Palestine Journal of Mathematics

Vol 15(2)(2026) , 842–855

SENSITIVITY OF MULTIPLE EIGENVALUES IN QUATERNIONIC HERMITIAN SYSTEMS

S. Parveen and I. Ali

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 12E15; 34L15; Secondary 15A18; 15A66.

Keywords and phrases: Multiple eigenvalue, generalized eigenvalue problem, quaternionic matrix polynomials, Hermitian matrix, Perturbation bounds.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper

Acknowledgement: The authors would like to thank Integral University, Lucknow, India, for providing the manuscript number IU/R & D/2025-MCN0003671 to present research work.

Corresponding Author: I. Ali

Abstract. This paper establishes perturbation bounds for multiple right eigenvalues of quaternionic Hermitian matrices under off-diagonal perturbations with non-uniform column norms. We show that when $A_{22} - \mu I$ is definite, the i-th eigenvalue perturbation scales as $O(\epsilon_i^2)$, where ϵ_i is the i-th column norm, while indefinite cases exhibit $O(\epsilon_i \epsilon)$ dependence due to non-commutative effects. Our results generalize classical Bauer-Fike-type theorems to the quaternionic setting, with applications to degenerate quantum systems and quaternionic signal processing. Numerical experiments validate the theoretical bounds and reveal conditions under which commutativity assumptions fail.

1 Introduction

The perturbation theory of eigenvalues for Hermitian matrices is a well-established topic in linear algebra, with numerous applications in numerical analysis, quantum mechanics, and engineering. Classical results, such as those presented in, provide bounds for the perturbation of multiple eigenvalues under off-diagonal perturbations. However, these results are primarily confined to the complex (or real) case. With the growing interest in quaternionic linear algebra due to its applications in quantum mechanics, signal processing, and robotics, there is a need to extend these perturbation bounds to the quaternions. The study of eigenvalue problems in quaternionic linear algebra has gained significant attention in recent decades due to its applications in quantum mechanics [1], signal processing [2], and computer graphics. While perturbation theory for complex Hermitian matrices is well-established [3, 4], the non-commutative nature of quaternion multiplication introduces fundamental differences that require careful reconsideration of these classical results. This paper extends the perturbation theory for multiple eigenvalues of Hermitian matrices to the quaternionic setting, with particular attention to the case where perturbation matrices have columns of widely varying magnitudes.

Quaternionic matrices present unique challenges compared to their complex counterparts. The lack of commutativity in $\mathbb{H}$ means we must distinguish between left and right eigenvalues, with right eigenvalues being more amenable to analysis due to their behavior under similarity transformations [5]. The determinant theory for quaternionic matrices, developed by Dieudonné [6], provides essential tools for our analysis, though it requires different approaches than the classical determinant.

Recent work by Ahmad and Ali [7, 8] has established foundational results for quaternionic matrix analysis, including perturbation bounds for single eigenvalues. However, the behavior of multiple eigenvalues under structured perturbations remains largely unexplored in the quaternionic setting. Our work fills this gap by developing perturbation bounds that account for the graded structure of perturbation matrices, where different columns may have substantially different magnitudes.

The key motivation for our study comes from several observations

- Right eigenvalues of quaternionic Hermitian matrices are real and maintain many properties of complex Hermitian matrices [9].
- The Bauer-Fike theorem and related perturbation results have quaternionic analogs [8].
- Multiple eigenvalues in physical systems often correspond to degenerate states whose splitting under perturbation requires precise quantification.

Our main contributions include

- Quaternionic versions of perturbation bounds for multiple eigenvalues under column-graded perturbations (Theorems 3.3-3.5).
- Extension to the quaternionic generalized eigenvalue problem (Theorem 5.1).
- New results for quaternionic matrix polynomials (Theorem 6.1).
- Numerical verification of our theoretical bounds through carefully constructed examples.

Our work builds on the classical perturbation theory of [10] while incorporating the quaternionic framework of [5] and recent advances from [8]. The fixed-point arguments we employ extend the work of [11] to the quaternionic case. The results have immediate applications in quantum systems with quaternionic Hamiltonians and in the analysis of algorithms for quaternionic matrix computations. We gratefully acknowledge the foundational insights derived from the works of [12, 13, 14, 15]. Their research in the area of quaternionic matrices and matrix bounds was instrumental in guiding the theoretical framework and analysis presented in this paper.

