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An Accelerated Adomian Decomposition–Laplace Transform Framework for Nonlinear Fractional Differential Equations

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Abstract This paper presents a hybrid method for solving nonlinear fractional differential equations by combining the Laplace transform with the accelerated Adomian decomposition method (AADM). The proposed Laplace-accelerated Adomian decomposition method (LAADM) improves computational efficiency by generating recursive polynomials without derivatives, simplifying implementation and reducing computation time. Numerical results show that LAADM provides fast-converging and highly accurate solutions, closely matching exact results across various test cases. This confirms the method's potential for broader application in solving complex fractional differential equations.

1 Introduction

The exploration of differential equations stands as a fundamental pillar within the field of mathematics, playing a pivotal role in understanding and resolving diverse real-world challenges. These mathematical constructs serve to depict the complex relation between a function and its derivatives.Contemporary focus has been directed towards fractional differential equations (FDEs) as they hold the remarkable capacity to model complex systems encountered in Mathematics, Engineering, Physics, Mechanics, Biology and Signal Processing.

Not long after classical calculus was developed, in 1695, fractional calculus was developed. Since fractional-order differential equations accurately represent many mathematical models, it is directly related to the dynamics of complicated real-world issues. The first comprehensive investigations in this field are attributed to figures such as Liouville, Riemann, and Leibniz. Oriti [1] applied fractional calculus to Quantum Gravity. Historically, fractional calculus was viewed as a purely mathematical discipline with few real-world applications.

However, in recent decades, the field has experienced significant development, proving that Fractional calculus can be effective and beneficial. A brief history of fractional calculus, especially in terms of its applications, is presented in Magin [2], who discusses its role in bioengineering. Today, fractional calculus and its applications are rapidly advancing, with increasing practical applications in various fields. Mishra et al. [3] highlight some technological applications of fractional calculus.

Fractional differentiation and integration research is by its very nature interdisciplinary, with applications spanning multiple areas such as well as other areas of pure and applied mathematics, such as turbulence, population dynamics, landscape evolution, medical imaging, bioengineering, biomedicine, financial systems, social systems, continuum mechanics, elasticity, signal analysis, quantum mechanics, and other fields.

Nonlinear phenomena have a profound influence on fields such as Mathematics, Engineering, Physics, Mechanics, Biology, and Signal Processing. These phenomena are often described via differential equations that are nonlinear. Differential equations continue to be an essential subject in both mathematics and science, necessitating creative methods to identify exact or approximate solutions. Numerical techniques are frequently employed to solve new linear and nonlinear equations because the majority of them lack precise analytic solutions. Singh et al. [4] presented a computationally efficient approach to the local fractional Fokker-Planck equation.

Recently, fractional differential equations (FDEs) have gained increasing prominence across various research fields, providing a more accurate representation of many important phenomena.

Solving these fractional differential equations has received a lot of attention.

The growing interest in fractional calculus applications has driven the exploration and development of numerical techniques specifically aimed at solving fractional differential equations (FDEs). It is more difficult to find analytical solutions for FDEs than for conventional ordinary differential equations (ODEs), and typically, solutions can only be approximated numerically. Jassim and Hussein [5] presented a novel formulation of the fractional derivative.

Abuteen [6] provides an analytical solution for fractional Riccati differential equations. Mohammed [7] introduces computational methods for resolving fractional-order delay differential equations. Pushpam and Lydia [8] propose the solution of nonlinear fractional differential equations with Caputo derivatives numerically using the Mahgoub Adomian Decomposition method. Ghomanjani and Bahmani [9] present a novel approach to solve nonlinear multi-order differential equations with fractional order.

Al-Mdallal [10] interduces the powerful Adomian decomposition method (ADM), which is used to solve nonlinear differential equations, adapted for fractional-order equations. Ibrahim and Murad [11] explore the solution of fractional differential equations with certain stability and existence findings. Mohammad and Trounev [12] apply an efficient numerical method based on Euler wavelets to solve a class of neutral delay differential equations numerically. Chen and Liu [13] present the homotopy perturbation method (HPM), adapted for fractional-order equations to effectively address nonlinear problems.

Mohammad and Cattani [14] present a collocation technique to address Volterra-Fredholm integral equations of the weakly singular type using quasi-affine biorthogonal systems. Lastly, Alzaki and Jassim [15] introduced the Sumudu homotopy perturbation method in order to provide approximate analytical solutions for nonlinear fractional ordinary differential equations.

This paper introduces a novel hybrid analytical technique for solving nonlinear fractional differential equations (FDEs) by integrating the accelerated Adomian decomposition method (AADM) with the Laplace integral transform. The key innovation lies in enhancing the convergence and efficiency of existing decomposition approaches through this combination.

Unlike standard methods, the proposed technique leverages fractional power series expansions within the Laplace domain, enabling more accurate and efficient handling of nonlinear terms and memory effects inherent in fractional systems. Before presenting the method in detail, we first provide a foundational overview of essential concepts and definitions in fractional calculus to establish the necessary context.

2 Definitions and Preliminaries

The fractional calculus hypothesis's fundamental definitions and characteristics, as outlined by Miller and Ross [16], introduced fractional differential equations and fractional calculus. Jumarie [17] presented a table of essential fractional calculus formulas, while Debnath and Bhatta [18] discussed integral transforms and their applications

In this section, we provide an overview of essential definitions and mathematical concepts related to fractional calculus that will be utilized in this work. This covers the meaning of the Mittag-Leffler function, Gamma function, Laplace transform, Riemann-Liouville (R-L) fractional order integration, Riemann-Liouville fractional order derivative, modified Riemann-Liouville derivative, and Caputo fractional order derivative.

Definition 2.1. The Gamma Function

The extension of a factorial function for the nonnegative integers is known as a gamma function which is denoted by the symbol Γ, Abramowitz, Stegun [[19] Chapter 6] A guide to mathematical functions and their formulations. For $\alpha > 0$, the gamma function $\Gamma(\alpha)$ defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \text{where} \quad \alpha > 0. \tag{2.1}$$

Some characteristics of the gamma function

$$\begin{aligned} &\text{(i)} && \Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1) \\ &\text{(ii)} && \Gamma(1/2) = \sqrt{\pi} \\ &\text{(iii)} && \Gamma(\alpha+1) = \alpha\,! \\ &\text{(iv)} && \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \\ &\text{(v)} && \Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}. \end{aligned} \tag{2.2}$$

Definition 2.2. The Riemann-Liouville fractional integrals

The Riemann–Liouville fractional integral, $I^{\alpha} f$, of order $\alpha \in$ R, $\alpha \geq 0$ of function $f(x) \in C_{\mu}$, $\mu \geq -1$, is defined as follows

$$I_t^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\mathrm{t})^{\alpha-1} f(\mathrm{t})\,\mathrm{dt}, \alpha > 0,\, x > 0 \tag{2.3}$$

$$I^0 \mathrm{f}(x) = \mathrm{f}(x) \tag{2.4}$$

Properties of the operator I^{α} is given by

$$\begin{aligned} &\text{(i)} && I^{\alpha} I^{\beta} f(x) = I^{\alpha+\beta} f(x) \\ &\text{(ii)} && I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x) \\ &\text{(iii)} && I^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \end{aligned} \tag{2.5}$$

Definition 2.3. [20] **The fractional order derivative of Riemann-Liouville**

The fractional order derivative of Riemann-Liouville of the function $f(x)$ is given as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)dt, \tag{2.6}$$

where $\mathrm{n}-1 \leq \alpha < n$, $\mathrm{n} \in Z^{+}$.

Definition 2.4. The fractional order derivative of modified Riemann-Liouville

A variant of the normal Riemann-Liouville fractional derivative is the modified Riemann-Liouville fractional order derivative. It is used to address some issues with initial conditions and is more suitable in some contexts, especially when dealing with physical problems. The fractional order derivative of modified Riemann-Liouville of a function $f(x)$ of order α (where $\alpha > 0$) is defined as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_t^x (t-x)^{m-\alpha} (f(x) - f(0))dx, \tag{2.7}$$

where $x \in [0,1]$, $\mathrm{m} - 1 \leq \alpha < m$ and where $m \geq 1$.

Definition 2.5. [21,22] **Laplace transform**

A powerful integral transform, the **Laplace transform** changes a function of time (usually denoted $f(x)$ into a complex variable's function. It is widely used in engineering, physics, and mathematics, particularly for solving differential equations. Assuming that the function f is defined for $x \geq 0$, the improper integral defines the Laplace transform of f, represented by $\mathcal{L}\{f\}$

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^{\infty} e^{-sx} f(x)\,dx, \tag{2.8}$$

Provided the integral is convergent.

The inverse Laplace transform is defined by

$$\mathcal{L}^{-1} F(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} F(s)\,ds. \tag{2.9}$$

Next, we provide the Laplace transform for the fractional derivative of Caputo and the Riemann-Liouville fractional integral operator.

Lemma 2.1. [23] The form of the Laplace transform of the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is as follows:

$$\mathcal{L}\left[I^{\alpha} f(x)\right] = \frac{F(s)}{s^{\alpha}} . \tag{2.10}$$

It is easy to prove the lemma since

$$\begin{aligned} \mathcal{L}\left[I^{\alpha} f(x)\right] &= \mathcal{L}\left[\frac{1}{\Gamma(\alpha)} \int_0^x (x-\mathrm{t})^{\alpha-1} f(\mathrm{t})\, \mathrm{dt}\right] \\ &= \frac{1}{\Gamma(\alpha)} F(s)\, G(s), \end{aligned} \tag{2.11}$$

where,

$$G(s) = \mathcal{L}\left[x^{\alpha-1}\right] . \tag{2.12}$$

Lemma 2.2. [23] The Caputo fractional derivative's Laplace transform for $m-1 < \alpha \leq \mathrm{m}$, $\mathrm{m} \in \mathbb{N}$, is given in the form

$$\mathcal{L}\left[D^{\alpha} f(x)\right] = \frac{s^{\mathrm{m}} F(s) - s^{\mathrm{m}-1} f(0) - s^{\mathrm{m}-2} f'(0) - \cdots - f^{(\mathrm{m}-1)}(0)}{s^{\mathrm{m}-\alpha}} . \tag{2.13}$$

It is straightforward to demonstrate the proof since

$$\begin{aligned} \mathcal{L}\left[D^{\alpha} f(x)\right] &= \mathcal{L}\left[I^{\mathrm{m}-\alpha} f^{(\mathrm{m})}(x)\right] \\ &= \frac{\mathcal{L}\left[f^{(\mathrm{m})}(x)\right]}{s^{\mathrm{m}-\alpha}} . \end{aligned} \tag{2.14}$$

Definition 2.6. [20] The Caputo fractional derivative of a function $f(x) \in C_{\mu}, \mu \geq -1$, of order $\alpha \in R$, $\alpha > 0$ defined as

$$\begin{aligned} {}_0^cD_x^{\alpha} f(x) &= I^{n-\alpha} D^n f(x) \\ &= \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^n(\tau) d\tau, & x > 0, (n-1) < \alpha < \mathrm{n},\ \mathrm{n} \in N \\ \frac{d^n f(x)}{dx^n}, & \alpha = n . \end{cases} \end{aligned} \tag{2.15}$$

The Laplace transform to Caputo's fractional derivative gives:

$$\begin{aligned} \mathcal{L}\left\{{}_0^cD_x^{\alpha} f(x)\right\} &= \mathcal{L}\left\{\frac{d^m}{dx^m} f(x)\right\} \\ &= s^{\alpha} F(s) - \sum_{m=0}^{n-1} s^{\alpha-m-1} f^{(m)}(0), \qquad (n-1 < \alpha < n) \end{aligned} \tag{2.16}$$

A few characteristics of the fractional derivative of Caputo

$$\begin{array}{ll} \text{(i)} & {}_0^cD_x^{\alpha}\left(I_x^{\alpha} f(x)\right) = f(x) \\ \text{(ii)} & I_x^{\alpha}\left({}_0^cD_x^{\alpha} f(x)\right) = f(x) - \sum_{k=0}^{n-1} f^{(k)}\left(0^{+}\right) \frac{x^k}{k!} \\ \text{(iii)} & {}_0^cD_x^{\alpha}(c) = 0, \quad c \in \mathbb{R} \\ \text{(iv)} & {}_0^cD_x^{\alpha}\left(x^{\gamma}\right) = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, & \gamma \in \{0,1,2,3,\ldots\},\ \gamma \geq [\alpha] \\ 0 & \gamma \in \{0,1,2,3,\ldots\},\ \gamma < [\alpha] \end{cases} \end{array} \tag{2.17}$$

where the floor function of α is $[\alpha]$.

Solving fractional differential equations has been a unique application of the Mittag-Leffler function and its generalized versions. As a generalization of the Mittag-Leffler function, Shukla [24] presented the so-called Mittag-Leffler function with two parameters$E_{\alpha,\beta}\left(Z\right)$, which has the following form:

$$E_{\alpha,\beta}\left(Z\right)=\sum_{j=0}^{\infty}\frac{Z^{j}}{\Gamma\left(\alpha \mathrm{j}+\beta\right)},\qquad \alpha>0,\beta>0. \tag{2.18}$$

Its k^{th} derivative is defined as,

$$E_{\alpha,\beta}^{(k)}(Z)=\sum_{k=0}^{\infty}\frac{(j+k)!Z^{j}}{j!\Gamma(\alpha j+\alpha k+\beta)},\qquad (k=0,1,2\cdots) \tag{2.19}$$

So, it is convenient to introduce the function

$$\epsilon_{k}\left(t,a:\ \alpha,\beta\right)=t^{\alpha \mathrm{k}+\beta-1}E_{\alpha,\beta}^{(k)}\left(\pm at^{\alpha}\right). \tag{2.20}$$

Its Laplace transform is

$$\int_{0}^{\infty}e^{-st}\epsilon_{k}\left(t,a:\ \alpha,\beta\right)dt=\frac{k!s^{\alpha-\beta}}{\left(s^{\alpha}\mp a\right)^{k+1}}\qquad\left(Re\left(s\right)>\left|a\right|^{1/\alpha}\right). \tag{2.21}$$

Hence

$$\mathcal{L}^{-1}\left[\frac{k!s^{\alpha-\beta}}{\left(s^{\alpha}\ \mp\ a\right)^{k+1}}\right]=t^{\alpha \mathrm{k}+\beta-1}E_{\alpha,\beta}^{(k)}\left(\pm at^{\alpha}\right). \tag{2.22}$$

Another convenient property of $\epsilon_{k}\left(t,a:\ \alpha,\beta\right)$, its simple fractional differentiation

$$D_{t}^{\lambda}\epsilon_{k}\left(t,a:\ \alpha,\beta\right)=\epsilon_{k}\left(t,a:\ \alpha,\beta-\lambda\right),\qquad (\lambda<\beta). \tag{2.23}$$

The basic characteristics of fractional calculus derived from the Riemann–Liouville definition are not satisfied by fractional derivatives with nonsingular kernels, such as the Caputo–Fabrizio or Atangana–Baleanu types. Many mathematical and physical models depend on the power-law memory effect, which they are unable to replicate. Additionally, they disrupt the relationship between fractional integration and differentiation as well as the semigroup property. Diethelm et al. [25] views the reasons to avoid using fractional derivatives with nonsingular kernels in Communications in Nonlinear Science and Numerical Simulation.

