-(x) = 0$, then x has only positive satisfaction for B. Conversely, if $f^+(x) = 0$ and $f^-(x) \neq 0$, then x does not satisfy the property of B but somewhat satisfies the counter-property. An element x may have $f^+(x) = 0$ and $f^-(x) = 0$ when the membership function of the property overlaps with that of its counter-property over some part of X.

For simplicity, we shall use the notation $f = (f^+, f^-)$ for the bipolar fuzzy set $B = \{(x, f^-(x), f^+(x)) \mid x \in X\}$.

Lemma 2.7. *[25] Let $r_1, r_2, r_3 \in \mathbb{R}$. The following statements are true:*

(i) $r_1 - \min\{r_2, r_3\} = \max\{r_1 - r_2, r_1 - r_3\}$,

(ii) $r_1 - \max\{r_2, r_3\} = \min\{r_1 - r_2, r_1 - r_3\}$.

3 Bipolar fuzzy (α, β)-translations in SBG-algebras

In this section, we explore various aspects of bipolar fuzzy sets within SBG-algebras. We begin by defining the inclusion of bipolar fuzzy sets, establishing conditions under which one set is a bipolar fuzzy extension or intension of another. We introduce the concepts of lower and upper bounds, $\perp$ and $\top$, for these sets. The concept of bipolar fuzzy (α, β)-translations of two types, Type I and Type II, is presented, where these translations adjust the membership functions by constants α and β. Several theorems follow, asserting that if a bipolar fuzzy set is an SBG-subalgebra (or SBG-ideal) of an algebra, then its (α, β)-translations are also SBG-subalgebras (or SBG-ideals). Further, the conditions under which the converse holds are also established. The section also includes remarks emphasizing that these translations serve as extensions of the original bipolar fuzzy sets. Definitions for complements of bipolar fuzzy sets and specific cuts (negative and positive) are provided. Lastly, it is shown that for a bipolar fuzzy set to be an SBG-subalgebra (or SBG-ideal), certain conditions on these cuts must hold. The section concludes with a discussion on homomorphisms between SBG-algebras, and the preservation of bipolar fuzzy SBG-ideals and SBG-subalgebras under such mappings.

Definition 3.1. The inclusion "$\subseteq$" is defined for any bipolar fuzzy sets $f = (A; f^-, f^+)$ and $\psi = (A; \psi^-, \psi^+)$ in A by setting:

$$f \subseteq \psi \ \Leftrightarrow \ f^-(x) \geq \psi^-(x) \text{ and } f^+(x) \leq \psi^+(x) \text{ for all } x \in A.$$

Then the structure $\psi = (A; \psi^-, \psi^+)$ is called a bipolar fuzzy extension of $f = (A; f^-, f^+)$, and the structure $f = (A; f^-, f^+)$ is called a bipolar fuzzy intension of $\psi = (A; \psi^-, \psi^+)$.

Now, we give a pseudocode for the above definition as follows.

Algorithm 3.2 (H). Confirming Bipolar Fuzzy Extension or Intension
Set A, bipolar fuzzy sets $f = (A; f^-, f^+)$ and $\psi = (A; \psi^-, \psi^+)$ Whether ψ is a bipolar fuzzy extension of f or is f a bipolar fuzzy intension of ψ

$IsBipolarFuzzyExtension(f, \psi)$

$x \in A$ $f^-(x) < \psi^-(x)$ **or** $f^+(x) > \psi^+(x)$ Return $False$
Return $True$

Algorithm 3.2 is designed to verify whether a given bipolar fuzzy set $\psi = (A; \psi^-, \psi^+)$ is a bipolar fuzzy extension of another bipolar fuzzy set $f = (A; f^-, f^+)$, or equivalently, whether f is a bipolar fuzzy intension of ψ. The algorithm systematically checks the following conditions for all elements $x \in A$:

- The algorithm verifies that the negative membership degree $f^-(x)$ is greater than or equal to the negative membership degree $\psi^-(x)$ and that the positive membership degree $f^+(x)$ is less than or equal to the positive membership degree $\psi^+(x)$. If either condition is violated for any element x, the algorithm concludes that ψ is not a bipolar fuzzy extension of f and returns *False*.
- If both conditions are satisfied for all elements x, the algorithm concludes that ψ is indeed a bipolar fuzzy extension of f and returns *True*.

This algorithm provides a reliable and efficient method for determining whether one bipolar fuzzy set satisfies the necessary conditions to be considered a bipolar fuzzy extension or intension within the framework of a given set A.

Example 3.3. In order to illustrate the concept of a bipolar fuzzy SBG-ideal in the context of an SBG-algebra, consider the set $H = \{0, a, b, c, d, e\}$ together with the binary operation $|$ defined by the Cayley table below:

$\mid$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	e	d	c	b
b	b	e	0	a	e	d
c	c	d	a	0	b	e
d	d	c	e	b	0	a
e	e	b	d	e	a	0

The operation $|$ satisfies the axioms of an SBG-algebra, namely (SBG_1) and (SBG_2). In particular, for every $x \in H$, it holds that $x^x|x^x = 0$. Moreover, the operation is constructed such that for all $x, y \in H$, the equation $0^y|(x^y|x^y) = x|x$ is fulfilled.