In this paper, we generalize the perturbation bounds for multiple eigenvalues of Hermitian matrices to the quaternionic matrices. Specifically, we consider a quaternionic Hermitian matrix $A = \text{diag}(\mu I, A_{22})$, where μ is a multiple eigenvalue, and study its perturbation under an off-diagonal quaternionic matrix E. Our work addresses the following key questions:

- How does the non-commutativity of quaternions affect the perturbation bounds for multiple eigenvalues?
- What are the analogues of the definite and indefinite cases in the quaternionic setting, and how do they influence the perturbation bounds?
- Can the results be extended to the quaternionic generalized eigenvalue problem?

The perturbation of multiple eigenvalues in quaternionic matrices remains unexplored despite its significance in systems with degenerate states (e.g., quaternionic quantum mechanics [11]). Unlike the complex case, where the spectral theorem fully diagonalizes Hermitian matrices, quaternionic matrices require symplectic decompositions [1,4], complicating the analysis of eigenvalue splitting under perturbations. Our work addresses this gap by providing the first systematic study of how non-uniform perturbations (quantified by column norms ϵ_j) affect multiple eigenvalues in $\mathbb{H}$, revealing fundamentally new scaling laws tied to non-commutativity.

The paper is organized as follows. Section 2 reviews notations and some existing results. In Section 3, we present our main perturbation theorems, separating the definite and indefinite cases. In Section 4, numerical examples to illustrate our results are purposed. In Section 5, we discuss about the extensions to the generalized eigenvalue problem. Finally in Section 6, we explores connections with quaternionic polynomials.

2 Notation and preliminaries

Notation: Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. $\mathbb{R}^+$ denotes the set of nonnegative real numbers. The set of real quaternions is defined by

$$\mathbb{H} = \{q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. The conjugate of $q \in \mathbb{H}$ is $\overline{q} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ and the modulus of q is $|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. $\Im(a)$ denotes the imaginary part of $a \in \mathbb{C}$. The collection of all n-column vectors with elements in $\mathbb{H}$ is denoted by $\mathbb{H}^n$. For $x \in \mathcal{K}^n$, where $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the transpose of x is x^T. If $x = [x_1, \ldots, x_n]^T$, the conjugate of x is defined as $\overline{x} = [\overline{x}_1, \ldots, \overline{x}_n]^T$ and the conjugate transpose of x is defined as $x^H = [\overline{x}_1, \ldots, \overline{x}_n]$. For $x, y \in \mathbb{H}^n$, the inner product is defined as $\langle x, y\rangle = y^H x$ and the norm of x is defined as $\|x\|_2 = \sqrt{\langle x, x\rangle}$. Let $A = [a_{ij}] \in M_{m\times n}$ and define $|A| = [|a_{ij}|]$ (entrywise absolute value).

The sets of $m \times n$ real, complex, and quaternionic matrices are denoted by $M_{m\times n}(\mathbb{R})$, $M_{m\times n}(\mathbb{C})$, and $M_{m\times n}(\mathbb{H})$, respectively. When $m = n$, these sets are denoted by $M_n(\mathcal{K})$, $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For $A \in M_{m\times n}(\mathcal{K})$, the transpose, conjugate and conjugate transpose of A are defined as $A^T = (a_{ji}) \in M_{n\times m}(\mathcal{K})$, $\overline{A} = (\overline{a_{ij}})$ and $A^H = (\overline{A})^T \in M_{n\times m}(\mathcal{K})$, respectively.

The space of all $p\times p$ quaternionic matrices is denoted by $M_p(\mathbb{H})$. The expressions $||A||_2, ||A||_\infty, ||A||_1$ stand for the spectral norm, maximum row sum norm and maximum column sum norm of $A \in M_p(\mathbb{H})$, respectively. We define the 2-norm on $A \in M_n(\mathbb{H})$ by

$$||A||_2 = \sup_{x\neq 0}\left\{\frac{\|Ax\|_2}{\|x\|_2} : x \in \mathbb{H}^n\right\} = \|A^H\|_2.$$

This paper examines the perturbation theory for multiple right eigenvalues of quaternionic Hermitian matrices, where a right eigenvalue μ satisfies $Ax = x\mu$ for some nonzero quaternionic vector x.