Consequently, the Riemann–Liouville derivative continues to be more consistent and suitable for characterizing real fractional dynamics from both mathematical and physical perspectives.

3 Method of Solution using Laplace Accelerated Adomian Method (LAADM)

The Laplace-Adomian decomposition method (LADM) is a sophisticated analytical methodology that solves nonlinear integro-differential equations with fractional order by combining the Adomian decomposition method and the Laplace transform. LADM's versatility enables it to be used in a variety of mathematical models, including algebraic, differential, integral, and integro-differential equations. Gaxiola [26] applied it to the Kundu–Eckhaus equation. A more efficient version, the Laplace-accelerated Adomian decomposition method (LAADM) combines the accelerated Adomian decomposition approach with the Laplace transform, fractional derivatives can be simplified to solve nonlinear fractional-order differential equations efficiently and addressing nonlinearity. Ramadan et al. [27] introduced a numerical and analytical solution of time-fractional nonlinear partial differential equations by employing the accelerated Adomian decomposition approach with the Laplace transform, while Arafa et al. [28], present a variant of accelerated Ramadan group Adomian decomposition approach to solve fractional Riccati differential equations. El-Kalla et al. [29] used it for differential equations with nonlinear delays.

In this study, we apply both LADM and LAADM for solving non-linear fractional ordinary differential equation in form:

$$D_t^{\alpha} y(t) + Ry(t) + Ny(t) = f(t), \quad \mathrm{n}-1 < \alpha \leq \mathrm{n}, \tag{3.1}$$

where $D_t^{\alpha} = {}_0^cD_t^{\alpha}$ is a Caputo fractional derivative operator, R, N are linear and nonlinear operators, respectively, f is a function that shows the homogeneity of the differential equation, and y depends on t, which may be found using the initial conditions listed below.

$$y(0) = w,\ y^{'}(0) = w_1,\ y^{''}(0) = w_2,\ \ldots, y^{(\mathrm{n}-1)}(0) = w_{(\mathrm{n}-1)}. \tag{3.2}$$

3.1 Solution Approach via the Laplace Adomian Method (LADM)

To discuss the solution of Equation (3.1), using LADM we apply the following steps:

Step 1: Utilizing the linearity of the Laplace transform and applying it to both sides of Eq. (3.1), we obtain:

$$\mathcal{L}[D_t^{\alpha} y(t)] = \mathcal{L}[f(t)] - \mathcal{L}[Ny(t)] - \mathcal{L}[Ry(t)]. \tag{3.3}$$

Step 2: Considering the Caputo-sense definition of the Laplace transform for fractional derivatives (2.6), we obtain:

$$s^{\alpha}\mathcal{L}[y(t)] - \sum_{i=0}^{n-1} s^{\alpha-\mathrm{i}-1}\left(D_t^{\mathrm{i}} y\right)(0) = \mathcal{L}[f(t)] - \mathcal{L}[Ry(t)] - \mathcal{L}[Ny(t)]. \tag{3.4}$$

Hence, we have

$$Y(s) = \frac{1}{s^{\alpha}}\sum_{i=0}^{n-1} s^{\alpha-\mathrm{i}-1}\left(D_t^{\mathrm{i}} y\right)(0) + \frac{1}{s^{\alpha}}\mathcal{L}[f(t)] - \frac{1}{s^{\alpha}}\mathcal{L}[Ry(t) + Ny(t)]. \tag{3.5}$$

Step 3: At this point, the solution $y(t)$, is defined by the series

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{3.6}$$

as well as the nonlinear term's decomposition as

$$Ny(t) = \sum_{n=0}^{\infty} A_n, \tag{3.7}$$

Where, $A_0, A_1, A_2, A_3, \ldots$ are Adomian polynomials, and A_n can be computed as:

$$A_n = \frac{1}{n\,!}\left(\frac{d^n}{d\lambda^n}\left[N\sum_{i=0}^{\infty}\left(\lambda^i y_i\right)\right]\right)_{\lambda=0}, \quad n = 0, 1, 2, 3\ldots. \tag{3.8}$$

Step 4: Taking inverse L-transform to Eq. (3.5), then by substituting Eqs. (3.6) and (3.7) in Eq. (3.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(t) =& \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\sum_{i=0}^{n-1} s^{\alpha-\mathrm{i}-1}\left(D_t^{\mathrm{i}} y\right)(0)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[f(t)]\right] \\ & - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[R\left(\sum_{n=0}^{\infty} y_n(t)\right) + \sum_{n=0}^{\infty} A_n]\right].\end{aligned} \tag{3.9}$$

Step 5: And by comparing the both sides of Eq. (3.9) yield the iterative algorithm that follows:

$$y_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\sum_{i=0}^{n-1} s^{\alpha-\mathrm{i}-1} D_t^{\mathrm{i}} y(0)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[f(t)]\right], \tag{3.10}$$

$$y_1(t) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[Ry_0(t) + A_0\right]\right], \tag{3.11}$$

$$y_2(t) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[Ry_1(t) + A_1\right]\right], \tag{3.12}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n\left(t\right) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[Ry_{(n-1)}\left(t\right) + A_{n-1}\right]\right], \quad n \geq 1. \tag{3.13}$$

To improve the clarity and comprehension of the suggested method, we will present a brief, sequentially explanation of our computational approach for solving the FDEs. The structure of the proposed technique is as follows:

3.2 Solution Approach via the Laplace accelerated Adomian Method (LAADM)

Step 1: Utilizing the linearity of the Laplace transform and applying it to both sides of Eq. (3.1), we obtain:

$$\mathcal{L}[D_t^{\alpha}y\left(t\right)] = \mathcal{L}\left[f\left(t\right)\right] - \mathcal{L}\left[Ny\left(t\right)\right] - \mathcal{L}\left[Ry\left(t\right)\right]. \tag{3.14}$$

Step 2: Considering the Caputo-sense definition of the Laplace transform for fractional derivatives (2.6), we obtain:

$$s^{\alpha}\mathcal{L}\left[y\left(t\right)\right] - \sum_{i=0}^{n-1} s^{\alpha-\mathrm{i}-1}\left(D_t^{\mathrm{i}}y\right)(0) = \mathcal{L}\left[f\left(t\right)\right] - \mathcal{L}\left[Ry\left(t\right)\right] - \mathcal{L}\left[Ny\left(t\right)\right]. \tag{3.15}$$

Hence, we have

$$Y\left(s\right) = \frac{1}{s^{\alpha}}\sum_{i=0}^{n-1} s^{\alpha-\mathrm{i}-1}\left(D_t^{\mathrm{i}}y\right)(0) + \frac{1}{s^{\alpha}}\mathcal{L}\left[f\left(t\right)\right] - \frac{1}{s^{\alpha}}\mathcal{L}\left[Ry\left(t\right) + Ny\left(t\right)\right]. \tag{3.16}$$

Step 3: At this point, the solution $y(t)$, is defined by the series

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{3.17}$$

as well as the nonlinear term's decomposition as

$$Ny = \sum_{n=0}^{\infty} \overline{A}_n\ , \tag{3.18}$$

where, $\overline{A}_0$, $\overline{A}_1$, $\overline{A}_2, \overline{A}_3, \ldots$ are accelerated Adomian polynomial, $\overline{A}_n$ can be computed as

$$\overline{A}_n = N\left(s_n\right) - \sum_{i=0}^{n-1}\overline{A}_i, \qquad \text{where} \qquad s_n = \sum_{i=0}^{n} y_i. \tag{3.19}$$

Step 4: Taking inverse L-transform to Eq. (3.16), then by substituting Eqs. (3.17) and (3.18) in Eq. (3.16), we have

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(t) =& \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\sum_{i=0}^{n-1} s^{\alpha-\mathrm{i}-1}\left(D_t^{\mathrm{i}}y\right)(0)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[f\left(t\right)\right]\right] \\ &- \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[R(\sum_{n=0}^{\infty} y_n\left(t\right)) + \sum_{n=0}^{\infty}\overline{A}_n]\right].\end{aligned} \tag{3.20}$$

Step 5: And by comparing the both sides of Eq. (3.20) yield the iterative algorithm that follows:

$$y_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\sum_{i=0}^{n-1} s^{\alpha-\mathrm{i}-1} D_t^{\mathrm{i}} y(0)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[f\left(t\right)\right]\right], \tag{3.21}$$

$$y_1(t) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[Ry_0(t) + \overline{A}_0\right]\right], \tag{3.22}$$

$$y_2(t) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[Ry_1(t) + \overline{A}_1\right]\right], \tag{3.23}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n(t) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[Ry_{n-1}(t) + \overline{A}_{n-1}\right]\right] \qquad n \geq 1. \tag{3.24}$$

Next, the forms of the Adomian and accelerated Adomian polynomials for functions with nonlinear terms are presented. For instance, Table 1 presents the Adomian and the accelerated Adomian polynomials for the nonlinear term y^2. Similarly, Table 2 shows the Adomian and accelerated Adomian polynomials for the nonlinear termy^3, It is evident that the terms involving accelerated Adomian appear more quickly than those using the standard Adomian.

Adomian polynomials of y^2	**Accelerated Adomian polynomials of y^2**
$A_0 = y_0^2$	$\overline{A}_0 = y_0^2$
$A_1 = 2y_0y_1$	$\overline{A}_1 = 2y_0y_1 + y_1^2$
$A_2 = y_1^2 + 2y_0y_2$	$\overline{A}_2 = 2y_0y_2 + 2y_1y_2 + y_2^2$
$A_3 = 2y_1y_2 + 2y_0y_3$	$\overline{A}_3 = 2y_0y_3 + 2y_1y_3 + 2y_2y_3 + y_3^2$
$A_4 = y_2^2 + 2y_1y_3 + 2y_0y_4$	$\overline{A}_4 = 2y_0y_4 + 2y_1y_4 + 2y_2y_4 + 2y_3y_4 + y_4^2$

Table 1: Adomian and accelerated Adomian polynomials of the nonlinear term y^2

Adomian polynomials of y^3	**Accelerated Adomian polynomials of y^3**
$A_0 = y_0^3$	$\overline{A}_0 = y_0^3$
$A_1 = 3y_1y_0^2$	$\overline{A}_1 = 3y_1y_0^2 + 3y_0y_1^2 + y_1^3$
$A_2 = 3y_0y_1^2 + 3y_2y_0^2$	$\overline{A}_2 = 3y_2y_0^2 + 6y_0y_1y_2 + 3y_0y_2^2 + 3y_2y_1^2 + 3y_1y_2^2 + y_2^3$
$A_3 = y_1^3 + 3y_3y_0^2 + 6y_0y_1y_2$	$\overline{A}_3 = 3y_3y_0^2 + 6y_0y_1y_3 + 6y_0y_2y_3 + 3y_0y_3^2 + 3y_3y_1^2 + 6y_1y_2y_3 + 3y_1y_3^2 + 3y_3y_2^2 + 3y_2y_3^2 + y_3^3$

Table 2: Adomian and accelerated Adomian polynomials of the nonlinear term y^3

4 Convergence Analysis

In this part, the convergence of the solution is discussed and the necessary condition that guarantees the existence of a unique solution is introduced. Obeidat et al. [30] provided the J-transform decomposition method theoretically analyzed and applied it to nonlinear ordinary differential equations.