Now, we define two bipolar fuzzy sets $f = (A; f^-, f^+)$ and $\psi = (A; \psi^-, \psi^+)$ over the set A as follows:

$$f^-(0) = -0.7,\ f^-(a) = -0.6,\ f^-(b) = -0.5,\ f^-(c) = -0.4,\ f^-(d) = -0.3,\ f^-(e) = -0.2,$$

with

$$f^+(0) = 0.8,\ f^+(a) = 0.7,\ f^+(b) = 0.6,\ f^+(c) = 0.5,\ f^+(d) = 0.4,\ f^+(e) = 0.3,$$

and

$$\psi^-(0) = -0.6,\ \psi^-(a) = -0.5,\ \psi^-(b) = -0.4,\ \psi^-(c) = -0.3,\ \psi^-(d) = -0.2,\ \psi^-(e) = -0.1,$$

along with

$$\psi^+(0) = 0.9,\ \psi^+(a) = 0.8,\ \psi^+(b) = 0.7,\ \psi^+(c) = 0.6,\ \psi^+(d) = 0.5,\ \psi^+(e) = 0.4.$$

We check that $f \subseteq \psi$ by verifying the following conditions for every element $x \in A$:

$$f^-(x) \geq \psi^-(x) \quad \text{and} \quad f^+(x) \leq \psi^+(x).$$

For each $x \in A$, these inequalities are satisfied, confirming that $f = (A; f^-, f^+)$ is a bipolar fuzzy intension of $\psi = (A; \psi^-, \psi^+)$, and likewise, $\psi = (A; \psi^-, \psi^+)$ is a bipolar fuzzy extension of $f = (A; f^-, f^+)$.

Definition 3.4. For any bipolar fuzzy set $f = (A; f^-, f^+)$ in A, we denote

$$\bot = -1 - \inf\{f^-(x) : x \in A\},$$

$$\top = 1 - \sup\{f^+(x) : x \in A\}.$$

Let $f = (A; f^-, f^+)$ be a bipolar fuzzy set in A and let $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. A bipolar fuzzy (α, β)-translation of $f = (A; f^-, f^+)$ of type I is defined as a bipolar fuzzy set $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$, where

$$f^-_{(\alpha,T_1)} : A \to [-1, 0], \quad x \mapsto f^-(x) + \alpha,$$

and

$$f^+_{(\beta,T_1)} : A \to [0, 1], \quad x \mapsto f^+(x) + \beta.$$

Theorem 3.5. *If a bipolar fuzzy set $f = (A; f^-, f^+)$ in A is a bipolar fuzzy SBG-subalgebra of A, then for all $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ of $f = (A; f^-, f^+)$ is also a bipolar fuzzy SBG-subalgebra of A.*

Proof. Assume that $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-subalgebra of A. For any $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ and for all $x, y \in A$, we have:

$$\begin{aligned} f^-_{(\alpha,T_1)}(x^y|x^y) &= f^-(x^y|x^y) + \alpha \\ &\leq \max\{f^-(x), f^-(y)\} + \alpha \\ &= \max\{f^-(x) + \alpha, f^-(y) + \alpha\} \\ &= \max\{f^-_{(\alpha,T_1)}(x), f^-_{(\alpha,T_1)}(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^+_{(\beta,T_1)}(x^y|x^y) &= f^+(x^y|x^y) + \beta \\ &\geq \min\{f^+(x), f^+(y)\} + \beta \\ &= \min\{f^+(x) + \beta, f^+(y) + \beta\} \\ &= \min\{f^+_{(\beta,T_1)}(x), f^+_{(\beta,T_1)}(y)\}. \end{aligned}$$

Hence, $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ is a bipolar fuzzy SBG-subalgebra of A. □

Theorem 3.6. *If there exists $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ of $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-subalgebra of A, then $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-subalgebra of A.*

Proof. Assume that $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ is a bipolar fuzzy SBG-subalgebra of A for some $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. For all $x, y \in A$, we have:

$$\begin{aligned} f^-(x^y|x^y) + \alpha &= f^-_{(\alpha,T_1)}(x^y|x^y) \\ &\leq \max\{f^-_{(\alpha,T_1)}(x), f^-_{(\alpha,T_1)}(y)\} \\ &= \max\{f^-(x) + \alpha, f^-(y) + \alpha\} \\ &= \max\{f^-(x), f^-(y)\} + \alpha, \end{aligned}$$

and

$$\begin{aligned} f^+(x^y|x^y) + \beta &= f^+_{(\beta,T_1)}(x^y|x^y) \\ &\geq \min\{f^+_{(\beta,T_1)}(x), f^+_{(\beta,T_1)}(y)\} \\ &= \min\{f^+(x) + \beta, f^+(y) + \beta\} \\ &= \min\{f^+(x), f^+(y)\} + \beta. \end{aligned}$$

Thus, we obtain:

$$f^-(x^y|x^y) \leq \max\{f^-(x), f^-(y)\}$$

and

$$f^+(x^y|x^y) \geq \min\{f^+(x), f^+(y)\}.$$

Hence, $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-subalgebra of A. □

Theorem 3.7. *If a bipolar fuzzy set $f = (A; f^-, f^+)$ in A is a bipolar fuzzy SBG-ideal of A, then for all $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ of $f = (A; f^-, f^+)$ is also a bipolar fuzzy SBG-ideal of A.*

Proof. Assume that $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A. For any $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ and for all $x, y \in A$, we have $f^-(0) \leq f^-(x)$ and $f^+(0) \geq f^+(x)$. Then we obtain:

$$\begin{aligned} f^-_{(\alpha,T_1)}(0) &= f^-(0) + \alpha \\ &\leq f^-(x) + \alpha \\ &= f^-_{(\alpha,T_1)}(x), \end{aligned}$$

and

$$\begin{aligned} f^+_{(\beta,T_1)}(0) &= f^+(0) + \beta \\ &\geq f^+(x) + \beta \\ &= f^+_{(\beta,T_1)}(x). \end{aligned}$$

On the other hand, for $x, y \in A$, we have $f^-(x) \leq \max\{f^-(x^y|x^y), f^-(y)\}$ and $f^+(x) \geq \min\{f^+(x^y|x^y), f^+(y)\}$. Additionally, we get:

$$\begin{aligned} f^-_{(\alpha,T_1)}(x) &= f^-(x) + \alpha \\ &\leq \max\{f^-(x^y|x^y), f^-(y)\} + \alpha \\ &= \max\{f^-(x^y|x^y) + \alpha, f^-(y) + \alpha\} \\ &= \max\{f^-_{(\alpha,T_1)}(x^y|x^y), f^-_{(\alpha,T_1)}(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^+_{(\beta,T_1)}(x) &= f^+(x) + \beta \\ &\geq \min\{f^+(x^y|x^y), f^+(y)\} + \beta \\ &= \min\{f^+(x^y|x^y) + \beta, f^+(y) + \beta\} \\ &= \min\{f^+_{(\beta,T_1)}(x^y|x^y), f^+_{(\beta,T_1)}(y)\}. \end{aligned}$$