Consider the eigenvalue problem for a quaternionic Hermitian matrix $\tilde{A}$:

$$\tilde{A} = \begin{pmatrix} A_{11} & E^* \\ E & A_{22} \end{pmatrix}, \quad A_{11} = \mu I_m, \tag{2.1}$$

where

- $\tilde{A} \in \mathbb{H}^{(m+n)\times(m+n)}$ is a quaternionic Hermitian matrix ($\tilde{A}^* = \tilde{A}$).
- $E \in \mathbb{H}^{n\times m}$ is the perturbation matrix with quaternionic entries.
- I_m is the $m \times m$ identity matrix over $\mathbb{H}$.
- The superscript $*$ denotes the conjugate transpose (quaternionic adjoint).

If E is a zero block, then μ is a right eigenvalue with multiplicity m. For small E, $\tilde{A}$ has m right eigenvalues close to μ.

Define the eigenvalue gap η between $A_{11} = \mu I$ and A_{22} as

$$\eta = \min_{\nu\in \mathrm{eig}(A_{22})} |\mu - \nu|, \tag{2.2}$$

where $\mathrm{eig}(A_{22})$ is the set of right eigenvalues of A_{22}, and $|\cdot|$ is the quaternionic modulus. Let

$$\epsilon = \|E\|_2, \tag{2.3}$$

where $\|\cdot\|_2$ is the spectral norm or the ℓ_2-norm for quaternionic vectors.

The quaternionic version of the main result from [10] states that $\tilde{A}$ has m right eigenvalues $\theta_1, \ldots, \theta_m$ such that

$$|\mu - \theta_j| \le \frac{2\epsilon^2}{\eta + \sqrt{\eta^2 + 4\epsilon^2}} \quad \text{for } 1 \le j \le m. \tag{2.4}$$

The right-hand side of (2.4) is second-order in ϵ when $\eta > 0$ and never exceeds ϵ.

When columns of E have disparate magnitudes, define

$$\epsilon_j = \|E_{(:,j)}\|_2 \quad \text{for } 1 \le j \le m, \tag{2.5}$$

where $E_{(:,j)}$ denotes the jth column of E and $\{i_1, \ldots, i_m\}$ is a permutation of $\{1, \ldots, m\}$ such that

$$\epsilon_1 \le \epsilon_2 \le \cdots \le \epsilon_m. \tag{2.6}$$

The eigenvalues τ_j of E^*E (which are real since E^*E is Hermitian) satisfy

$$0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_m. \tag{2.7}$$

We use the notation $X \prec Y$ ($X \preceq Y$) for quaternionic Hermitian matrices to mean $Y - X$ is positive definite (semi-definite), and $X \succ Y$ ($X \succeq Y$) to mean $Y \prec X$ ($Y \preceq X$).

The paper develops perturbation bounds that reflect the effect of column magnitude disparity in E, improving upon (2.4) by establishing individual bounds for each θ_j. The key results show that

- When $A_{22} - \mu I$ is definite, $|\theta_j - \mu|$ scales with ϵ_j^2.
- In the indefinite case, $|\theta_j - \mu|$ scales with $\epsilon_j \epsilon$.

The non-commutativity of quaternions introduces several technical challenges

- Right eigenvalues behave differently than left eigenvalues under similarity transformations.
- The spectral theorem requires careful adaptation to the quaternionic case.
- Matrix decompositions must account for the non-commutative nature of quaternions.

3 Perturbation Bounds for Quaternionic Hermitian Matrices

Our results show that when $A_{22} - \mu I$ is definite (either positive or negative), the perturbation bounds maintain a quadratic dependence on column norms similar to the complex case. However, in the indefinite case, the non-commutativity leads to bounds that depend on products of column norms, reflecting the more intricate interaction between perturbation directions.

The following theorems establish precise bounds that capture these phenomena, providing a complete picture of multiple eigenvalue perturbation in the quaternionic matrices. We separate our analysis into definite and indefinite cases, as the behaviour differs fundamentally between these situations.

Throughout this section, $\tilde{A}$ is a quaternionic Hermitian matrix given by

$$\tilde{A} = \begin{pmatrix} \mu I_m & E^* \\ E & A_{22} \end{pmatrix} \in \mathbb{H}^{(m+n)\times(m+n)} \tag{3.1}$$

where $\mathbb{H}$ denotes the quaternion algebra, $\mu \in \mathbb{R}$ is a multiple right eigenvalue, I_m is the identity matrix, $E \in \mathbb{H}^{n\times m}$, and $A_{22} = A_{22}^* \in \mathbb{H}^{n\times n}$. Without loss of generality, we assume $\mu = 0$.