Theorem 4.1. (Uniqueness theorem): There is a unique solution to Equation (3.13) whenever

$$0 < k < 1 \qquad \text{for} \qquad k = \frac{(L_1 + L_2)t^{\alpha}}{\Gamma(\alpha{+}1)}.$$

Proof. Let X be the Banach space for every continuous function on $I = [0, T]$ with the norm

$$\|y(t)\| = \max_{t\in I} |y(t)| . \tag{4.1}$$

A mapping $F : X \to X$ is defined, where

$$y_n\,(t) = \mathcal{L}^{-1}\,\left[\frac{1}{s^\alpha}\mathcal{L}\,[(Ry_{n-1}\,(t)) + N\,(y_{n-1}\,(t))]\right],\;\; n \geq 1. \tag{4.2}$$

Now, consider R and N to be Lipschitzian with $|R\,(y) - R(\overline{y})| < L_1\,|y - \overline{y}|$ and$|N\,(y) - N(\overline{y})| < L_2\,|y - \overline{y}|$ where constant L_1 and L_2 are Lipchitz constant respectively and y, $\overline{y}$ is different values of the function.

$$\begin{aligned}
&\|Fy - \; F\,\overline{y}\| = \\
&\max_{t\in I}\left|\mathcal{L}^{-1}\,\left[\frac{1}{s^\alpha}\mathcal{L}\,[(Ry_{n-1}\,(t)) + N(y_{n-1}\,(t))]\right] - \mathcal{L}^{-1}\,\left[\frac{1}{s^\alpha}\mathcal{L}\,[(R\,\overline{y}_{\,n-1}\,(t)) + N(\overline{y}_{\,n-1}\,(t))]\right]\right| \\
&\qquad \leq \max_{t\in I}\,[\,\mathcal{L}^{-1}\,[\frac{1}{s^\alpha}\mathcal{L}\,|Ry\,(t) - R\overline{y}\,(t)| + \frac{1}{s^\alpha}\mathcal{L}\;|Ny\,(t) - N\overline{y}\,(t)|]\,] \\
&\qquad \leq \max_{t\in I}\,[\,L_1\mathcal{L}^{-1}\,[\frac{1}{s^\alpha}\mathcal{L}\,|y\,(t) - \overline{y}\,(t)|] + \frac{1}{s^\alpha}L_2\mathcal{L}^{-1}\;\mathcal{L}\;|y\,(t) - \overline{y}\,(t)|] \\
&\qquad \leq \max_{t\in I}\,(\,L_1 + L_2)[\mathcal{L}^{-1}\,[\frac{1}{s^\alpha}\mathcal{L}\,|y\,(t) - \overline{y}\,(t)| + \frac{1}{s^\alpha}\mathcal{L}^{-1}\;\mathcal{L}\;|y\,(t) - \overline{y}\,(t)|] \\
&\qquad \leq \max_{t\in I}\,(\,L_1 + L_2)[\mathcal{L}^{-1}\,[\frac{1}{s^\alpha}\mathcal{L}\,\|\mathrm{y(t)}-\bar{\mathrm{y}}\mathrm{(t)}\|]
\end{aligned} \tag{4.3}$$

$$\|Fy - \; F\,\overline{y}\| = (L_1 + L_2)\;\frac{t^\alpha}{\Gamma(\alpha{+}1)}\;\;\|y\,(t) - \;\overline{y}\,(t)\|$$

$$\|Fy - \; F\,\overline{y}\| \leq k\,\|y\,(t) - \;\overline{y}\,(t)\| .$$

Under condition $0 < \frac{(L_1+L_2)t^\alpha}{\Gamma(\alpha+1)} < 1$, according to the Banach fixed point theorem for contraction, there is a unique solution to Eq. (3.13) since the mapping F is contraction. This concludes proof of the Theorem 4.1. □

Theorem 4.2. (Convergence Theorem): Generally, the solution of Equation (3.1) will be convergence.

Proof. Let s_n represent n^{th} partial sum, i. e, $s_n = \sum_{i=0}^{n} y_i\,(t)$. We will show that in Banach space X, $\{s_n\}$ is a Cauchy sequence.
Consider

$$\begin{aligned}
\|s_n - s_p\| &= \max_{t\in I} |s_n - s_p| \\
&= \max_{t\in I}\left|\sum_{i=p+1}^{n} y_i(t)\right|,\; n\; = 1,\; 2\;,3\;,\ldots \\
&\leq \max_{t\in I}\left|\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\sum_{i=p+1}^{n} R\,(y_{i-1}\,(t))\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\;\mathcal{L}\left[\sum_{i=p+1}^{n}\overline{A}_{i-1}\,(t)\right]\right]\right|
\end{aligned}$$

$$\begin{aligned}
&= \max_{t\in I}\left|\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathcal{L}\left[\sum_{i=p}^{n-1} R\left(y_i\left(t\right)\right)\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{i=p}^{n-1}\overline{A}_i\left(t\right)\right]\right]\right| \\
&\leq \max_{t\in I}\left|\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathcal{L}\left[L_1\left(s_{n-1}\right)-L_1\left(s_{p-1}\right)\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathcal{L}\left[L_2\left(s_{n-1}\right)-L_2\left(s_{p-1}\right)\right]\right]\right| \\
&\leq L_1\max_{t\in I}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathcal{L}\left[\left\|s_{n-1}-s_{p-1}\right\|\right]\right] + L_2\max_{t\in I}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathcal{L}\left[\left\|s_{n-1}-s_{p-1}\right\|\right]\right] \\
&= \left(L_1+L_2\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\left\|s_{n-1}-s_{p-1}\right\|.
\end{aligned} \tag{4.4}$$

Thus,

$$\left\|s_n - s_p\right\| \leq k\left\|s_{n-1}-s_{p-1}\right\|. \tag{4.5}$$

Choose $n = p + 1$, then

$$\left\|s_{n+1}-s_p\right\| \leq k\left\|s_n - s_{p-1}\right\| \leq k^2\left\|s_{n-1}-s_{p-2}\right\| \leq \cdots \leq k^p\left\|s_1 - s_0\right\|. \tag{4.6}$$

Similarly, using the triangle inequality

$$\begin{aligned}
\left\|s_{n+1}-s_p\right\| &\leq \left\|s_{n+1}-s_p\right\| + \left\|s_{n+2}-s_{p+1}\right\| + \cdots + \left\|s_n - s_{n-1}\right\| \\
&\leq \left[k^p + k^{p+1} + \cdots + k^{m-1}\right]\left\|s_1 - s_0\right\| \\
&\leq k^{\mathrm{p}}\left[\frac{1-k^{\mathrm{n-p}}}{1-k}\right]\left\|y_1\right\|
\end{aligned} \tag{4.7}$$

But $0 < k < 1$, then $1 - k^{\mathrm{n-p}} < 1$, then

$$\left\|s_n - s_p\right\| \leq \frac{k^{\mathrm{p}}}{1-k}\max_{t\in I}\left|y_1\right|. \tag{4.8}$$

Since $y(t)$ is bounded, then $\|y_1\| < \infty$. So, as $p \to \infty$, $\|s_n - s_p\| \to 0$. Hence $\{s_n\}$ is a Cauchy sequence in X. Therefore, the series $y\left(t\right) = \sum_{i=0}^{\infty} y_i(t)$ converges.

Hence, theorem 4.2 has been fully proved. □

Theorem 4.3. (Error estimate): The series solution's maximum absolute truncation error, to the equation provided by Eq.

$$\begin{aligned}
\left|E_m(t)\right| &= \max_{t\in I}\left|y\left(t\right) - \sum_{n=0}^{m} y_n\left(t\right)\right| \\
&\leq \frac{k^m}{1-k}\max_{t\in I}\left|y_1\right|.
\end{aligned} \tag{4.9}$$

Proof. From Eq. (3.8) and theorem 4.2, we have

$$\left\|s_n - s_m\right\| \leq \frac{k^{\mathrm{m}}}{1-k}\max_{t\in I}\left|y_1\right|, \tag{4.10}$$

as $n \to \infty$, then $s_n \to y\left(t\right)$ so, we have

$$\left\|y\left(t\right) - s_m\right\| \leq \frac{k^m}{1-k}\max_{t\in I}\left|y_1(t)\right|. \tag{4.11}$$

Ultimately, the interval I's maximum absolute truncation error, as estimated, is

$$\begin{aligned}
\max_{t\in I}\left|y\left(t\right) - \sum_{n=0}^{m} y_n(t)\right| &\leq \frac{k^m}{1-k}\max_{t\in I}\left|y_1\left(t\right)\right| \\
&= \frac{k^m}{1-k}\left\|y_1(t)\right\|
\end{aligned} \tag{4.12}$$

Therefore, theorem 4.3 has been fully proved. □

5 Application of the LAADM and Numerical Discussions

This section contains test problems that show the effectiveness and accuracy of the proposed hybrid technique (LAADM). In each case, we developed a new MATHEMATICA program to solve issues raised by the suggested combination strategy.

Example 5.1. [6, 31] Consider the fractional logistic differential equations that follow

$$D_t^{\alpha} y(t) = y(t) - y^2(t), \qquad 0 \leq \mathrm{t} \leq 1, \ \ 0 < \alpha \leq 1, \tag{5.1}$$

under the initial condition $y(0) = \frac{1}{2}$, whose exact solution for $\alpha = 1$ is

$$y(t) = \frac{1}{e^{-t} + 1}. \tag{5.2}$$

This example is treated by many researchers. The most recent ones are by Abuteen [6] utilizing the Caputo-Fabrizio fractional nonhomogeneous differential equations solution. Khirsariya and Rao [31] presented an analytic solution using hybrid technique, which combines the homotopy perturbation method and the Sawi transform.

For this example, we use Laplace with the normal Adomian decomposition method and the suggested accelerated Adomian to compare the accuracy of both methods for this illustration.

5.1 Solution using Laplace Transform coupled with Adomian decomposition method

Applying Laplace transform for Eq. (5.1) and by using the Laplace transform's fractional derivatives properties, we have

$$s^{\alpha} Y(s) - s^{\alpha-1} y(0) = \mathcal{L}[y(t)] - \mathcal{L}\left[y^2(t)\right], \tag{5.3}$$

$$Y(s) - \frac{s^{\alpha-1}}{2s^{\alpha}} = \frac{1}{s^{\alpha}}\mathcal{L}[y(t)] - \frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right], \tag{5.4}$$

$$Y(s) - \frac{1}{2s} = \frac{1}{s^{\alpha}}\mathcal{L}[y(t)] - \frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right], \tag{5.5}$$

$$Y(s) = \frac{1}{2s} + \frac{1}{s^{\alpha}}\mathcal{L}[y(t)] - \frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right], \tag{5.6}$$

For Eq. (5.6), take the inverse Laplace transform to both sides, we obtain:

$$y(t) = \frac{1}{2} + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[y(t)]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right]\right]. \tag{5.7}$$

After that the solution is represented as an infinite series, and the nonlinear term can be defined as follows, respectively

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n(t), \\ y^2(t) &= \sum_{n=0}^{\infty} A_n(y), \end{aligned} \tag{5.8}$$

where, A_n are Adomian polynomial and it may be determined using formula provided below

$$A_n = \frac{1}{n\,!}\left(\frac{d^n}{d\lambda^n}\left[N\sum_{i=0}^{\infty}\left(\lambda^i y_i\right)\right]\right)_{\lambda=0}, \qquad n = 0\,,\ 1\,,\ 2\,, 3 \ldots\,. \tag{5.9}$$

To substitute Eq. (5.8) in Eq. (5.7), we get:

$$\sum_{n=0}^{\infty} y_n = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{n=0}^{\infty} y_n\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{5.10}$$

Then from Eq. (5.10) we get:

$$y_0 = \frac{1}{2}\,, \tag{5.11}$$

$$y_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y_0\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[A_0\right]\right], \tag{5.12}$$

$$y_2 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y_1\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[A_1\right]\right], \tag{5.13}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y_{n-1}\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[A_{n-1}\right]\right],\ n\ =\ 1,\ 2\ ,\ 3, \ldots. \tag{5.14}$$

Where the first few components of A_n are given by

$$A_0 = y_0^2, \tag{5.15}$$

$$A_1 = 2y_0y_1, \tag{5.16}$$

$$A_2 = y_1^2 + 2y_0y_2, \tag{5.17}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Then, from Eq. (5.12) we get

$$\begin{aligned} y_1 &= \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y_0\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[A_0]\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{1}{2}\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[\frac{1}{4}]\right] \\ &= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] \\ &= \frac{t^{\alpha}}{4\Gamma(1+\alpha)}\,. \end{aligned} \tag{5.18}$$

Similarly, we get $y_{\,2},\ y_3,\ \ldots$ as follows:

$$y_2 = 0, \tag{5.19}$$

$$y_3 = -\frac{t^{3\alpha}\Gamma(1+2\alpha)}{16\Gamma\left([1+\alpha]\right)^2\Gamma(1+3\alpha)}\,, \tag{5.20}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$\begin{aligned} y\left(t\right) &= \sum_{n=0}^{\infty} y_n \\ &= y_0 + y_1 + y_2 + y_3 + \ldots \\ &= \frac{1}{2} + \frac{t^{\alpha}}{4\Gamma(1+\alpha)} - \frac{t^{3\alpha}\Gamma\left(1+2\alpha\right)}{16\Gamma\left([1+\alpha]\right)^2\Gamma\left(1+3\alpha\right)} + \ldots. \end{aligned} \tag{5.21}$$

After four iterations and computation at $\alpha = 1$, the approximate solution has the following form.

$$y\left(t\right) \approx \frac{1}{2} + \frac{t}{4} - \frac{t^3}{48}. \tag{5.22}$$

5.2 Solution using Laplace Transform coupled with accelerated Adomian decomposition Method

Similarly, applying Laplace transform for Eq. (5.1) and by using the Laplace transform's fractional derivatives properties, we have

$$s^{\alpha}Y(s) - s^{\alpha-1}y(0) = \mathcal{L}[y(t)] - \mathcal{L}\left[y^2(t)\right], \tag{5.23}$$

$$Y(s) - \frac{s^{\alpha-1}}{2s^{\alpha}} = \frac{1}{s^{\alpha}}\mathcal{L}[y(t)] - \frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right], \tag{5.24}$$

$$Y(s) - \frac{1}{2s} = \frac{1}{s^{\alpha}}\mathcal{L}[y(t)] - \frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right], \tag{5.25}$$

$$Y(s) = \frac{1}{2s} + \frac{1}{s^{\alpha}}\mathcal{L}[y(t)] - \frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right]. \tag{5.26}$$

And for Eq. (5.26), take the inverse Laplace transform to both sides, we obtain:

$$y(t) = \frac{1}{2} + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[y(t)]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right]\right]. \tag{5.27}$$

By the same way the solution is represented as an infinite series, and the nonlinear term can be defined as follows, respectively

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n(t), \\ y^2(t) &= \sum_{n=0}^{\infty} \overline{A}_n(y), \end{aligned} \tag{5.28}$$

Where $\overline{A}_n$ are the accelerated Adomian polynomials and they may be determined using formula provided below:

$$\overline{A}_n = N(s_n) - \sum_{i=0}^{n-1} \overline{A}_i \qquad \text{where} \quad s_n = y_0^2 + y_1^2 + \dots y_n^2. \tag{5.29}$$

To substitute Eq. (5.28) in Eq. (5.27), we get:

$$\sum_{n=0}^{\infty} y_n = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{n=0}^{\infty} y_n\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{n=0}^{\infty} \overline{A}_n\right]\right]. \tag{5.30}$$