Hence, $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ is a bipolar fuzzy SBG-ideal of A. □

Theorem 3.8. *If there exists $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ of $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A, then $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A.*

Proof. Assume that $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ is a bipolar fuzzy SBG-ideal of A for some $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. For all $x, y \in A$, we have $f^-(0) \leq f^-(x)$ and $f^+(0) \geq f^+(x)$. Then, we obtain:

$$\begin{aligned} f^-(0) + \alpha &= f^-_{(\alpha,T_1)}(0) \\ &\leq f^-_{(\alpha,T_1)}(x) \\ &= f^-(x) + \alpha, \end{aligned}$$

and

$$\begin{aligned} f^+(0) + \beta &= f^+_{(\beta,T_1)}(0) \\ &\geq f^+_{(\beta,T_1)}(x) \\ &= f^+(x) + \beta. \end{aligned}$$

Furthermore, let $x, y \in A$. Then, we achieve:

$$\begin{aligned} f^-(x) + \alpha &= f^-_{(\alpha,T_1)}(x) \\ &\leq \max\{f^-_{(\alpha,T_1)}(x^y|x^y), f^-_{(\alpha,T_1)}(y)\} \\ &= \max\{f^-(x^y|x^y) + \alpha, f^-(y) + \alpha\} \\ &= \max\{f^-(x^y|x^y), f^-(y)\} + \alpha, \end{aligned}$$

and

$$\begin{aligned} f^+(x) + \beta &= f^+_{(\beta,T_1)}(x) \\ &\geq \min\{f^+_{(\beta,T_1)}(x^y|x^y), f^+_{(\beta,T_1)}(y)\} \\ &= \min\{f^+(x^y|x^y) + \beta, f^+(y) + \beta\} \\ &= \min\{f^+(x^y|x^y), f^+(y)\} + \beta. \end{aligned}$$

Hence, $f^-(x) \leq \max\{f^-(x^y|x^y), f^-(y)\}$ and $f^+(x) \geq \min\{f^+(x^y|x^y), f^+(y)\}$. Therefore, $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A. □

Remark 3.9. If $f = (A; f^-, f^+)$ is a bipolar fuzzy set of A, then for all $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$, we have $f^-_{(\alpha,T_1)}(x) = f^-(x) + \alpha \leq f^-(x)$ and $f^+_{(\beta,T_1)}(x) = f^+(x) + \beta \geq f^+(x)$ for all $x \in A$. Hence, the bipolar fuzzy (α, β)-translation $f^{T_1}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_1)}, f^+_{(\beta,T_1)})$ of $f = (A; f^-, f^+)$ is a bipolar fuzzy extension of $f = (A; f^-, f^+)$ for all $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$.

Definition 3.10. For any bipolar fuzzy set $f = (A; f^-, f^+)$ in A, we denote:

$$\pm = \sup\{f^-(x) : x \in A\},$$
$$\mp = \inf\{f^+(x) : x \in A\}.$$

Let $f = (A; f^-, f^+)$ be a bipolar fuzzy set in A and let $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$. A bipolar fuzzy (α, β)-translation of $f = (A; f^-, f^+)$ of Type II is defined as a bipolar fuzzy set $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$, where:

$$f^-_{(\alpha,T_2)} : A \to [-1, 0], \quad x \mapsto f^-(x) - \alpha,$$
$$f^+_{(\beta,T_2)} : A \to [0, 1], \quad x \mapsto f^+(x) - \beta.$$

Theorem 3.11. *If a bipolar fuzzy set $f = (A; f^-, f^+)$ in A is a bipolar fuzzy SBG-subalgebra of A, then for all $(\alpha, \beta) \in [\pm, 0]\times[0, \mp]$, a bipolar fuzzy (α, β)-translation $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ of $f = (A; f^-, f^+)$ is also a bipolar fuzzy SBG-subalgebra of A.*

Proof. Assume that $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-subalgebra of A. For any $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and for all $x, y \in A$, we have:

$$\begin{aligned} f^-_{(\alpha,T_2)}(x^y|x^y) &= f^-(x^y|x^y) - \alpha \\ &\le \max\{f^-(x), f^-(y)\} - \alpha \\ &= \max\{f^-(x) - \alpha, f^-(y) - \alpha\} \\ &= \max\{f^-_{(\alpha,T_2)}(x), f^-_{(\alpha,T_2)}(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^+_{(\beta,T_2)}(x^y|x^y) &= f^+(x^y|x^y) - \beta \\ &\ge \min\{f^+(x), f^+(y)\} - \beta \\ &= \min\{f^+(x) - \beta, f^+(y) - \beta\} \\ &= \min\{f^+_{(\beta,T_2)}(x), f^+_{(\beta,T_2)}(y)\}. \end{aligned}$$

Hence, $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ is a bipolar fuzzy SBG-subalgebra of A. □

Theorem 3.12. *If there exists $(\alpha, \beta) \in [\pm, 0]\times[0, \mp]$ such that the bipolar fuzzy (α, β)-translation $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ of $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-subalgebra of A, then $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-subalgebra of A.*

Proof. Assume that $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ is a bipolar fuzzy SBG-subalgebra of A for some $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$. For all $x, y \in A$, we have:

$$\begin{aligned} f^-(x^y|x^y) - \alpha &= f^-_{(\alpha,T_2)}(x^y|x^y) \\ &\le \max\{f^-_{(\alpha,T_2)}(x), f^-_{(\alpha,T_2)}(y)\} \\ &= \max\{f^-(x) - \alpha, f^-(y) - \alpha\} \\ &= \max\{f^-(x), f^-(y)\} - \alpha, \end{aligned}$$

and

$$\begin{aligned} f^+(x^y|x^y) - \beta &= f^+_{(\beta,T_2)}(x^y|x^y) \\ &\ge \min\{f^+_{(\beta,T_2)}(x), f^+_{(\beta,T_2)}(y)\} \\ &= \min\{f^+(x) - \beta, f^+(y) - \beta\} \\ &= \min\{f^+(x), f^+(y)\} - \beta. \end{aligned}$$