Since μ is not a right eigenvalue of A_{22}, A_{22} is invertible, and the gap η is defined as:

$$\eta = 1/\|A_{22}^{-1}\|_2 = \min_{\nu \in \sigma_r(A_{22})} |\nu| \tag{3.2}$$

where $\sigma_r(A_{22})$ denotes the set of right eigenvalues of A_{22}.

For any $\lambda \notin \sigma_r(A_{22})$, define the symplectic transformation

$$X = \begin{pmatrix} I_m & -E^*(A_{22} - \lambda I_n)^{-1} \\ 0 & I_n \end{pmatrix} \in \mathbb{H}^{(m+n)\times(m+n)} \tag{3.3}$$

Then we have the quaternionic congruence transformation:

$$X(\tilde{A} - \lambda I_{m+n})X^* = \begin{pmatrix} -\lambda I_m - E^*(A_{22} - \lambda I_n)^{-1}E & 0 \\ 0 & A_{22} - \lambda I_n \end{pmatrix} \tag{3.4}$$

Using the Dieudonné determinant for quaternionic matrices, we obtain:

$$\mathrm{Det}(\tilde{A} - \lambda I_{m+n}) = \mathrm{Det}\left(-E^*(A_{22} - \lambda I_n)^{-1}E - \lambda I_m\right) \cdot \mathrm{Det}(A_{22} - \lambda I_n) \tag{3.5}$$

Thus, any right eigenvalue $\tilde{\lambda}$ of $\tilde{A}$ not in $\sigma_r(A_{22})$ is a root of

$$\operatorname{Det}\left(-E^*(A_{22}-\lambda I_n)^{-1}E-\lambda I_m\right)=0 \tag{3.6}$$

For $|\lambda|/\eta<1$, we have the quaternionic Neumann series expansion:

$$(A_{22}-\lambda I_n)^{-1}=\sum_{j=0}^{\infty}\lambda^j A_{22}^{-j-1} \tag{3.7}$$

where the order of multiplication matters in the quaternionic case.

Substituting into (3.6) gives the key expansion

$$-E^*(A_{22}-\lambda I_n)^{-1}E-\lambda I_m=-\sum_{j=0}^{\infty}\lambda^j E^* A_{22}^{-j-1}E-\lambda I_m \tag{3.8}$$

Theorem 3.1. *Let $\tilde{A}\in\mathbb{H}^{N\times N}$ be a quaternionic Hermitian matrix of form (2.1) with $\mu=0$.*

1. Assume $\epsilon<\sqrt{3/4}\eta$. Then (a) $\tilde{A}$ has exactly m right eigenvalues $\theta_j\in\mathbb{R}$ in the open interval $(-\eta/2,\eta/2)$, and moreover

$$|\theta_j|\le\frac{2\epsilon^2}{\eta+\sqrt{\eta^2+4\epsilon^2}},$$

for $1\le j\le m$; (b) The function (3.6) has exactly m zeros in $(-\eta/2,\eta/2)$ and these zeros are precisely the eigenvalues θ_j of $\tilde{A}$.

2. $\tilde{A}$ has m eigenvalues $\theta_j=\vartheta_j+O(\epsilon^4/\eta^2)$, where ϑ_j for $1\le j\le m$ are the eigenvalues of $-E^*A_{22}^{-1}E$.

Proof: We prove each part separately

Part 1(a): Since $4t^2/(1+\sqrt{1+4t^2})<1$ if $t^2<3/4$, we have

$$\frac{2\epsilon^2}{\eta+\sqrt{\eta^2+4\epsilon^2}}<\frac{\eta}{2}\quad\text{if}\quad\frac{\epsilon}{\eta}<\sqrt{\frac{3}{4}}.$$

By the extension of the main result of the quaternionic matrices, $\tilde{A}$ has exactly m right eigenvalues θ_j in $(-\eta/2,\eta/2)$ and the inequality holds. The eigenvalues are real because $\tilde{A}$ is quaternionic Hermitian.

Part 1(b) This follows from Part 1(a) and the quaternionic version of the determinant formula (3.5), noting that $\det(A_{22}-\lambda I)\neq 0$ for $\lambda\in(-\eta/2,\eta/2)$ since η is the gap between 0 and the spectrum of A_{22}.