Then from Eq. (5.30) we get:

$$y_0 = \frac{1}{2}, \tag{5.31}$$

$$y_1 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[y_0]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_0\right]\right], \tag{5.32}$$

$$y_2 = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[y_1]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_1\right]\right], \tag{5.33}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mathrm{y}_n = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[y_{n-1}]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_{n-1}\right]\right], \quad n = 1, 2, 3, \dots . \tag{5.34}$$

where the first few components of $\overline{A}_n$ are given by

$$\overline{A}_0 = {y_0}^2, \tag{5.35}$$

$$\overline{A}_1 = 2y_0y_1 + {y_1}^2, \tag{5.36}$$

$$\overline{A}_2 = 2y_0y_2 + 2y_1y_2 + {y_2}^2, \tag{5.37}$$

$$\vdots \qquad \vdots \qquad \vdots$$

from Eq. (5.32), it is easy to show that

$$y_1 = \frac{t^{\alpha}}{4\Gamma(1+\alpha)}\,, \tag{5.38}$$

by similar way, we get $y\ _2,\ y_3,\ \ldots$ as follows:

$$y_2 = -\frac{t^{3\alpha}\Gamma\left[1+2\alpha\right]}{16\Gamma\left(1+\alpha\right)^2\Gamma(1+3\alpha)}\,, \tag{5.39}$$

$$y_3 = \frac{t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{32\Gamma\left(1+\alpha\right)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} - \frac{t^{7\alpha}\Gamma\left(1+2\alpha\right)^2\Gamma(1+6\alpha)}{256\Gamma\left(1+\alpha\right)^4\Gamma\left(1+3\alpha\right)^2\Gamma(1+7\alpha)}, \tag{5.40}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$\begin{aligned} y\left(t\right) &= \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + y_3 + \ldots \\ &= \frac{1}{2} + \frac{t^{\alpha}}{4\Gamma(1+\alpha)} - \frac{t^{3\alpha}\Gamma\left[1+2\alpha\right]}{16\Gamma\left(1+\alpha\right)^2\Gamma\left(1+3\alpha\right)} + \frac{t^{5\alpha}\Gamma\left(1+2\alpha\right)\Gamma\left(1+4\alpha\right)}{32\Gamma\left(1+\alpha\right)^3\Gamma\left(1+3\alpha\right)\Gamma\left(1+5\alpha\right)} \\ &\quad - \frac{t^{7\alpha}\Gamma\left(1+2\alpha\right)^2\Gamma\left(1+6\alpha\right)}{256\Gamma\left(1+\alpha\right)^4\Gamma\left(1+3\alpha\right)^2\Gamma\left(1+7\alpha\right)} + \ldots . \end{aligned} \tag{5.41}$$

After four iterations and computation at $\alpha = 1$, the approximate solution has the following form

$$y\left(t\right) \approx \frac{1}{2} + \frac{t}{4} - \frac{t^3}{48} + \frac{t^5}{480} - \frac{t^7}{16128}. \tag{5.42}$$

It is evident that the suggested approach of Laplace transform and accelerated Adomian converges to the exact solution faster after four iterations than the hybrid approach of Laplace transform and regular Adomian, where

$$y\left(t\right) = \frac{1}{e^{-t}+1} \;=\; \frac{1}{2} + \frac{t}{4} - \frac{t^3}{48} + \frac{t^5}{480} - \frac{17t^7}{80640} + \frac{31t^9}{1451520} + \ldots . \tag{5.43}$$

Taking into account the same number of iterations (3 iterations), it is necessary to observe that the suggested method approximates better than the method presented in [31] applying the homotopy perturbation approach with the Sawi transform. In addition, after applying four iterations, our suggested method achieves higher accuracy and better convergence compared to Adomian decomposition approach coupled with Laplace transform.

Table 3: Present a comparison between the approximate solution using the Laplace transform coupled with the Accelerated Adomian Decomposition Method (LAADM) and that using the homotopy perturbation approach with the Sawi transform (HPSTM), with the same number of iterations (3 iterations), for different fractional orders α, for Example 5.1.

t	Approximate solution (LAADM)					Exact solution
	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.04$	$\alpha = 0.05$	$\alpha = 1$
LAADM						
0.1	0.7115634989	0.7083293041	0.7051426180	0.7020037720	0.6989122063	0.5249791875
0.2	0.7128640206	0.7108793452	0.7088936696	0.7069101934	0.7049314503	0.5498339973
0.3	0.7136306707	0.7123934734	0.7111359163	0.7098613175	0.7085725834	0.5744425168
0.4	0.7141773057	0.7134781928	0.7127495253	0.7119941754	0.7112147793	0.5986876601
0.5	0.7146028644	0.7143256763	0.7140146053	0.7136719558	0.7132999138	0.6224593312
HPSTM						
0.1	0.7279291460	0.7246049271	0.7212790875	0.7179533744	0.7146295346	0.5249791875
0.2	0.7292915794	0.7273104082	0.7253074187	0.7232835548	0.7212397720	0.5498339973
0.3	0.7300911275	0.7289034084	0.7276874339	0.7264438047	0.7251731306	0.5744425168
0.4	0.7306595666	0.7300382954	0.7293865849	0.7287048401	0.7279934728	0.5986876601
0.5	0.7311011391	0.7309212251	0.7307105298	0.7304693304	0.7301979088	0.6224593312

Table 3: Approximate solutions of LAADM and HPSTM for three iterations $y(t) = y_0 + y_1 + y_2 + y_3$ [31]

Table 4 Shows the absolute errors investigated by applying the Laplace transform combined with accelerated Adomian (LAADM), using three iterations, since α=1, for Example 5.1.

T	Exact solution	Approx. solution	Absolute error
0.1	0.5249791875	0.5249791875	1.486×10^{-11}
0.2	0.5498339973	0.5498339992	1.894×10^{-9}
0.3	0.5744425168	0.5744425489	3.213×10^{-8}
0.4	0.5986876601	0.5986878984	2.383×10^{-7}
0.5	0.6224593312	0.6224604531	1.122×10^{-6}

Table 4: LAADM results: Approximate solution and absolute error (3 iterations)

Table 5 Comparison of the approximate solutions and absolute errors produced by applyingLaplace transform combined with Adomian (LADM) and with accelerated Adomian (LAADM), for Example 5.1.

T	Exact solution	LAADM Approx.	LAADM Error	LADM Approx.	LADM Error
0.1	0.524 979 187 478 94	0.524 979 187 478 94	2.133×10^{-17}	0.524 979 187 478 94	2.106×10^{-11}
0.2	0.549 833 997 312 478	0.549 833 997 312 486	7.587×10^{-15}	0.549 833 997 312 478	2.688×10^{-9}
0.3	0.574 442 516 811 659	0.574 442 516 812 311	6.522×10^{-13}	0.574 442 516 811 659	4.569×10^{-8}
0.4	0.598 687 660 112 452	0.598 687 660 127 654	1.520×10^{-11}	0.598 687 660 112 452	3.399×10^{-7}
0.5	0.622 459 331 201 854 6	0.622 459 331 375 293	1.734×10^{-10}	0.622 459 331 201 854 6	1.606×10^{-6}
0.6	0.645 656 306 225 795 4	0.645 656 307 483 362	1.258×10^{-9}	0.645 656 306 225 795 4	5.694×10^{-6}
0.7	0.668 187 772 168 166 2	0.668 187 778 828 971	6.661×10^{-9}	0.668 187 772 168 166 2	1.654×10^{-5}
0.8	0.689 974 481 127 612 5	0.689 974 509 134 461	2.801×10^{-8}	0.689 974 481 127 612 5	4.152×10^{-5}
0.9	0.710 949 502 625 003 9	0.710 949 601 270 060	9.865×10^{-8}	0.710 949 502 625 003 9	9.318×10^{-5}
1.0	0.731 058 578 630 004 9	0.731 058 880 566 517	3.019×10^{-7}	0.731 058 578 630 004 9	1.914×10^{-4}

Table 5: Approximate solutions and absolute errors of LAADM and LADM (4 iterations)

For Example 5.1 ($\alpha = 1,\ t \in [0, 0.1]$)		
Iterations (M)	**Actual Error**	**Theoretical Bound**
0	2.50×10^{-2}	9.29×10^{-2}
1	2.50×10^{-2}	4.65×10^{-3}
2	2.50×10^{-2}	2.32×10^{-4}

Note 5.1. The bound is conservative but holds for $m \geq 1$.

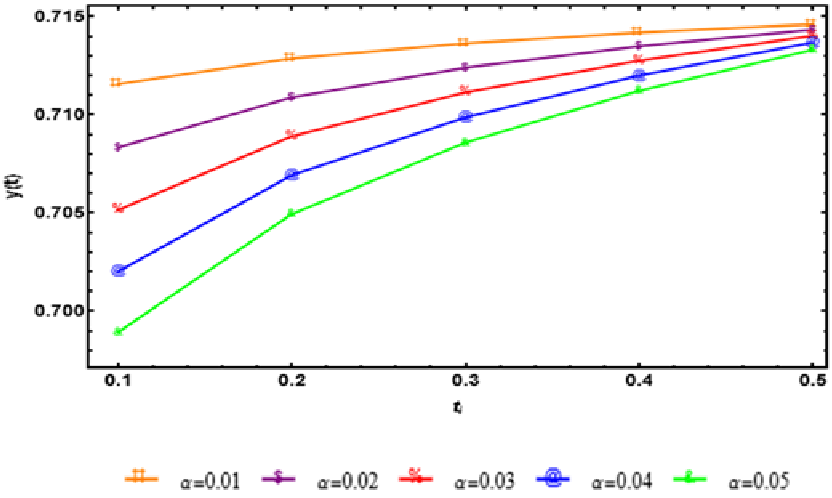

Figure 1: This graph shows how the approximate solutions for y(t) change over time (t) for five different fractional values of $\alpha(0.01, 0.02, 0.03, 0.04, and 0.05)$. The solutions were obtained using the Laplace transform coupled with the accelerated Adomian method after three iterations.

For Example 5.1 $\left(D^{\alpha}y = y - y^2,\ y\,(0) = 0.5\right)$, solution from different initial guesses $(\text{y}\,(0) = 0.5 \pm 0.1)$ converge to the same trajectory (Figure 1), validating uniqueness.

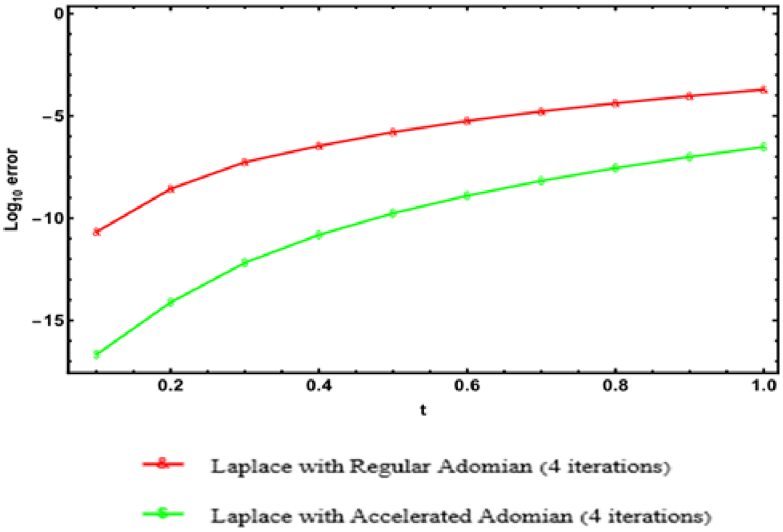

Figure 2: Comparison of the absolute errors of two methods, Laplace with Regular Adomian and Laplace with Accelerated Adomian, both using four iterations, as a function of time (t). The graph shows that the Accelerated Adomian method consistently produces a lower error for Example 5.1.

Table 3 Demonstrates that, the approximate solution becomes closer to the exact solution as the value of α increases. In particular, when $\alpha = 1$, the approximate solution almost coincides with the exact solution, which highlights the accuracy, flexibility, and strong convergence of the LAADM. Furthermore, compared with the HPSTM, the proposed approach provides more reliable results with faster convergence toward the exact solution.

It is clear from Figure 1 that, as the value of α increases, the approximate solution decreases and becomes closer to the exact solution, which demonstrates the accuracy of proposed approach compared with the homotopy perturbation approach with the Sawi transform (HPSTM).

Table 3 and Table 4 illustrate that the LAADM, as the number of iterations increases, results in a reduction of the absolute error and an improvement in both accuracy and convergence.

Table 5 and Figure 2 show that the suggested method of combining Laplace transform with accelerated Adomian provides superior accuracy than the Laplace with normal Adomian decomposition method.

Table 5 shows LAADM achieves 10^{-4} error in 4 iterations, while LADM requires 6 iterations. The theoretical error bound (theorem 4.3) is satisfied (Figure 2).

$$\|E_m\| \sim \varphi\left(k^m\right) \qquad \text{with} \qquad k = 0.462 \qquad (\text{estimated from} \qquad |dN/dy|\ ).$$

Example 5.2. [7, 32] Consider the fractional order nonlinear delay differential equations

$$D_t^{\alpha} y\left(t\right) \ = \ 1 - 2y^2\left(\frac{t}{2}\right), 0 \leq \text{t} \leq 1,\ 0\ < \alpha \leq 1, \tag{5.44}$$

with initial condition $y\left(0\right) = 0$, and exact solution for $\alpha = 1$ is

$$y\left(t\right) = \text{sin}t\ \ . \tag{5.45}$$

This example has been studied by several researchers. The most recent work by Mohammed [7] applied a computational method for solving differential equations with fractional-order delays. Sharmila and Lydia [32] solving fractional delay differential equations using the Mahgoub Adomian decomposition method.

For this example, we employ Laplace with the normal Adomian decomposition method and the suggested accelerated Adomian decomposition method to compare the accuracy of the equation using both methods that we are studying in this paper.