Thus, we attain:

$$f^-(x^y|x^y) \le \max\{f^-(x), f^-(y)\}$$

and

$$f^+(x^y|x^y) \ge \min\{f^+(x), f^+(y)\}.$$

Therefore, we conclude that $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-subalgebra of A. □

Theorem 3.13. *If a bipolar fuzzy set $f = (A; f^-, f^+)$ in A is a bipolar fuzzy SBG-ideal of A, then for all $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$, a bipolar fuzzy (α, β)-translation $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ of $f = (A; f^-, f^+)$ is also a bipolar fuzzy SBG-ideal of A.*

Proof. Assume that $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A. For any $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and for all $x \in A$, we have $f^-(0) \leq f^-(x)$ and $f^+(0) \geq f^+(x)$. Additionally, we get:

$$\begin{aligned} f^-_{(\alpha,T_2)}(0) &= f^-(0) - \alpha \\ &\leq f^-(x) - \alpha \\ &= f^-_{(\alpha,T_2)}(x), \end{aligned}$$

and

$$\begin{aligned} f^+_{(\beta,T_2)}(0) &= f^+(0) - \beta \\ &\geq f^+(x) - \beta \\ &= f^+_{(\beta,T_2)}(x). \end{aligned}$$

Furthermore, let $x, y \in A$. Then, we get $f^-(x) \leq \max\{f^-(x^y|x^y), f^-(y)\}$ and $f^+(x) \geq \min\{f^+(x^y|x^y), f^+(y)\}$. Additionally, we attain:

$$\begin{aligned} f^-_{(\alpha,T_2)}(x) &= f^-(x) - \alpha \\ &\leq \max\{f^-(x^y|x^y), f^-(y)\} - \alpha \\ &= \max\{f^-(x^y|x^y) - \alpha, f^-(y) - \alpha\} \\ &= \max\{f^-_{(\alpha,T_2)}(x^y|x^y), f^-_{(\alpha,T_2)}(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^+_{(\beta,T_2)}(x) &= f^+(x) - \beta \\ &\geq \min\{f^+(x^y|x^y), f^+(y)\} - \beta \\ &= \min\{f^+(x^y|x^y) - \beta, f^+(y) - \beta\} \\ &= \min\{f^+_{(\beta,T_2)}(x^y|x^y), f^+_{(\beta,T_2)}(y)\}. \end{aligned}$$

Hence, we conclude that $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ is a bipolar fuzzy SBG-ideal of A. □

Theorem 3.14. *If there exists $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ such that the bipolar fuzzy (α, β)-translation $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ of $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A, then $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A.*

Proof. Assume that $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ is a bipolar fuzzy SBG-ideal of A for some $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$. For all $x \in A$, we have $f^-(0) \leq f^-(x)$ and $f^+(0) \geq f^+(x)$. We also get:

$$\begin{aligned} f^-(0) - \alpha &= f^-_{(\alpha,T_2)}(0) \\ &\leq f^-_{(\alpha,T_2)}(x) \\ &= f^-(x) - \alpha, \end{aligned}$$

and

$$\begin{aligned} f^+(0) - \beta &= f^+_{(\beta,T_2)}(0) \\ &\geq f^+_{(\beta,T_2)}(x) \\ &= f^+(x) - \beta. \end{aligned}$$

Furthermore, let $x, y \in A$. Then, we achieve:

$$\begin{aligned} f^-(x) - \alpha &= f^-_{(\alpha,T_2)}(x) \\ &\leq \max\{f^-_{(\alpha,T_2)}(x^y|x^y), f^-_{(\alpha,T_2)}(y)\} \\ &= \max\{f^-(x^y|x^y) - \alpha, f^-(y) - \alpha\} \\ &= \max\{f^-(x^y|x^y), f^-(y)\} - \alpha, \end{aligned}$$

and

$$\begin{aligned} f^+(x) - \beta &= f^+_{(\beta,T_2)}(x) \\ &\geq \min\{f^+_{(\beta,T_2)}(x^y|x^y), f^+_{(\beta,T_2)}(y)\} \\ &= \min\{f^+(x^y|x^y) - \beta, f^+(y) - \beta\} \\ &= \min\{f^+(x^y|x^y), f^+(y)\} - \beta. \end{aligned}$$

Hence, we obtain $f^-(x) \leq \max\{f^-(x^y|x^y), f^-(y)\}$ and $f^+(x) \geq \min\{f^+(x^y|x^y), f^+(y)\}$. Therefore, we conclude that $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A. □

Remark 3.15. If $f = (A; f^-, f^+)$ is a bipolar fuzzy set of A, then for all $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$, we have $f^-_{(\alpha,T_2)}(x) = f^-(x) - \alpha \leq f^-(x)$ and $f^+_{(\beta,T_2)}(x) = f^+(x) - \beta \geq f^+(x)$ for all $x \in A$. Hence, the bipolar fuzzy (α, β)-translation $f^{T_2}_{(\alpha,\beta)} = (A; f^-_{(\alpha,T_2)}, f^+_{(\beta,T_2)})$ of $f = (A; f^-, f^+)$ is a bipolar fuzzy extension of $f = (A; f^-, f^+)$ for all $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$.

Definition 3.16. Let $f = (A; f^-, f^+)$ be a bipolar fuzzy set of A. The bipolar fuzzy set $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is defined by:

$$\overline{f^-}(x) = -1 - f^-(x),$$

$$\overline{f^+}(x) = 1 - f^+(x),$$

for all $x \in A$. Then, $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is called the complement of $f = (A; f^-, f^+)$ in A.