Part 2 For $|\lambda|=O(\epsilon^2/\eta)$, we expand $(A_{22}-\lambda I)^{-1}$ using the quaternionic power series

$$(A_{22}-\lambda I)^{-1}=\sum_{j=0}^{\infty}\lambda^j A_{22}^{-j-1},$$

which converges for $|\lambda|<\eta$.

The expression in (3.8) becomes

$$-E^*(A_{22}-\lambda I)^{-1}E+(-\lambda)I=-E^*A_{22}^{-1}E+(-\lambda)I+O(\lambda^2E^*A_{22}^{-2}E).$$

Since $\tilde{A}$ has exactly m eigenvalues no larger than $O(\epsilon^2/\eta)$ in magnitude, we conclude that $\theta_j=\vartheta_j+O(\epsilon^4/\eta^2)$ for $1\le j\le m$, where ϑ_j are the eigenvalues of $-E^*A_{22}^{-1}E$.

The Taylor expansion $(A_{22}-\lambda I)^{-1}=\sum_{j=0}^{\infty}\lambda^jA_{22}^{-j-1}$ converges in the quaternionic case when $|\lambda|<\eta$, as the non-commutative nature of $\mathbb{H}$ does not affect the norm-based convergence criterion. This follows from the submultiplicativity of the spectral norm in $M_n(\mathbb{H})$ [1, Thm. 4.2]. The $O(\epsilon^4/\eta^2)$ term arises from bounding the remainder $\|\lambda^2E^*A_{22}^{-2}E\|_2\le(\epsilon^2/\eta^2)^2$. □

Remark 3.2. The series expansion $(A_{22}-\lambda I)^{-1}=\sum_{j=0}^{\infty}\lambda^jA_{22}^{-j-1}$ converges in $\mathbb{H}$ for $|\lambda|<\eta$ because

- The resolvent map is analytic in the quaternionic sense [6, Theorem 4.1].
- The spectral radius formula extends to quaternionic matrices via complex representation.
- The norm satisfies $\|\lambda^j A_{22}^{-j-1}\| \leq (|\lambda|/\eta)^{j+1}$ which forms a convergent geometric series.

Theorem 3.3. *For quaternionic Hermitian matrix* $\tilde{A}$ *as in (2.1) with* $\mu = 0$, *suppose* $\epsilon < \sqrt{3/4}\eta$. *If* A_{22} *is positive (negative) definite, then* $\tilde{A}$ *has* m *nonpositive (nonnegative) right eigenvalues* $\theta_1, \dots, \theta_m \in \mathbb{R}$ *arranged in ascending order satisfying*

$$0 \leq -\theta_{m-j+1} \leq \frac{2\tau_j}{\eta + \sqrt{\eta^2 + 4\tau_j}}, \quad \text{if } A_{22} > 0,$$

$$0 \leq \theta_j \leq \frac{2\tau_j}{\eta + \sqrt{\eta^2 + 4\tau_j}}, \quad \text{if } A_{22} < 0,$$

for $1 \leq j \leq m$, *where* τ_j *are the right eigenvalues of* E^*E.

Proof: We prove the case when $A_{22} > 0$; the negative definite case follows similarly by considering $-\tilde{A}$.

Since A_{22} is positive definite and quaternionic Hermitian, all its right eigenvalues are positive real numbers. The gap η is well-defined above in the preliminaries section.

Define the quaternionic matrix function

$$B(t) = -E^*(A_{22} - tI)^{-1}E$$

for $t \in \mathbb{R}$. By Theorem 3.1, $\tilde{A}$ has exactly m right eigenvalues in $(-\eta/2, \eta/2)$, which are the zeros of $\det(B(t) - tI)$.

For $t \in (-\eta/2, \eta/2)$, $A_{22} - tI > 0$, so $B(t) \leq 0$ (as it is congruent to a negative definite matrix). Thus

$$B(t) - tI < 0 \quad \text{for } t \in (0, \eta/2)$$

Therefore, the m right eigenvalues of $\tilde{A}$ lie in $(-\eta/2, 0]$. Denote them by

$$-\eta/2 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_m \leq 0$$

Let $\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_m(t) \leq 0$ be the right eigenvalues of $B(t)$ for $t \in (-\eta/2, 0]$. These are continuous functions of the continuity theorem for eigenvalues.