5.3 Solution using Laplace Transform coupled with Adomian decomposition method

Applying Laplace transform for Eq. (5.44) and using the fractional derivatives properties of Laplace transform we have

$$s^{\alpha} Y\left(s\right) - s^{\alpha-1} y\left(0\right) = \ \frac{1}{s} - 2\mathcal{L}[y^2\left(\frac{t}{2}\right)], \tag{5.46}$$

$$Y\left(s\right) = \ \frac{1}{s^{\alpha+1}} - \frac{2}{s^{\alpha}}\mathcal{L}[y^2\left(\frac{t}{2}\right)]. \tag{5.47}$$

And for Eq. (5.47), take the inverse Laplace transform to both sides, we obtain:

$$y\left(t\right) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \ 2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y^2\left(\frac{t}{2}\right)\right]\right]. \tag{5.48}$$

After that the solution is represented as an infinite series, and the nonlinear term can be defined as follows, respectively

$$\begin{gathered} y\left(t\right) = \sum_{n=0}^{\infty} y_n, \\ y^2\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} A_n(y), \end{gathered} \tag{5.49}$$

where A_n are Adomian polynomial and they may be determined using formula provided below:

$$A_n \ = \ \frac{1}{n\,!}\Big(\frac{d^n}{d\lambda^n}\left[N\sum_{i=0}^{\infty}\left(\lambda^i y_i\right)\right]\Big)_{\lambda=0} \quad , \ n = \ 0\ ,\ 1\ ,\ 2\ , 3, \ldots. \tag{5.50}$$

To substitute Eq. (5.49) in Eq. (5.48), we get:

$$\sum_{n=0}^{\infty} y_n = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \ 2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{5.51}$$

Then from Eq. (5.51) we get:

$$y_0=\frac{t^{\alpha}}{\Gamma(1+\alpha)}\ , \tag{5.52}$$

$$y_1= -2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\,[\,A_0\,]\right], \tag{5.53}$$

$$y_2= -2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\,[\,A_1\,]\right], \tag{5.54}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n=-\,2\mathcal{L}^{-1}\ \left[\frac{1}{s^{\alpha}}\mathcal{L}\,[A_{n-1}]\right],\ n\ =\ 1\,2\,,\ldots. \tag{5.55}$$

Where the first few components of A_n are given by

$$A_0=y_0^2\left(\frac{t}{2}\right)=\left[\frac{2^{-\alpha}t^{\alpha}}{\Gamma(1+\alpha)}\right]^2=\frac{2^{-2\alpha}t^{2\alpha}}{\Gamma(1+\alpha)^2}, \tag{5.56}$$

$$A_1=2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right), \tag{5.57}$$

$$A_2=y_1^2\left(\frac{t}{2}\right)+y_0\left(\frac{t}{2}\right)y_2\left(\frac{t}{2}\right), \tag{5.58}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, from Eqs. (5.55) and (5.56) we get:

$$\begin{aligned} y_1&= -2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\,[\,A_0\,]\right] \\ &=-2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{2^{-2\alpha}t^{2\alpha}}{\Gamma\,(1+\alpha)^2}\right]\right] \\ &=-2\mathcal{L}^{-1}\left[\frac{s^{-1-3\alpha}\Gamma\,(1+2\alpha)}{\Gamma\,(1+\alpha)^2}\right] \\ &=-\frac{2^{1-2\alpha}t^{3\alpha}\Gamma\,(1+2\alpha)}{\Gamma\,(1+\alpha)^2\Gamma\,(1+3\alpha)}. \end{aligned} \tag{5.59}$$

Similarly, from Eqs. (5.55), (5.47) and (5.58) we get $y_{\,2},\ y_3,\ \ldots$ as follows:

$$y_2=\frac{2^{3-6\alpha}t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma\,(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)}\ , \tag{5.60}$$

$$y_3=-2(\frac{2^{2-10\alpha}t^{7\alpha}\Gamma\,(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)}+\frac{2^{4-12\alpha}t^{7\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma\,(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)}, \tag{5.61}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$\begin{aligned} y\left(t\right) &= \sum_{n=0}^{\infty} y_n \\ &= y_0 + y_1 + y_2 + y_3 + \cdots \\ &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2^{1-2\alpha}t^{3\alpha}\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+\alpha\right)^2\Gamma\left(1+3\alpha\right)} + \frac{2^{3-6\alpha}t^{5\alpha}\Gamma\left(1+2\alpha\right)\Gamma\left(1+4\alpha\right)}{\Gamma\left(1+\alpha\right)^3\Gamma\left(1+3\alpha\right)\Gamma\left(1+5\alpha\right)} \\ &\quad - 2(\frac{2^{2-10\alpha}t^{7\alpha}\Gamma\left(1+2\alpha\right)^2\Gamma\left(1+6\alpha\right)}{\Gamma\left(1+\alpha\right)^4\Gamma\left(1+3\alpha\right)^2\Gamma\left(1+7\alpha\right)} \\ &\quad + \frac{2^{4-12\alpha}t^{7\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma\left(1+\alpha\right)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)} + \ldots. \end{aligned} \tag{5.62}$$

After three iterations and computation at $\alpha = 1$, the approximate solution has the following form

$$y\left(t\right) \ \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}\ . \tag{5.63}$$

5.4 Solution using Laplace Transform coupled with accelerated Adomian decomposition method

Similarly, applying Laplace transform for Eq. (5.44) and using the fractional derivatives properties of Laplace transform we have

$$s^{\alpha}Y\left(s\right) - s^{\alpha-1}y\left(0\right) = \frac{1}{s} - 2\mathcal{L}\left[y^2\left(\frac{t}{2}\right)\right], \tag{5.64}$$

$$Y\left(s\right) = \frac{1}{s^{\alpha+1}} - \frac{2}{s^{\alpha}}\mathcal{L}\left[y^2\left(\frac{t}{2}\right)\right]. \tag{5.65}$$

And for Eq. (5.65), take the inverse Laplace transform to both sides, we obtain:

$$y\left(t\right) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \ 2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y^2\left(\frac{t}{2}\right)\right]\right]. \tag{5.66}$$

By the same way the solution is represented as an infinite series, and the nonlinear term can be defined as follows, respectively

$$\begin{aligned} y\left(t\right) &= \sum_{n=0}^{\infty} y_n, \\ y^2\left(\frac{t}{2}\right) &= \sum_{n=0}^{\infty} \overline{A}_n(y), \end{aligned} \tag{5.67}$$

where $\overline{A}_n$ are the accelerated Adomian polynomials and they may be determined using formula provided below:

$$\overline{A}_n = N\left(s_n\right) - \sum_{i=0}^{n-1}\overline{A}_i, \quad \text{where} \quad s_n = y_0^2 + y_1^2 + \cdots + y_n^2. \tag{5.68}$$

By substituting Eq. (5.67) in Eq. (5.66), we get:

$$\sum_{n=0}^{\infty} y_n = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - 2\mathcal{L}^{-1}\ \left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{n=0}^{\infty}\overline{A}_n\right]\right]. \tag{5.69}$$

Then from Eq. (5.69), we get:

$$y_0 = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.70}$$

$$y_1 = -2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[\overline{A}_0]\right], \tag{5.71}$$

$$y_2 = -2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[\overline{A}_1]\right], \tag{5.72}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mathrm{y}_n = -2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_{n-1}\right]\right], \quad n = 1, 2, 3, \ldots . \tag{5.73}$$

where the first few components of $\overline{A}_n$ are given by

$$\overline{A}_0 = {y_0}^2\left(\frac{t}{2}\right) = \left[\frac{2^{-\alpha}t^{\alpha}}{\Gamma(1+\alpha)}\right]^2 = \frac{2^{-2\alpha}t^{2\alpha}}{\Gamma(1+\alpha)^2}, \tag{5.74}$$

$$\overline{A}_1 = 2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right) + {y_1}^2\left(\frac{t}{2}\right), \tag{5.75}$$

$$\overline{A}_2 = 2y_0\left(\frac{t}{2}\right)y_2\left(\frac{t}{2}\right) + 2y_1\left(\frac{t}{2}\right)y_2\left(\frac{t}{2}\right) + {y_2}^2\left(\frac{t}{2}\right), \tag{5.76}$$

$$\vdots \qquad \vdots \qquad \vdots$$

from Eq. (5.71) and Eq. (5.74), we can show that

$$\begin{aligned} \mathrm{y}_1 &= -2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_0\right]\right] \\ &= -2\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{2^{-2\alpha}t^{2\alpha}}{\Gamma(1+\alpha)^2}\right]\right] \\ &= -2\mathcal{L}^{-1}\left[\frac{s^{-1-3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right] \\ &= -\frac{2^{1-2\alpha}t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}. \end{aligned} \tag{5.77}$$

Similarly, from Eqs. (5.73), (5.75) and (5.76) we get $y_2, y_3, \ldots$ as follows:

$$\mathrm{y}_2 = \frac{2^{3-6\alpha}t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)}, \tag{5.78}$$

$$\begin{aligned} \mathrm{y}_3 = -2\Bigg(&\frac{2^{4-12\alpha}t^{7\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)} \\ &-\frac{2^{5-16\alpha}t^{9\alpha}\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)\Gamma(1+9\alpha)} \\ &+\frac{2^{6-22\alpha}t^{11\alpha}\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)^2\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)^2\Gamma(1+11\alpha)}\Bigg), \end{aligned} \tag{5.79}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$
\begin{aligned}
y\left(t\right) &= \sum_{n=0}^{\infty} y_n \\
&= y_0 + y_1 + y_2 + y_3 + \ldots \\
&= \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2^{1-2\alpha} t^{3\alpha} \Gamma(1+2\alpha)}{\Gamma\left(1+\alpha\right)^2 \Gamma(1+3\alpha)} + \frac{2^{3-6\alpha} t^{5\alpha} \Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma\left(1+\alpha\right)^3 \Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&\quad - 2\left(\frac{2^{4-12\alpha} t^{7\alpha} \Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma\left(1+\alpha\right)^4 \Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)}\right. \\
&\quad - \frac{2^{5-16\alpha} t^{9\alpha} \Gamma\left(1+2\alpha\right)^2 \Gamma\left(1+4\alpha\right) \Gamma\left(1+8\alpha\right)}{\Gamma\left(1+\alpha\right)^5 \Gamma\left(1+3\alpha\right)^2 \Gamma\left(1+5\alpha\right) \Gamma\left(1+9\alpha\right)} \\
&\quad \left. + \frac{2^{6-22\alpha} t^{11\alpha} \Gamma\left(1+2\alpha\right)^2 \Gamma\left(1+4\alpha\right)^2 \Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6 \Gamma(1+3\alpha)^2 \Gamma(1+5\alpha)^2 \Gamma(1+11\alpha)}\right).
\end{aligned}
\tag{5.80}
$$

After three iterations and computation at $\alpha = 1$, the approximate solution has the following form

$$
y\left(t\right) \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{13440} + \frac{t^9}{414720} - \frac{t^{11}}{81100800}\,. \tag{5.81}
$$

It is evident that the suggested approach of Laplace transform and accelerated Adomian converges to the exact solution faster after three iterations than the hybrid approach of Laplace transform and regular Adomian, where:

$$
\begin{aligned}
y\left(t\right) &= \sin t \\
&= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}}{39916800} \\
&\quad + \frac{t^{13}}{6227020800} - \frac{t^{15}}{1307674368000} + O\left[t\right]^{16}.
\end{aligned}
\tag{5.82}
$$

Taking into account the same number of terms (2 iterations), it is necessary to observe that the suggested method approximates better than the methods presented in [7] using the Laplace transform with the Adomian method, and by using three iterations, the proposed method achieves superior accuracy and convergence compared to the approach in [32] employing the Mahgoub Adomian decomposition method (MADM).

Table 6: Demonstrates a comparison of the approximate solutions and absolute errors produced by LAADM and LADM, for Example 5.2.

t	Exact solution	LAADM (2 iterations)		LADM (2 iterations) [7]	
		Approx. solution	Absolute error	Approx. solution	Absolute error
0.1	0.09983341664	0.09983341665	7.438×10^{-12}	0.09983341667	1.984×10^{-11}
0.2	0.19866933080	0.19866933175	9.51×10^{-10}	0.19866933333	2.538×10^{-9}
0.3	0.29552020666	0.29552022288	1.622×10^{-8}	0.29552025000	4.334×10^{-8}
0.4	0.38941834231	0.38941846349	1.212×10^{-7}	0.38941866667	3.244×10^{-7}
0.5	0.47942553860	0.47942611452	5.759×10^{-7}	0.47942708333	1.545×10^{-6}
0.6	0.56464247340	0.56464452857	2.055×10^{-6}	0.56464799910	5.527×10^{-6}
0.7	0.64421768724	0.64422370408	6.017×10^{-6}	0.64423391667	1.623×10^{-5}
0.8	0.71735609090	0.71737132698	1.524×10^{-5}	0.71739733333	4.124×10^{-5}
0.9	0.78332690963	0.78336143739	3.453×10^{-5}	0.78342075000	9.384×10^{-5}

Table 6: Comparison between exact solution, LAADM and LADM (two iterations)

Table 7: Comparison of the approximate solutions and absolute errors produced by applying the Mahgoub Adomian approach (MADM), the Laplace transform combined with Adomian

(LADM), and the Laplace transform combined with Accelerated Adomian (LAADM), for Example 5.2.

t	Approximate solution of MADM					
	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$	Exact solution	Absolute error $(\alpha = 1)$
0.1	0.328593	0.191049	0.114217	0.0998334	0.0998334	1.984×10^{-11}
0.2	0.430593	0.313936	0.219133	0.198669	0.198669	2.538×10^{-9}
0.3	0.492718	0.412922	0.318584	0.295520	0.295520	4.334×10^{-8}
0.4	0.537160	0.494487	0.412504	0.389419	0.389418	3.244×10^{-7}
0.5	0.574228	0.561583	0.500393	0.479427	0.479426	1.545×10^{-6}
0.6	0.610263	0.616108	0.581656	0.564648	0.564642	5.527×10^{-6}
0.7	0.649765	0.659578	0.655701	0.644234	0.644218	1.623×10^{-5}
0.8	0.696221	0.693376	0.721989	0.717397	0.717356	4.124×10^{-5}
0.9	0.752484	0.718866	0.780060	0.783421	0.783327	9.384×10^{-5}
1.0	0.820985	0.737442	0.829546	0.841667	0.841471	1.957×10^{-4}

Table 7: Approximate solution of MADM for three iterations $y(t) = y_0 + y_1 + y_2 + y_3$ [32]

t	Approximate solution of LADM					
	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$	Exact solution	Absolute error $(\alpha = 1)$
0.1	0.328593	0.191049	0.114217	0.099833	0.0998334	2.748×10^{-15}
0.2	0.430593	0.313936	0.412922	0.198669	0.198669	1.41×10^{-12}
0.3	0.492718	0.412922	0.318584	0.295520	0.295520	5.42×10^{-11}
0.4	0.537160	0.494487	0.412504	0.389419	0.389418	7.213×10^{-10}
0.5	0.574228	0.561583	0.500393	0.479427	0.479426	5.37×10^{-9}
0.6	0.610263	0.616108	0.581656	0.564648	0.564642	2.768×10^{-8}
0.7	0.649765	0.659578	0.655701	0.644234	0.644218	1.107×10^{-7}
0.8	0.696221	0.693376	0.721989	0.717397	0.717356	3.677×10^{-7}
0.9	0.752484	0.718866	0.780060	0.783421	0.783327	1.06×10^{-6}
1.0	0.820985	0.737442	0.829546	0.841667	0.841471	2.731×10^{-6}

Table 8: Approximate solution of LADM for three iterations $y(t) = y_0 + y_1 + y_2 + y_3$

t	Approximate solution of LAADM					
	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$	Exact solution	Absolute error $(\alpha = 1)$
0.1	0.114217	0.191049	0.114217	0.099833	0.099833	3.331×10^{-16}
0.2	0.219133	0.313934	0.219133	0.198669	0.198669	1.761×10^{-13}
0.3	0.318584	0.412905	0.318584	0.295520	0.295520	6.758×10^{-12}
0.4	0.412503	0.494412	0.412503	0.389418	0.389418	8.977×10^{-11}
0.5	0.500389	0.561340	0.500389	0.479426	0.479426	6.666×10^{-10}
0.6	0.581641	0.615479	0.581641	0.564642	0.564642	3.425×10^{-9}
0.7	0.655660	0.658174	0.655660	0.644218	0.644218	1.365×10^{-8}
0.8	0.721891	0.690568	0.721891	0.717356	0.717356	4.515×10^{-8}
0.9	0.779844	0.713696	0.779844	0.783327	0.783327	1.295×10^{-7}
1.0	0.829113	0.728530	0.829113	0.841471	0.841471	3.319×10^{-7}

Table 9: Approximate solution of LAADM for three iterations $y(t) = y_0 + y_1 + y_2 + y_3$

Table 6 and Figure 3 demonstrate that the suggested technique attains superior accuracy with lower absolute error than the Laplace Adomian technique.