Definition 3.17. Let $f = (f^+, f^-)$ be a bipolar fuzzy set in a Hilbert algebra H. For $(t^-, t^+) \in [-1, 0] \times [0, 1]$, the sets

$$N_L(f, t^-) = \{x \in H \mid f^-(x) \leq t^-\}$$

and

$$N_U(f, t^-) = \{x \in H \mid f^-(x) \geq t^-\}$$

are called the negative lower t^--cut and the negative upper t^--cut of $f = (f^+, f^-)$, respectively. The sets

$$P_L(f, t^+) = \{x \in H \mid f^+(x) \leq t^+\}$$

and

$$P_U(f, t^+) = \{x \in H \mid f^+(x) \geq t^+\}$$

are called the positive lower t^+-cut and the positive upper t^+-cut of $f = (f^+, f^-)$, respectively.

Theorem 3.18. *Let $f = (A; f^-, f^+)$ be a bipolar fuzzy set in A. Then $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-subalgebra of A if and only if, for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, the sets $N_U(f, t^-)$ and $P_L(f, t^+)$ are SBG-subalgebras of A, provided that $N_U(f, t^-)$ and $P_L(f, t^+)$ are nonempty.*

Proof. Assume that $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-subalgebra of A. Let $(t^-, t^+) \in [-1, 0] \times [0, 1]$ be such that $N_U(f, t^-)$ and $P_L(f, t^+)$ are nonempty. Let $x, y \in N_U(f, t^-)$. Then we have $f^-(x) \geq t^-$ and $f^-(y) \geq t^-$. So t^- is a lower bound of $\{f^-(x), f^-(y)\}$. Since $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-subalgebra of A, we attain

$$\overline{f^-}(x^y|x^y) \leq \max\{\overline{f^-}(x), \overline{f^-}(y)\}.$$

By Lemma 2.7 (1), we have

$$-1 - f^-(x^y|x^y) \leq \max\{-1 - f^-(x), -1 - f^-(y)\} = -1 - \min\{f^-(x), f^-(y)\}.$$

Thus, we obtain

$$f^-(x^y|x^y) \geq \min\{f^-(x), f^-(y)\} \geq t^-,$$

which means that $x^y|x^y \in N_U(f, t^-)$. Therefore, $N_U(f, t^-)$ is an SBG-subalgebra of A.

Let $x, y \in P_L(f, t^+)$. Then we attain $f^+(x) \leq t^+$ and $f^+(y) \leq t^+$. So, it follows that t^+ is an upper bound of $\{f^+(x), f^+(y)\}$. Since $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-subalgebra of A, we achieve

$$\overline{f^+}(x^y|x^y) \geq \min\{\overline{f^+}(x), \overline{f^+}(y)\}.$$

By Lemma 2.7 (2), we have

$$1 - f^+(x^y|x^y) \geq \min\{1 - f^+(x), 1 - f^+(y)\} = 1 - \max\{f^+(x), f^+(y)\}.$$

Thus, we acquire $f^+(x^y|x^y) \leq \max\{f^+(x), f^+(y)\} \leq t^+$, which means $x^y|x^y \in P_L(f, t^+)$. Therefore, we derive that $P_L(f, t^+)$ is an SBG-subalgebra of A.

Conversely, assume that for all (t^-, t^+) in $[-1, 0] \times [0, 1]$, $N_U(f, t^-)$ and $P_L(f, t^+)$ are SBG-subalgebras of A if $N_U(f, t^-)$ and $P_L(f, t^+)$ are nonempty. We examine two cases as below:
(i) Let $x, y \in A$. Then we have $f^-(x), f^-(y) \in [-1, 0]$. Choose $t^- = \min\{f^-(x), f^-(y)\}$. Thus, $f^-(x) \geq t^-$ and $f^-(y) \geq t^-$, and so $x, y \in N_U(f, t^-) \neq \emptyset$. By the assumption, $N_U(f, t^-)$ is an SBG-subalgebra of A, and so $x^y|x^y \in N_U(f, t^-)$. Thus, we get

$$f^-(x^y|x^y) \geq t^- = \min\{f^-(x), f^-(y)\}.$$

By Lemma 2.7 (1), we have

$$\begin{aligned}
\overline{f^-}(x^y|x^y) &= -1 - f^-(x^y|x^y) \\
&\leq -1 - \min\{f^-(x), f^-(y)\} \\
&= \max\{-1 - f^-(x), -1 - f^-(y)\} \\
&= \max\{\overline{f^-}(x), \overline{f^-}(y)\}.
\end{aligned}$$

(ii) Let $x, y \in A$. Then $f^+(x), f^+(y) \in [0, 1]$. Choose $t^+ = \max\{f^+(x), f^+(y)\}$. Thus, we attain $f^+(x) \leq t^+$ and $f^+(y) \leq t^+$. So, we extract $x, y \in P_L(f, t^+) \neq \emptyset$. By assumption, $P_L(f, t^+)$ is an SBG-subalgebra of A, and so $x^y|x^y \in P_L(f, t^+)$. Thus, we conclude

$$f^+(x^y|x^y) \leq t^+ = \max\{f^+(x), f^+(y)\}.$$

By Lemma 2.7 (2), we have

$$\begin{aligned}
\overline{f^+}(x^y|x^y) &= 1 - f^+(x^y|x^y) \\
&\geq 1 - \max\{f^+(x), f^+(y)\} \\
&= \min\{1 - f^+(x), 1 - f^+(y)\} \\
&= \min\{\overline{f^+}(x), \overline{f^+}(y)\}.
\end{aligned}$$

Hence, $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-subalgebra of A. □

Theorem 3.19. *Let $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ be a bipolar fuzzy set in A. Then $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A if and only if, for all $(t^-, t^+) \in [-1, 0] \times [0, 1]$, the sets $N_U(f, t^-)$ and $P_L(f, t^+)$ are SBG-ideals of A, provided that $N_U(f, t^-)$ and $P_L(f, t^+)$ are nonempty.*

Proof. Assume that $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-ideal of A. Let (t^-, t^+) be in $[-1, 0] \times [0, 1]$ such that $N_U(f, t^-)$ and $P_L(f, t^+)$ are nonempty.