Tables 7, 8 and 9 and Figure 4 show that the suggested method of combining Laplace transform with accelerated Adomian provides superior accuracy than the two others methods Laplace with normal Adomian decomposition and Mahgoub Adomian decomposition.

As shown in Tables 6, 7, 8 and 9 increasing the number of iterations leads to remarkable improvement in the accuracy of the suggested approach, accompanied by a reduction in the absolute error, thereby confirming its efficiency and advantage over other approaches.

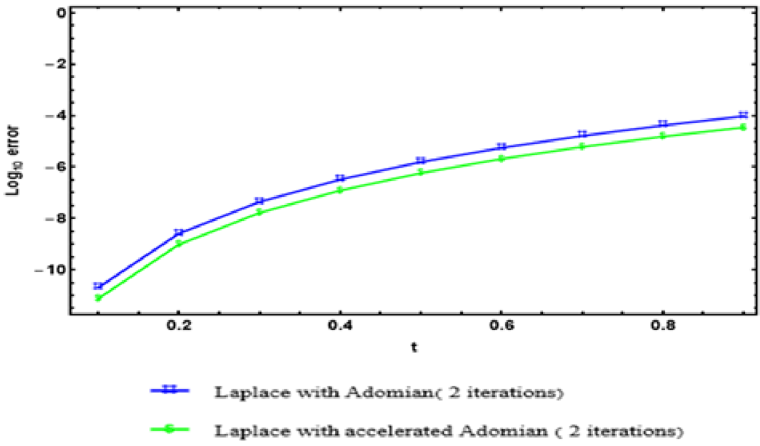

Figure 3: Comparison of the absolute errors of the Laplace transform paired with the Adomian method and the accelerated Adomian method (both with 2 iterations) for Example 5.2. The graph shows that the accelerated Adomian method results in a lower absolute error over time.

For Example 5.2 $\left(D^{\alpha} y = y^2,\ y(0) = 0\right)$, the series converges uniformly (Table 5.4). The residual $\|R_m(t)\| = \left\|y(t) - \sum_{i=0}^{m} y_i(t)\right\|$, for $m \geq 0$, decreases geometrically (Figure 3).

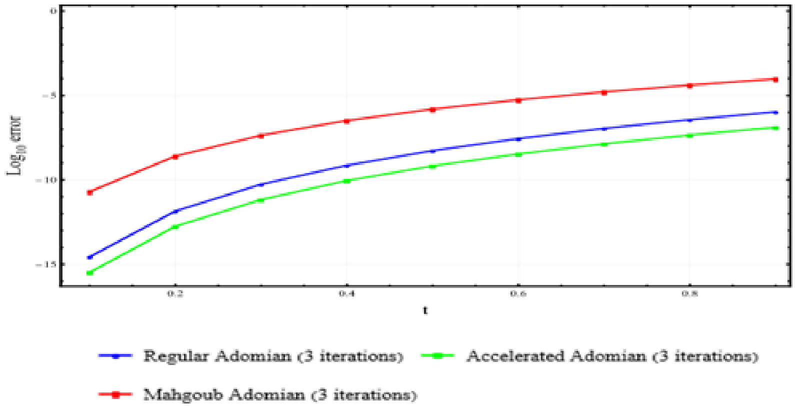

Figure 4: Comparison of absolute errors for Example 5.2. The graph plots the base-10 logarithm of the error over time (\(t\)) for three methods, each using three iterations: Regular Adomian, Accelerated Adomian, and Mahgoub Adomian. It is observed that the Accelerated Adomian method has the lowest error, and the Mahgoub Adomian method has the highest error.

Example 5.3. [8, 33] The following FBIVP is considered

$$D_t^{\alpha} y(t) = 2y(t) - y^2(t) + 1, 0 \leq t \leq 1, 0 < \alpha \leq 1, \tag{5.83}$$

with initial condition $y(0) = 0$, and exact solution for $\alpha = 1$ is

$$y(t) = 1 + \sqrt{2}\text{Tanh}\left[\sqrt{2}t + \frac{1}{2}\text{Ln}\left[\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]\right]. \tag{5.84}$$

This example is treated by many researchers. The most recent one is by Pushpam, Lydia [8] used Mahgoub Adomian to solve fractional differential equations that are nonlinear. Lydia et al. [33] used Kharrat-Toma iterative method to solve fractional differential equations that are nonlinear.

For this example, we use Laplace with the suggested accelerated Adomian decomposition method to compare the accuracy for the equation that we are studying in this paper of this method and Mahgoub Adomian decomposition method.

5.5 Solution using Laplace Transform coupled with accelerated Adomian decomposition method

Applying Laplace transform for Eq. (5.83) and using the fractional derivatives properties of Laplace transform we have

$$s^{\alpha}Y(s) = 2\mathcal{L}[y(t)] - \mathcal{L}\left[y^2(t)\right] + \frac{1}{s}, \tag{5.85}$$

$$Y(s) = \frac{1}{s^{\alpha+1}} + \frac{2}{s^{\alpha}}\mathcal{L}[y(t)] - \frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right]. \tag{5.86}$$

For Eq. (5.86), take inverse Laplace transform to both sides, we obtain:

$$y(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \mathcal{L}^{-1}\left[\frac{2}{s^{\alpha}}\mathcal{L}[y(t)]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[y^2(t)\right]\right]. \tag{5.87}$$

By the same way the solution is represented as an infinite series, and the nonlinear term can be defined as follows, respectively

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n(t), \\ y^2(t) &= \sum_{n=0}^{\infty} \overline{A}_n(y), \end{aligned} \tag{5.88}$$

where $\overline{A}_n$ are the accelerated Adomian polynomials and they may be determined using formula provided below:

$$\overline{A}_n = N(s_n) - \sum_{i=0}^{n-1} \overline{A}_i \tag{5.89}$$

where

$$s_n = y_0^2 + y_1^2 + \cdots + y_n^2.$$

Substitute Eq. (5.86) in Eq. (5.85), we get:

$$\sum_{n=0}^{\infty} y_n = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \mathcal{L}^{-1}\left[\frac{2}{s^{\alpha}}\mathcal{L}[\sum_{n=0}^{\infty} y_n]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\sum_{n=0}^{\infty} \overline{A}_n\right]\right]. \tag{5.90}$$

Then from Eq. (5.90) we get:

$$y_0 = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.91}$$

$$y_1 = \mathcal{L}^{-1}\left[\frac{2}{s^{\alpha}}\mathcal{L}[y_0]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_0\right]\right], \tag{5.92}$$

$$y_2 = \mathcal{L}^{-1}\left[\frac{2}{s^{\alpha}}\mathcal{L}[y_1]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_1\right]\right], \tag{5.93}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n = \mathcal{L}^{-1}\left[\frac{2}{s^{\alpha}}\mathcal{L}[y_{n-1}]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_{n-1}\right]\right], \quad n = 1, 2, 3, \ldots. \tag{5.94}$$

where the first few components of $\overline{A}_n$ are given by

$$\overline{A}_0 = y^2 = [\frac{t^{\alpha}}{\Gamma(1+\alpha)}]^2 = \frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}, \tag{5.95}$$

$$\overline{A}_1 = 2\, y_0\, y_1 + y_1^2, \tag{5.96}$$

$$\overline{A}_2 = 2\, y_0\, y_2 + 2\, y_1\, y_2 + y_2^2, \tag{5.97}$$

$$\vdots \qquad \vdots \qquad \vdots$$

from Eq. (5.91), Eq. (5.92) and Eq. (5.95), it is easy to show that

$$\begin{aligned} y_1 &= \mathcal{L}^{-1}\left[\frac{2}{s^{\alpha}}\mathcal{L}[y_0]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\overline{A}_0\right]\right] \\ &= \mathcal{L}^{-1}\left[\frac{2}{s^{\alpha}}\mathcal{L}[\frac{t^{\alpha}}{\Gamma(1+\alpha)}]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}\right]\right] \\ &= \mathcal{L}^{-1}\left[\frac{2}{s^{2\alpha+1}}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{3\alpha+1}}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right] \\ &= \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\ , \end{aligned} \tag{5.98}$$

Similarly, from Eqs. (5.94) and (5.96), we get $y_{\,2}, \ldots$ as follows:

$$\begin{aligned} y_2 =& 2\left(\frac{2t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{t^{4\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)}\right) - \frac{4t^{4\alpha}\Gamma\left(1+3\alpha\right)}{\Gamma\left(1+\alpha\right)\Gamma\left(1+2\alpha\right)\Gamma\left(1+4\alpha\right)} \\ & - \frac{4t^{5\alpha}\Gamma\left(1+4\alpha\right)}{\Gamma\left(1+\alpha\right)^2\Gamma\left(1+5\alpha\right)} + \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\ & + \frac{4t^{6\alpha}\Gamma(1+5\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)\Gamma(1+6\alpha)} - \frac{t^{7\alpha}\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+\alpha)^2\Gamma(1+7\alpha)}, \end{aligned} \tag{5.99}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$\begin{aligned} y\left(t\right) &= \sum_{n=0}^{\infty} y_n \\ &= y_0 + y_1 + y_2 + \ldots \\ &= \frac{t^{\alpha}}{\Gamma\left(1+\alpha\right)} + \frac{2t^{2\alpha}}{\Gamma\left(1+2\alpha\right)} - \frac{t^{3\alpha}\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+\alpha\right)^2\Gamma\left(1+3\alpha\right)} \\ &+ 2\left(\frac{2t^{3\alpha}}{\Gamma\left(1+3\alpha\right)} - \frac{t^{4\alpha}\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+\alpha\right)^2\Gamma\left(1+4\alpha\right)}\right) - \frac{4t^{4\alpha}\Gamma\left(1+3\alpha\right)}{\Gamma\left(1+\alpha\right)\Gamma\left(1+2\alpha\right)\Gamma\left(1+4\alpha\right)} \\ &- \frac{4t^{5\alpha}\Gamma\left(1+4\alpha\right)}{\Gamma\left(1+\alpha\right)^2\Gamma\left(1+5\alpha\right)} + \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\ &+ \frac{4t^{6\alpha}\Gamma(1+5\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)\Gamma(1+6\alpha)} - \frac{t^{7\alpha}\Gamma\left(1+2\alpha\right)^2\Gamma\left(1+6\alpha\right)}{\Gamma\left(1+\alpha\right)^4\Gamma\left(1+3\alpha\right)^2\Gamma\left(1+7\alpha\right)} + \ldots. \end{aligned} \tag{5.100}$$

After two iterations and computation at $\alpha = 1$, the approximate solution has the following form:

$$y\left(t\right) \approx t + t^2 + \frac{t^3}{3} - \frac{2t^4}{3} - \frac{t^5}{15} + \frac{t^6}{9} - \frac{t^7}{63}. \tag{5.101}$$

5.6 Solution using Mahgoub Transform coupled with Mahgoub Adomian decomposition method

Now, we derive some terms of the Mahgoub Adomian decomposition series, as follows [8]

$$y_0 = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.102}$$

$$y_1 = \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}, \tag{5.103}$$

$$\begin{aligned} y_2 =& \frac{4t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{2t^{4\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)} - \frac{4t^{4\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)} \\ &+ \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)}, \end{aligned} \tag{5.104}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n \\ &= y_0 + y_1 + y_2 + \ldots \\ &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + \frac{4t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &- \frac{2t^{4\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)} - \frac{4t^{4\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)} \\ &+ \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} + \ldots. \end{aligned} \tag{5.105}$$

After two iterations and computation at $\alpha = 1$, the approximate solution has the following form:

$$y(t) \approx t + t^2 + \frac{t^3}{3} - \frac{2t^4}{3} + \frac{2t^5}{15}. \tag{5.106}$$

It is evident that the suggested approach of the Laplace transform with accelerated Adomian converges to the exact solution faster after two iterations than the hybrid approach of Mahgoub Adomian, where:

$$y(t) = t + t^2 + 0.33333t^3 - 0.33333t^4 - 0.46667t^5 - 0.15556t^6 + 0.16825t^7 + \ldots. \tag{5.107}$$

5.7 Solution using Kharat-Toma Transform coupled with iterative method

Now, we derive some terms of the Mahgoub Adomian decomposition series, as follows [33]

$$y_0 = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.108}$$

$$y_1 = \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}, \tag{5.109}$$

$$\begin{aligned} y_2 =& \frac{4t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{2t^{4\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)} - \frac{4t^{4\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)} \\ &+ \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} - \frac{4t^{5\alpha}\Gamma(1+4\alpha)}{\Gamma(1+2\alpha)^2\Gamma(1+5\alpha)} \\ &- \left(\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)^2 \frac{\Gamma(6\alpha+1)\,t^{7\alpha}}{\Gamma(7\alpha+1)\Gamma(\alpha+1)^4} + \frac{4\Gamma[5\alpha+1]\,t^{6\alpha}}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2\Gamma(6\alpha+1)}, \end{aligned} \tag{5.110}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n \\ &= y_0 + y_1 + y_2 + \ldots \\ &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + \frac{4t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &- \frac{2t^{4\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)} - \frac{4t^{4\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)} \\ &+ \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} - \frac{4t^{5\alpha}\Gamma(1+4\alpha)}{\Gamma(1+2\alpha)^2\Gamma(1+5\alpha)} \\ &- \left(\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)^2 \frac{\Gamma(6\alpha+1)\,t^{7\alpha}}{\Gamma(7\alpha+1)\Gamma(\alpha+1)^4} + \frac{4\Gamma[5\alpha+1]\,t^{6\alpha}}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2\Gamma(6\alpha+1)} \\ &+ \ldots\ldots\ldots \end{aligned} \tag{5.111}$$

After two iterations and computation at $\alpha = 1$, the approximate solution has the following form:

$$y(t) \approx t + t^2 + \frac{t^3}{3} - \frac{2t^4}{3} - \frac{t^5}{15} + \frac{t^6}{9} - \frac{t^7}{63}. \tag{5.112}$$

It is observed that the numerical values using the same number of iterations (2 iterations), for $\alpha = 1$,which obtained by the proposed method and the method introduced in Lydia et al. [33] using Kharrat-Toma iterative method to solve nonlinear fractional differential equations are nearly identical in the initial conditions.

However, the proposed method is less computationally demanding, as it requires fewer mathematical operations to achieve the same level of accuracy.

Table 10: Presents a comparison of the approximate solutions and absolute errors produced by applying Mahgoub Adomian decomposition (MADM) and Laplace transform combined with accelerated Adomian (LAADM) for Example 5.3.

Table 10 shows that the suggested method of combining Laplace transform with accelerated Adomian achieves better accuracy with lower absolute error than the Mahgoub Adomian decomposition method [8].

From Table 11 an while the numerical outcomes of both approaches coincide in the initial iterations, the proposed technique attains this accuracy with a reduced computational burden. By requiring fewer algebraic manipulations and iterative evaluations, the suggested method achieves an equivalent level of precision more efficiently, making it particularly as advantageous for large-scale or computationally intensive problems and admits a more natural extension to broader classes of fractional differential models [33].

t	Approximate solution				Exact solution	Absolute error $(\alpha = 1)$
	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$		
MADM						
0.1	0.583071	0.244588	0.128749	0.110268	0.110295	2.72×10^{-5}
0.2	0.940433	0.471271	0.275152	0.241643	0.241977	3.341×10^{-4}
0.3	1.234620	0.705263	0.441811	0.393924	0.395105	1.181×10^{-3}
0.4	1.480480	0.942026	0.626664	0.565632	0.567812	2.18×10^{-3}
0.5	1.684740	1.175630	0.826259	0.754167	0.756014	1.848×10^{-3}
0.6	1.851730	1.400330	1.036230	0.955968	0.953566	2.402×10^{-3}
0.7	1.984780	1.610880	1.251570	1.166680	1.152950	1.373×10^{-2}
0.8	2.086730	1.802660	1.466830	1.381290	1.346360	3.493×10^{-2}
0.9	2.160090	1.971690	1.676310	1.594330	1.526910	6.742×10^{-2}
1.0	2.207160	2.114650	1.874210	1.800000	1.689500	1.105×10^{-1}
LAADM						
0.1	0.577431	0.244460	0.128744	0.110266	0.110295	2.909×10^{-5}
0.2	0.912655	0.469710	0.275038	0.241586	0.241977	3.912×10^{-4}
0.3	1.166250	0.698718	0.441079	0.393516	0.395105	1.589×10^{-3}
0.4	1.353550	0.924319	0.623976	0.564013	0.567812	3.799×10^{-3}
0.5	1.482630	1.137950	0.819004	0.749529	0.756014	6.486×10^{-3}
0.6	1.559660	1.331460	1.020130	0.945156	0.953566	8.411×10^{-3}
0.7	1.589980	1.497600	1.220350	1.144830	1.152950	8.122×10^{-3}
0.8	1.578560	1.630230	1.412050	1.341550	1.346360	4.811×10^{-3}
0.9	1.530030	1.724440	1.587310	1.527690	1.526910	7.797×10^{-4}
1.0	1.448810	1.776540	1.738230	1.695240	1.689500	5.740×10^{-3}

Table 10: Approximate solutions of MADM and LAADM for two iterations $y(t) = y_0 + y_1 + y_2$

Table 11: Presents a comparison of the approximate solutions and absolute errors produced by applying Kharat-Toma transform with iterative method (KTIM) and Laplace transform combined with accelerated Adomian (LAADM) for Example 5.3.

t	Approximate solution				Exact solution	Absolute error $(\alpha = 1)$
	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$		
KTIM						
0.1	0.5774312	0.2444602	0.1287442	0.1102661	0.1102952	2.909×10^{-5}
0.2	0.9126546	0.4697096	0.2750385	0.2415856	0.2419768	3.912×10^{-4}
0.3	1.1662530	0.6987185	0.4410789	0.3935155	0.3951048	1.589×10^{-3}
0.4	1.3535500	0.9243191	0.6239758	0.5640131	0.5678122	3.799×10^{-3}
0.5	1.4826340	1.1379530	0.8190042	0.7495288	0.7560144	6.486×10^{-3}
0.6	1.5596570	1.3314630	1.0201250	0.9451557	0.9535662	8.411×10^{-3}
0.7	1.5899850	1.4976010	1.2203450	1.1448270	1.1529490	8.122×10^{-3}
0.8	1.5785590	1.6302340	1.4120470	1.3415530	1.3463640	4.811×10^{-3}
0.9	1.5300280	1.7244390	1.5873120	1.5276910	1.5269110	7.797×10^{-4}
1.0	1.4488050	1.7765430	1.7382340	1.6952380	1.6894980	5.74×10^{-3}
LAADM						
0.1	0.5774312	0.2444602	0.1287442	0.1102661	0.1102952	2.909×10^{-5}
0.2	0.9126546	0.4697096	0.2750385	0.2415856	0.2419768	3.912×10^{-4}
0.3	1.1662530	0.6987185	0.4410789	0.3935155	0.3951048	1.589×10^{-3}
0.4	1.3535500	0.9243191	0.6239758	0.5640131	0.5678122	3.799×10^{-3}
0.5	1.4826340	1.1379530	0.8190042	0.7495288	0.7560144	6.486×10^{-3}
0.6	1.5596570	1.3314630	1.0201250	0.9451557	0.9535662	8.411×10^{-3}
0.7	1.5899850	1.4976010	1.2203450	1.1448270	1.1529490	8.122×10^{-3}
0.8	1.5785590	1.6302340	1.4120470	1.3415530	1.3463640	4.811×10^{-3}
0.9	1.5300280	1.7244390	1.5873120	1.5276910	1.5269110	7.797×10^{-4}
1.0	1.4488050	1.7765430	1.7382340	1.6952380	1.6894980	5.74×10^{-3}

Table 11: Approximate solutions of KTIM and LAADM for two iterations $y(t) = y_0 + y_1 + y_2$

Example 5.4. [34] The following non-linear FODE is considered

$$D_t^{\alpha} y\,(t) = 1 - y^2(\mathrm{t}), 0 \leq \mathrm{t} \leq 1, 0 \ < \alpha \ \leq \ 1 \tag{5.113}$$

with initial condition $y\,(0) = 0$, and exact solution for $\alpha = 1$ is

$$y\,(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{5.114}$$

This example is examined by many researchers. The most recent one is by Jassim and Hussein [34] introduce a novel technique to solve nonlinear ordinary differential equations with fractional-order.

For this example, we apply the Laplace transform with the suggested accelerated Adomian decomposition method to compare the accuracy for the equation that we are studying in this paper of this method with the Hussein–Jassim approach, and the Adomian decomposition method paired to Laplace transform.

5.8 Solution using Laplace Transform coupled with Adomian decomposition method

Applying Laplace transform for Eq. (5.113) and using the fractional derivatives properties of Laplace transform we have

$$\mathcal{L}[D_t^{\alpha} y\,(t)] = \mathcal{L}\,[1] - \mathcal{L}\left[y^2\,(\mathrm{t})\right], \tag{5.115}$$

$$s^{\alpha} Y\,(s) - s^{\alpha-1} y\,(0) = \frac{1}{s} - \mathcal{L}\left[y^2(t)\right], \tag{5.116}$$

$$Y\,(s) = \frac{1}{s^{\alpha+1}} - \frac{1}{s^{\alpha}} \mathcal{L}\left[y^2(t)\right]. \tag{5.117}$$

For Eq. (5.117), take the inverse Laplace transform to both sides, we obtain:

$$\mathcal{L}^{-1}\,[Y\,(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[y^2\,(t)\right]\right], \tag{5.118}$$

$$y\,(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[y^2\,(t)\right]\right]. \tag{5.119}$$

After that the solution is represented as an infinite series, and the nonlinear term can be defined as follows, respectively

$$\begin{gathered} y\,(t) = \sum_{n=0}^{\infty} y_n\,(t)\,, \\ y^2\,(t) = \sum_{n=0}^{\infty} A_n(y), \end{gathered} \tag{5.120}$$

where, A_n are Adomian polynomial and it may be determined using formula provided below

$$A_n = \frac{1}{n\,!} \left(\frac{d^n}{d\lambda^n} \left[N \sum_{i=0}^{\infty} \left(\lambda^i y_i\right) \right] \right)_{\lambda=0}, \qquad n = 0\,,\,1\,,\,2\,,3 \ldots . \tag{5.121}$$

To substitute Eq. (5.120) in Eq. (5.119), we get:

$$\sum_{n=0}^{\infty} y_n = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{5.122}$$

Then from Eq. (5.122) we get:

$$y_0 = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.123}$$

$$y_1 = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\,[A_0]\right], \tag{5.124}$$

$$y_2 = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[A_1\right]\right], \tag{5.125}$$

$$y_3 = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[A_2\right]\right], \tag{5.126}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[A_{n-1}\right]\right],\ n = 1,\ 2\ ,\ 3, \ldots. \tag{5.127}$$

Where the first few components of A_n are given by

$$A_0 = y_0^2, \tag{5.128}$$

$$A_1 = 2y_0y_1, \tag{5.129}$$

$$A_2 = y_1^2 + 2y_0y_2, \tag{5.130}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Then, from Eq. (5.123), Eq. (5.124) and Eq. (5.128) we get

$$\begin{aligned} y_1 &= -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[A_0]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{s^{-1-3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right] \\ &= -\frac{t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}. \end{aligned} \tag{5.131}$$

Similarly, from Eq. (5.127), Eq. (5.129) and Eq. (5.130), we can find $y_{\ 2},\ y_3,\ \ldots$ as follows:

$$y_2 = \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)}, \tag{5.132}$$

$$y_3 = -\frac{t^{7\alpha}\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)} - \frac{4t^{7\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)}, \tag{5.133}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n \\ &= y_0 + y_1 + y_2 + y_3 + \ldots. \\ &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)]\Gamma(1+5\alpha)} \\ &\quad - \frac{t^{7\alpha}\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)} - \frac{4t^{7\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)} + \ldots. \end{aligned} \tag{5.134}$$

After three iterations and computation at $\alpha = 1$, the approximate solution has the following form.

$$y(t) \approx t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315}. \tag{5.135}$$

5.9 Solution using Laplace Transform coupled with accelerated Adomian decomposition Method

Similarly, applying Laplace transform for Eq. (5.113) and by using the Laplace transform's fractional derivatives properties, we have

$$\mathcal{L}[D_t^{\alpha} y\,(t)] = \mathcal{L}\,[1] - \mathcal{L}\left[y^2\,(\mathrm{t})\right], \tag{5.136}$$

$$s^{\alpha} Y\,(s) - s^{\alpha-1} y\,(0) = \frac{1}{s} - \mathcal{L}\left[y^2(t)\right], \tag{5.137}$$

$$Y\,(s) = \frac{1}{s^{\alpha+1}} - \frac{1}{s^{\alpha}} \mathcal{L}\left[y^2(t)\right]. \tag{5.138}$$

For Eq. (5.138), take the inverse Laplace transform to both sides, we obtain:

$$\mathcal{L}^{-1}\left[Y\,(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[y^2\,(t)\right]\right], \tag{5.139}$$

$$y\,(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[y^2\,(t)\right]\right]. \tag{5.140}$$

After that the solution is represented as an infinite series, and the nonlinear term can be defined as follows, respectively

$$\begin{aligned} y\,(t) &= \sum_{n=0}^{\infty} y_n\,(t), \\ y^2\,(t) &= \sum_{n=0}^{\infty} \overline{A}_n(y), \end{aligned} \tag{5.141}$$

Where $\overline{A}_n$ are the accelerated Adomian polynomials and they may be determined using formula provided below:

$$\overline{A}_n = N\,(s_n) - \sum_{i=0}^{n-1} \overline{A}_i \quad \text{where} \quad s_n = y_0^2 + y_1^2 + \cdots + y_n^2. \tag{5.142}$$

To substitute Eq. (5.141) in Eq. (5.140), we get:

$$\sum_{n=0}^{\infty} y_n = \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[\sum_{n=0}^{\infty} \overline{A}_n\right]\right]. \tag{5.143}$$