First, consider $x \in A$ such that $x \in N_U(f, t^-)$. This implies $f^-(x) \geq t^-$. Since $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A, we have

$$\begin{aligned}
\overline{f^-}(0) \leq \overline{f^-}(x) &\Rightarrow 1 - f^+(0) \leq 1 - f^+(x) \\
&\Rightarrow f^+(0) \geq f^+(x) \geq t^-.
\end{aligned}$$

Thus, $0 \in N_U(f, t^-)$.

Moreover, let $x, y \in A$ such that $x^y|x^y, y \in N_U(f, t^-)$. Then we have $f^-(x^y|x^y) \geq t^-$ and $f^-(y) \geq t^-$. Since $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-ideal of A, we obtain

$$\begin{aligned}
\overline{f^-}(x) \leq \max\{\overline{f^-}(x^y|x^y), \overline{f^-}(y)\} &\Rightarrow 1 - f^-(x) \leq \max\{1 - f^-(x^y|x^y), 1 - f^-(y)\} \\
&\Rightarrow 1 - f^-(x) \leq 1 - \min\{f^-(x^y|x^y), f^-(y)\} \\
&\Rightarrow f^-(x) \geq \min\{f^-(x^y|x^y), f^-(y)\} \geq t^-.
\end{aligned}$$

Hence, $x \in N_U(f, t^-)$. Therefore, $N_U(f, t^-)$ is an SBG-ideal of A.

Now, consider $x \in A$ such that $x \in P_L(f,t^+)$. This implies $f^+(x) \leq t^+$. Since $f = (A; f^-, f^+)$ is a bipolar fuzzy SBG-ideal of A, we have

$$\begin{aligned}\overline{f^+}(0) \geq \overline{f^+}(x) \quad &\Rightarrow \quad 1 - f^+(0) \geq 1 - f^+(x) \\ &\Rightarrow \quad f^+(0) \leq f^+(x) \leq t^+.\end{aligned}$$

Thus, $0 \in P_L(f,t^+)$.

Furthermore, let $x, y \in A$ such that $x^y|x^y, y \in P_L(f,t^+)$. Then we have $f^+(x^y|x^y) \leq t^+$ and $f^+(y) \leq t^+$. Since $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-ideal of A, we obtain

$$\begin{aligned}\overline{f^+}(x) \geq \min\{\overline{f^+}(x^y|x^y), \overline{f^+}(y)\} \quad &\Rightarrow \quad 1 - f^+(x) \geq \min\{1 - f^+(x^y|x^y), 1 - f^+(y)\} \\ &\Rightarrow \quad 1 - f^+(x) \geq 1 - \max\{f^+(x^y|x^y), f^+(y)\} \\ &\Rightarrow \quad f^+(x) \leq \max\{f^+(x^y|x^y), f^+(y)\} \leq t^+.\end{aligned}$$

Hence, $x \in P_L(f,t^+)$. Therefore, $P_L(f,t^+)$ is an SBG-ideal of A.

Conversely, assume that for all (t^-, t^+) in $[-1,0] \times [0,1]$, $N_U(f,t^-)$ and $P_L(f,t^+)$ are SBG-ideals of A if $N_U(f,t^-)$ and $P_L(f,t^+)$ are nonempty.

First, let $x \in A$. Then $f^-(x) \in [-1,0]$. Choose $t^- = f^-(x)$. Thus, $f^-(0) \geq f^-(x) = t^-$, and so $0 \in N_U(f,t^-) \neq \emptyset$. By assumption, $N_U(f,t^-)$ is an SBG-ideal of A, so $0 \in N_U(f,t^-)$. Thus, we have

$$f^-(0) \geq t^- = f^-(x),$$

and therefore,

$$\overline{f^-}(0) = -1 - f^-(0) \leq -1 - f^-(x) = \overline{f^-}(x).$$

Also, let $x, y \in A$. Then $f^-(x^y|x^y), f^-(y) \in [-1,0]$. Choose $t^- = \min\{f^-(x^y|x^y), f^-(y)\}$. Thus, we have $f^-(x^y|x^y) \geq t^-$ and $f^-(y) \geq t^-$. Therefore, $x^y|x^y, y \in N_U(f,t^-) \neq \emptyset$. By assumption, $N_U(f,t^-)$ is an SBG-ideal of A, so $x \in N_U(f,t^-)$. Thus, we have $f^-(x) \geq t^- = \min\{f^-(x^y|x^y), f^-(y)\}$. By Lemma 2.7 (1), we have

$$\begin{aligned}\overline{f^-}(x) &= -1 - f^-(x) \\ &\leq -1 - \min\{f^-(x^y|x^y), f^-(y)\} \\ &= \max\{-1 - f^-(x^y|x^y), -1 - f^-(y)\} \\ &= \max\{\overline{f^-}(x^y|x^y), \overline{f^-}(y)\}.\end{aligned}$$

Let $x \in A$. Then $f^+(x) \in [0,1]$. Choose $t^+ = f^+(x)$. Thus, we have $f^+(0) \leq f^+(x) = t^+$, so $x \in P_L(f,t^+) \neq \emptyset$. By assumption, $P_L(f,t^+)$ is an SBG-ideal of A, so $0 \in P_L(f,t^+)$. Thus, we have

$$f^+(0) \leq t^+ = f^+(x),$$

and therefore,

$$\overline{f^+}(0) = 1 - f^+(0) \geq 1 - f^+(x) = \overline{f^+}(x).$$

Finally, let $x, y \in A$. Then $f^+(x^y|x^y), f^+(y) \in [0,1]$. We choose $t^+ = \max\{f^+(x^y|x^y), f^+(y)\}$. Thus, we have $f^+(x^y|x^y) \leq t^+$ and $f^+(y) \leq t^+$, so $x^y|x^y, y \in P_L(f,t^+) \neq \emptyset$. By assumption, $P_L(f,t^+)$ is an SBG-ideal of A, so $x \in P_L(f,t^+)$. Thus, we have $f^+(x) \geq t^+ = \max\{f^+(x^y|x^y), f^+(y)\}$. By Lemma 2.7 (2), we have

$$\begin{aligned}\overline{f^+}(x) &= 1 - f^+(x) \\ &\leq 1 - \max\{f^+(x^y|x^y), f^+(y)\} \\ &= \min\{1 - f^+(x^y|x^y), 1 - f^+(y)\} \\ &= \min\{\overline{f^+}(x^y|x^y), \overline{f^+}(y)\}.\end{aligned}$$

Hence, we conclude that $\overline{f} = (A; \overline{f^-}, \overline{f^+})$ is a bipolar fuzzy SBG-ideal of A. □

Definition 3.20. Let $\langle A; |_A, 0_A \rangle$ and $\langle B; |_B, 0_B \rangle$ be SBG-algebras. Then, the mapping $f : A \to B$ is called a homomorphism if it satisfies:

(i) $\alpha(0_A) = 0_B$,

(ii) $\alpha(x|_A y) = \alpha(x)|_B\alpha(y)$

for all $x, y \in X$.

Lemma 3.21. *Let $\langle A; |_A, 0_A\rangle$ and $\langle B; |_B, 0_B\rangle$ be SBG-algebras. If the mapping $f : A \to B$ is a homomorphism, then $\alpha(x^y) = \alpha(x)|_B(\alpha(y)|_B\alpha(y))$ is verified for each $x, y \in A$.*

Proof. It is clearly obtained from Definition 3.20. □

Theorem 3.22. *Let $\langle A; |_A, 0_A\rangle$ and $\langle B; |_B, 0_B\rangle$ be SBG-algebras, let $\alpha : A \to B$ be a surjective homomorphism, and let f be a bipolar fuzzy set on B. Then f is a bipolar fuzzy SBG-ideal of B if and only if f^α is a bipolar fuzzy SBG-ideal of A, where $f^\alpha : A \to [-1, 0]$ is defined by $f^\alpha(x) = f(\alpha(x))$ for all $x \in A$.*

Proof. Assume that $\langle A; |_A, 0_A\rangle$ and $\langle B; |_B, 0_B\rangle$ are SBG-algebras, $\alpha : A \to B$ is a surjective homomorphism, and f is a bipolar fuzzy SBG-ideal of B. Let $x_1 \in A$. Then, we have

$$\begin{aligned} f^{-\alpha}(0_A) &= f^-(\alpha(0_A)) \\ &\leq f^-(\alpha(x_1)) \\ &= f^{-\alpha}(x_1), \end{aligned}$$

and

$$\begin{aligned} f^{-\alpha}(x_1) &= f^-(\alpha(x_1)) \\ &\leq \max\{f^-(\alpha(x_2)), f^-(\alpha(x_1^{x_2})|_B\alpha(x_1^{x_2}))\} \\ &= \max\{f^-(\alpha(x_2)), f^-(\alpha(x_1^{x_2}|_A x_1^{x_2}))\} \\ &= \max\{f^{-\alpha}(x_2), f^{-\alpha}(x_1^{x_2}|_A x_1^{x_2})\}. \end{aligned}$$

Also, we have

$$\begin{aligned} f^{+\alpha}(0_A) &= f^+(\alpha(0_A)) \\ &\geq f^+(\alpha(x_1)) \\ &= f^{+\alpha}(x_1), \end{aligned}$$

and

$$\begin{aligned} f^{+\alpha}(x_1) &= f^+(\alpha(x_1)) \\ &\geq \min\{f^+(\alpha(x_2)), f^+(\alpha(x_1^{x_2})|_B\alpha(x_1^{x_2}))\} \\ &= \min\{f^+(\alpha(x_2)), f^+(\alpha(x_1^{x_2}|_A x_1^{x_2}))\} \\ &= \min\{f^{+\alpha}(x_2), f^{+\alpha}(x_1^{x_2}|_A x_1^{x_2})\}. \end{aligned}$$

Hence f^α is a bipolar fuzzy SBG-ideal of A.

Conversely, let f^α be a bipolar fuzzy SBG-ideal of A. Let $y_1, y_2 \in B$ such that $\alpha(x_1^{x_2}) = y_1^{y_2}$ and for $x_1, x_2 \in A$. Then, we attain

$$\begin{aligned} f^-(0_B) &= f^-(\alpha(0_A)) \\ &\leq f^{-\alpha}(0_A) \\ &= f^-(\alpha(x_1)) \\ &= f^-(y_1), \end{aligned}$$

and

$$\begin{aligned} f^-(y_1) &= f^-(\alpha(x_1)) \\ &= f^{-\alpha}(x_1) \\ &\leq \max\{f^{-\alpha}(x_2), f^{-\alpha}(x_1^{x_2}|_A x_1^{x_2})\} \\ &= \max\{f^-(\alpha(x_2)), f^-(\alpha(x_1^{x_2}|_A x_1^{x_2}))\} \\ &= \max\{f^-(\alpha(x_2)), f^-(\alpha(x_1^{x_2})|_B\alpha(x_1^{x_2}))\} \\ &= \max\{f^-(y_2), f^-(y_1^{y_2}|_B y_1^{y_2})\}. \end{aligned}$$

Moreover, we get

$$\begin{aligned} f^+(0_B) &= f^+(\alpha(0_A)) \\ &\geq f^{+\alpha}(x_1) \\ &= f^+(\alpha(x_1)) \\ &= f^+(y_1), \end{aligned}$$

and

$$\begin{aligned} f^+(y_1) &= f^+(\alpha(x_1)) \\ &= f^{+\alpha}(x_1) \\ &\geq \min\{f^{+\alpha}(x_2), f^{+\alpha}(x_1^{x_2}|_A x_1^{x_2})\} \\ &= \min\{f^+(\alpha(x_2)), f^+(\alpha(x_1^{x_2}|_A x_1^{x_2}))\} \\ &= \min\{f^+(\alpha(x_2)), f^+(\alpha(x_1^{x_2})|_B \alpha(x_1^{x_2}))\} \\ &= \min\{f^+(y_2), f^+(y_1^{y_2}|_B y_1^{y_2})\}. \end{aligned}$$