Then from Eq. (5.143) we get:

$$y_0 = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.144}$$

$$y_1 = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[\overline{A}_0\right]\right], \tag{5.145}$$

$$y_2 = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[\overline{A}_1\right]\right], \tag{5.146}$$

$$y_3 = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[\overline{A}_2\right]\right], \tag{5.147}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[\overline{A}_{n-1}\right]\right], \; n \;=\; 1,\; 2\;,\; 3, \ldots. \tag{5.148}$$

Where the first few components of $\overline{A}_n$ are given by

$$\overline{A}_0 = \, {y_0}^2, \tag{5.149}$$

$$\overline{A}_1 = 2y_0y_1 + {y_1}^2, \tag{5.150}$$

$$\overline{A}_2 = 2y_0y_2 + 2y_1y_2 + {y_2}^2, \tag{5.151}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Then, from Eq. (5.144), Eq. (5.145) and Eq. (5.149) we get

$$\begin{aligned} y_1 &= -\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}[\overline{A}_0]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}[\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{s^{-1-3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right] \\ &= -\frac{t^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}. \end{aligned} \tag{5.152}$$

Similarly, from Eq. (5.148), Eq. (5.150) and Eq. (5.151), we can find $y_2, y_3, \ldots$ as follows:

$$y_2 = \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)]\Gamma(1+5\alpha)} - \frac{t^{7\alpha}\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)}, \tag{5.153}$$

$$\begin{aligned} y_3 = &-\frac{4t^{7\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)} \\ &+\frac{4t^{9\alpha}\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)\Gamma(1+9\alpha)} \\ &+\frac{2t^{9\alpha}\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)\Gamma(1+9\alpha)} \\ &-\frac{4t^{11\alpha}\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)^2\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)^2\Gamma(1+11\alpha)} \\ &-\frac{2t^{11\alpha}\Gamma(1+2\alpha)^3\Gamma(1+6\alpha)\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^3\Gamma(1+7\alpha)\Gamma(1+11\alpha)} \\ &+\frac{4t^{13\alpha}\Gamma(1+2\alpha)^3\Gamma(1+4\alpha)\Gamma(1+6\alpha)\Gamma(1+12\alpha)}{\Gamma(1+\alpha)^7\Gamma(1+3\alpha)^3\Gamma(1+5\alpha)\Gamma(1+7\alpha)\Gamma(1+13\alpha)} \\ &-\frac{t^{15\alpha}\Gamma(1+2\alpha)^4\Gamma(1+6\alpha)^2\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^8\Gamma(1+3\alpha)^4\Gamma(1+7\alpha)^2\Gamma(1+15\alpha)}, \end{aligned} \tag{5.154}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given

$$
\begin{aligned}
y\left(t\right) &= \sum_{n=0}^{\infty} y_n \\
&= y_0 + y_1 + y_2 + y_3 + \dots . \\
&= \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+\alpha\right)^2\Gamma\left(1+3\alpha\right)} + \frac{2t^{5\alpha}\Gamma\left(1+2\alpha\right)\Gamma\left(1+4\alpha\right)}{\Gamma\left(1+\alpha\right)^3\Gamma(1+3\alpha)]\Gamma(1+5\alpha)} \\
&\quad - \frac{t^{7\alpha}\Gamma\left(1+2\alpha\right)^2\Gamma\left(1+6\alpha\right)}{\Gamma\left(1+\alpha\right)^4\Gamma\left(1+3\alpha\right)^2\Gamma\left(1+7\alpha\right)} - \frac{4t^{7\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)} \\
&\quad + \frac{4t^{9\alpha}\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)\Gamma(1+9\alpha)} + \frac{2t^{9\alpha}\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)\Gamma(1+9\alpha)} \\
&\quad - \frac{4t^{11\alpha}\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)^2\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)^2\Gamma(1+11\alpha)} - \frac{2t^{11\alpha}\Gamma(1+2\alpha)^3\Gamma(1+6\alpha)\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^3\Gamma(1+7\alpha)\Gamma(1+11\alpha)} \\
&\quad + \frac{4t^{13\alpha}\Gamma(1+2\alpha)^3\Gamma(1+4\alpha)\Gamma(1+6\alpha)\Gamma(1+12\alpha)}{\Gamma(1+\alpha)^7\Gamma(1+3\alpha)^3\Gamma(1+5\alpha)\Gamma(1+7\alpha)\Gamma(1+13\alpha)} \\
&\quad - \frac{t^{15\alpha}\Gamma(1+2\alpha)^4\Gamma(1+6\alpha)^2\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^8\Gamma(1+3\alpha)^4\Gamma(1+7\alpha)^2\Gamma(1+15\alpha)} + \dots .
\end{aligned}
\tag{5.155}
$$

After three iterations and computation at $\alpha = 1$, the approximate solution has the following form.

$$y\left(t\right) \ \approx t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \frac{38t^9}{2835} - \frac{134t^{11}}{51975} + \frac{4t^{13}}{12285} - \frac{t^{15}}{59535}. \tag{5.156}$$

5.10 Solution using Hussein–Jassim Method

Now, we derive some terms of the Hussein–Jassim Method, as presented in [34]

$$y_0 = 0, \tag{5.157}$$

$$y_1 = \frac{t^{\alpha}}{\Gamma\left(1+\alpha\right)}, \tag{5.158}$$

$$y_2 = 0, \tag{5.159}$$

$$y_3 = -\frac{t^{3\alpha}\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+\alpha\right)^2\Gamma\left(1+3\alpha\right)}, \tag{5.160}$$

$$\vdots \qquad \vdots \qquad \vdots$$

then, the approximate series solution is given by

$$
\begin{aligned}
y\left(t\right) &= \sum_{n=0}^{\infty} y_n \\
&= y_0 + y_1 + y_2 + \dots \\
&= \frac{t^{\alpha}}{\Gamma\left(1+\alpha\right)} - \frac{t^{3\alpha}\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+\alpha\right)^2\Gamma\left(1+3\alpha\right)} + \dots .
\end{aligned}
\tag{5.161}
$$

After three iterations and computation at $\alpha = 1$, the approximate solution has the following form:

$$y\left(t\right) \approx t - \frac{t^3}{3}. \tag{5.162}$$

When considering the number of iterations (3 iterations), the proposed approach demonstrates superior accuracy and convergence rate compared to both the Hussein–Jassim Method (HJM), which presented in [34] to solve nonlinear ordinary differential equations with fractional-order, and the Adomian Decomposition Method with Laplace transform (LADM).

Table 12: Comparison of the approximate solutions and absolute errors produced by applying the Jassim-Hussein approach (HJM), the Laplace transform combined with Adomian (LADM), and the Laplace transform combined with Accelerated Adomian (LAADM), for Example 5.4 [34].

t	Approximate solution			Exact solution	Absolute error ($\alpha = 1$)
	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$		
HJM					
0.1	0.1679647	0.1300302	0.09966667	0.09966799	3.208×10^{-6}
0.2	0.2846595	0.2386287	0.19733330	0.19737530	4.1987×10^{-5}
0.3	0.3790594	0.3349978	0.29100000	0.29131260	3.1261×10^{-4}
0.4	0.4545372	0.4191991	0.37866670	0.37994900	1.2823×10^{-3}
0.5	0.5119251	0.4903133	0.45833330	0.46211720	3.7838×10^{-3}
0.6	0.5512656	0.5471360	0.52800000	0.53704960	9.0496×10^{-3}
0.7	0.5722823	0.5883650	0.58566670	0.60436780	1.8701×10^{-2}
0.8	0.5745534	0.6126692	0.62933330	0.66403680	3.4703×10^{-2}
0.9	0.5575884	0.6187162	0.65700000	0.71629790	5.9298×10^{-2}
1.0	0.5208654	0.6051853	0.66666670	0.76159420	9.4927×10^{-2}
LADM					
0.1	0.16800233	0.13003745	0.099667995	0.099667995	2.1781×10^{-11}
0.2	0.28523921	0.23878899	0.19737531	0.19737532	1.1019×10^{-8}
0.3	0.38183599	0.33595834	0.2913122	0.29131261	4.1531×10^{-7}
0.4	0.46270091	0.42254513	0.37994358	0.37994896	5.3838×10^{-6}
0.5	0.53011939	0.49891438	0.46207837	0.46211716	3.8784×10^{-5}
0.6	0.58493079	0.56525374	0.53685723	0.53704957	1.9233×10^{-4}
0.7	0.62633049	0.62137269	0.60363148	0.60436778	7.3629×10^{-4}
0.8	0.65128905	0.66622617	0.66170604	0.66403677	2.3307×10^{-3}
0.9	0.65379373	0.69723735	0.70991915	0.71629787	6.3787×10^{-3}
1.0	0.62397584	0.70943754	0.74603175	0.76159416	1.5562×10^{-2}
LAADM					
0.1	0.16800234	0.13003745	0.099667995	0.099667995	8.4031×10^{-12}
0.2	0.28524029	0.23878908	0.19737532	0.19737532	4.2083×10^{-9}
0.3	0.38185559	0.33596069	0.29131246	0.29131261	1.56×10^{-7}
0.4	0.46285213	0.42256884	0.37994699	0.37994896	1.9761×10^{-6}
0.5	0.53084951	0.4990555	0.46210333	0.46211716	1.3824×10^{-5}
0.6	0.58754892	0.56585481	0.53698338	0.53704957	6.6189×10^{-5}
0.7	0.63397209	0.62340322	0.60412447	0.60436778	2.433×10^{-4}
0.8	0.67046735	0.67201185	0.66403677	0.66403677	7.3585×10^{-4}
0.9	0.69668027	0.71170643	0.71629787	0.71629787	1.9155×10^{-3}
1.0	0.71154306	0.74207586	0.76159416	0.76159416	4.4279×10^{-3}

Table 12: Approximate solutions of HJM, LADM and LAADM for three iterations $y(t) = y_0 + y_1 + y_2 + y_3$

Table 12 and Figure 6 show that the suggested method of combining Laplace transform with accelerated Adomian investigates superior accuracy, accompanied by a reduction in the absolute error than the two others methods Jassim-Hussein method and Laplace with normal Adomian decomposition method.

It is clear from Figure 5 that, as the value of α increases, the approximate solution decreases and becomes closer to the exact solution, which demonstrates the accuracy, convergence and efficiency of the proposed approach accelerated Adomian combines with Laplace transform.

Table 12 confirms LAADM reduces error by 10× vs. Jassim-Hussein Method (3 iteration). The bound $\|E_m\| \leq 0.01$ holds for $m \geq 2$.

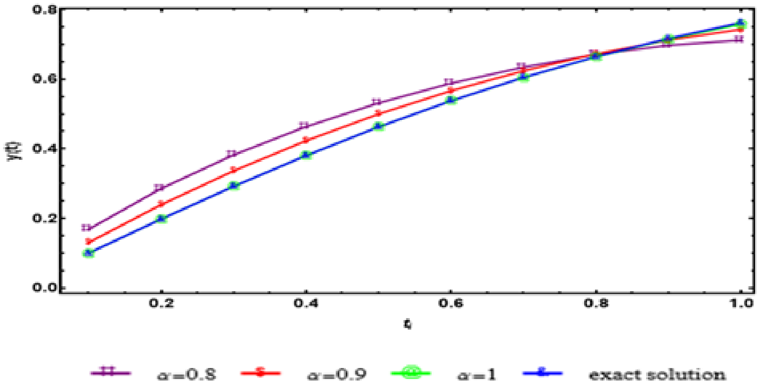

Figure 5: The graph compares the approximate solutions for y(t) with the exact solution for Example 5.4. The approximate solutions, generated using the Laplace transform and accelerated Adomian method (3 iterations), are shown for three fractional values of α (0.8, 0.9, and 1). The figure demonstrates that the approximate solutions get closer to the exact solution as α increases towards 1.

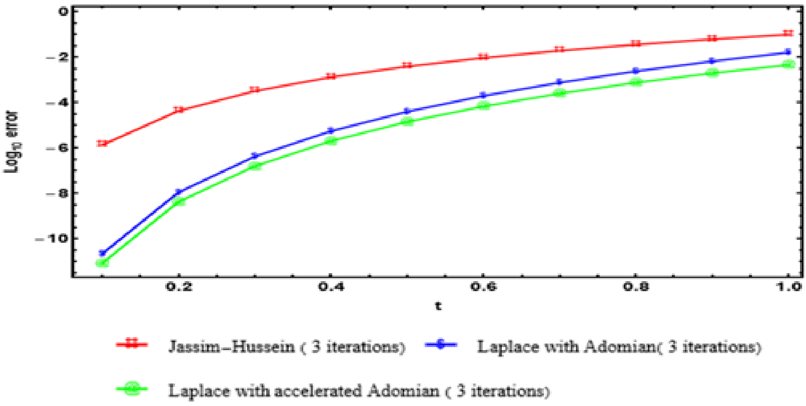

Figure 6: Comparison of the absolute errors of three methods for Example 5.4. The graph plots the base-10 logarithm of the error against time (t). The methods compared are the Jassim-Hussein method, the Laplace transform with Adomian, and the Laplace transform with accelerated Adomian, all using three iterations. The graph shows that the Laplace transform with accelerated Adomian method results in the lowest error.

6 Conclusions

In this research, we introduced a novel hybrid method for solving nonlinear fractional differential equations by combining the accelerated Adomian decomposition method (AADM) with the Laplace integral transform, termed as Laplace-accelerated Adomian decomposition method (LAADM). The primary novelty of our approach lies in its ability to generate recursive polynomials without derivatives, which significantly simplifies implementation while reducing computation time. Unlike traditional methods, LAADM leverages fractional power series expansions within the Laplace domain, enabling more accurate and efficient handling of nonlinear terms and memory effects inherent in fractional systems. Numerical results demonstrated that LAADM provides fast-converging and high accuracy with fewer iterations compared to existing methods, without requiring linearization or discretization. Despite its advantages, our method may face challenges in problems with highly complex boundary conditions or high-dimensional system.

Future research could enhance LAADM adaptive algorithms to automatically determine the optimal number of iterations required for specific problems and through the integration with machine learning techniques for improved handling even more complex fractional systems. Overall, LAADM offers a promising analytical tool for fractional systems across engineering, physics, biological and medical fields.

Statements and Declarations

Ethics-approved: None of the authors of this article have ever conducted investigations using humans or animals.

Conflict of Interest

The authors declare that they have no conflict of interest.

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