Hence f is a bipolar fuzzy SBG-ideal of B. □

Theorem 3.23. *Let $\langle A; |_A, 0_A\rangle$ and $\langle B; |_B, 0_B\rangle$ be SBG-algebras, let $\alpha : A \to B$ be a surjective homomorphism, and let f be a bipolar fuzzy set on B. Then f is a bipolar fuzzy SBG-subalgebra of B if and only if f^α is a bipolar fuzzy SBG-subalgebra of A, where $f^\alpha : A \to [-1, 0]$ is defined by $f^\alpha(x) = f(\alpha(x))$ for all $x \in A$.*

Proof. Assume that $\langle A; |_A, 0_A\rangle$ and $\langle B; |_B, 0_B\rangle$ are SBG-algebras, $f : A \to B$ is a surjective homomorphism and f is a bipolar fuzzy SBG-subalgebra of B. Let $x_1, x_2 \in A$. Then, we have

$$\begin{aligned} f^{-\alpha}(x_1^{x_2}|_A x_1^{x_2}) &= f^-(\alpha(x_1^{x_2}|_A x_1^{x_2}) \\ &= f^-(\alpha(x_1^{x_2})|_B \alpha(x_1^{x_2})) \\ &\leq \max\{f^-(\alpha(x_1)), f^-(\alpha(x_2))\} \\ &= \max\{f^{-\alpha}(x_1), f^{-\alpha}(x_2)\}, \end{aligned}$$

and

$$\begin{aligned} f^{+\alpha}(x_1^{x_2}|_A x_1^{x_2}) &= f^+(\alpha(x_1^{x_2}|_A x_1^{x_2})) \\ &= f^+(\alpha(x_1^{x_2})|_B \alpha(x_1^{x_2})) \\ &\geq \min\{f^+(\alpha(x_1)), f^+(\alpha(x_2))\} \\ &= \min\{f^{+\alpha}(x_1), f^{+\alpha}(x_2)\}. \end{aligned}$$

Hence, f^α is a bipolar fuzzy SBG-subalgebra of A.

Conversely, let f^f be a bipolar fuzzy SBG-subalgebra of A. Let $y_1, y_2 \in B$ such that $\alpha(x_1^{x_2}) = y_1^{y_2}$ for $x_1, x_2 \in A$. Then, we have

$$\begin{aligned} f^+(y_1^{y_2}|_B y_1^{y_2}) &= f^+((\alpha(x_1^{x_2})|_B \alpha(x_1^{x_2})) \\ &= f^+(\alpha(x_1^{x_2}|_A x_1^{x_2})) \\ &= f^{+\alpha}(x_1^{x_2}|_A x_1^{x_2}) \\ &\geq \min\{f^{+\alpha}(x_1), f^{+\alpha}(x_2)\} \\ &= \min\{f^+(\alpha(x_1)), f^+(\alpha(x_2))\} \\ &= \min\{f^+(y_1), f^+(y_2)\}, \end{aligned}$$

and also

$$\begin{aligned} f^-(y_1^{y_2}|_B y_1^{y_2}) &= f^-((\alpha(x_1^{x_2})|_B \alpha(x_1^{x_2})) \\ &= f^-(\alpha(x_1^{x_2}|_A x_1^{x_2})) \\ &= f^{-\alpha}(x_1^{x_2}|_A x_1^{x_2}) \\ &\leq \max\{f^{-\alpha}(x_1), f^{-\alpha}(x_2)\} \\ &= \max\{f^-(\alpha(x_1)), f^-(\alpha(x_2))\} \\ &= \max\{f^-(y_1), f^-(y_2)\}. \end{aligned}$$

Hence, f is a bipolar fuzzy SBG-subalgebra of B. □

4 Conclusion

In this paper, we have conducted a comprehensive study of bipolar fuzzy sets in the context of Sheffer stroke BG-algebras (SBG-algebras). We introduced the notions of bipolar fuzzy SBG-subalgebras and SBG-ideals, and provided detailed definitions, characterizations, and structural properties of these concepts. The inclusion relation, extension and intension of bipolar fuzzy sets were formalized, and two types of (α, β)-translations were defined and analyzed. Through a series of theorems, we established the preservation of SBG-subalgebra and SBG-ideal properties under these translations, and also proved the converse implications.

Algorithmic approaches were presented to verify whether a bipolar fuzzy set is an SBG-subalgebra or SBG-ideal, thereby facilitating computational applications. Theoretical results were supported by concrete examples, illustrating the practical aspects of the developed concepts. In addition, we explored the role of negative and positive cuts, showing their connection with classical SBG-subalgebras and SBG-ideals, and provided characterizations involving complements of bipolar fuzzy sets.

Moreover, the preservation of bipolar fuzzy SBG-ideals and subalgebras under homomorphisms between SBG-algebras was established, further enriching the algebraic framework. The results obtained not only generalize and unify prior work on fuzzy algebraic structures but also highlight the interplay between fuzzy and crisp algebraic systems within the bipolar setting.

For future research, several promising directions emerge. These include the investigation of quotient structures and congruence relations in the setting of bipolar fuzzy SBG-algebras, the study of more general fuzzy ideals and subalgebras, and the development of computational algorithms for large-scale or infinite SBG-algebras. Applications in fields such as artificial intelligence, decision theory, and information sciences also present exciting possibilities, particularly where nuanced modeling of uncertainty and duality is required.

In summary, this work advances the theory of bipolar fuzzy algebraic structures, provides a foundation for further exploration, and opens new perspectives for both theoretical development and practical applications in the broader area of fuzzy logic and algebra.

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