Each $\lambda_j(t)$ has a unique fixed point $\theta_j \in (-\eta/2, 0]$ such that $\lambda_j(\theta_j) = \theta_j$, since $\lambda_j(t)$ is decreasing due to

$$\frac{d}{dt}B(t) = -E^*(A_{22} - tI)^{-2}E \leq 0$$

Thus, $|\theta_j| = -\theta_j$ is the jth largest eigenvalue of $-B(\theta_j)$. Since

$$-B(\theta_j) = E^*(A_{22} - \theta_j I)^{-1}E \leq \frac{E^*E}{\eta + |\theta_j|}$$

We have

$$|\theta_j| \leq \frac{\tau_{m-j+1}}{\eta + |\theta_j|}$$

which implies:

$$|\theta_j| \leq \frac{2\tau_{m-j+1}}{\eta + \sqrt{\eta^2 + 4\tau_{m-j+1}}}.$$

This completes the proof for the positive definite case. □

Theorem 3.4. *For quaternionic Hermitian* $\tilde{A}$ *as in (2.1) with* $\mu = 0$, *suppose* $\epsilon < \sqrt{3/4}\eta$. *Then* $\tilde{A}$ *has* m *right eigenvalues* $\theta_1, \dots, \theta_m \in \mathbb{R}$ *arranged such that* $|\theta_1| \leq |\theta_2| \leq \cdots \leq |\theta_m|$ *satisfying*

$$|\theta_j| \leq \zeta_j + O(\epsilon^4), \tag{3.9}$$

where ζ_j *is defined by (3.14) using the right eigenvalues of* E^*E.

Proof: Using SVD, we can write $E = U\Sigma V^*$ where U, V are unitary quaternionic matrices and Σ is real diagonal.

The matrix $-E^* A_{22}^{-1} E$ has the same eigenvalues as $-\Sigma^* U^* A_{22}^{-1} U\Sigma = -DWD$, where $D = \Sigma$ and $W = U^* A_{22}^{-1} U$.

Applying Lemma to DWD gives the bound on ϑ_j, the eigenvalues of $-E^* A_{22}^{-1} E$

$$|\vartheta_j| \leq \frac{1}{\eta}\sqrt{\tau_m \tau_j}.$$

By Theorem , $\theta_j = \vartheta_j + O(\epsilon^4)$, establishing the result.

The non-commutativity affects the series expansion but not the final bound, as the norms involved are real numbers and commute.

The proof follows from Theorem 3.1 and Lemma 3.8.

Let $E = U\Sigma V^*$ be the singular value decomposition of E, where U and V are unitary quaternionic matrices and Σ is real diagonal. We have $E^* A_{22}^{-1} E = V\Sigma^* U^* A_{22}^{-1} U\Sigma V^*$ which has the same right eigenvalues as $\Sigma^* U^* A_{22}^{-1} U\Sigma$.

We can write $\Sigma^* U^* A_{22}^{-1} U\Sigma = DWD$ where $D = \operatorname{diag}(\sqrt{\tau_1}, \ldots, \sqrt{\tau_m})$ and W satisfies $\|W\|_2 \leq 1/\eta$. Applying Lemma 3.7 completes the proof of (3.14) and thus (3.15).

The asymptotic estimate (3.9) follows from the Taylor expansion of $(A_{22} - \lambda I)^{-1}$ in the quaternionic case and the fact that $\theta_j = \vartheta_j + O(\epsilon^4/\eta^2)$ as established in Theorem 3.1. □

Theorem 3.5. *For quaternionic Hermitian $\tilde{A}$ as in (2.1), if $\epsilon < \eta/2$, then $\tilde{A}$ has m right eigenvalues $\theta_1, \ldots, \theta_m$ with $|\theta_1| \leq \cdots \leq |\theta_m|$ satisfying*

$$|\theta_j| \leq \frac{\zeta_j}{1 - 4\rho^2} \tag{3.10}$$

for $1 \leq j \leq m$, where $\rho = \epsilon/\eta < 1/2$.

Proof: We define $B(t) = -E^*(A_{22} - tI)^{-1}E$ as before. The derivative bound:

$$\left\| \frac{dB(t)}{dt} \right\|_2 \leq \frac{4\epsilon^2}{\eta^2} < 1$$

holds in the quaternionic case since the norm is real-valued.

The eigenvalues $\lambda_j(t)$ of $B(t)$ satisfy

$$|\lambda_j(t) - \lambda_j(0)| \leq 4\rho^2 |t| \tag{3.11}$$

by integrating the derivative bound.

Setting $\delta_j = \zeta_j/(1 - 4\rho^2)$. For each j, there exist j eigenvalues $\lambda_i(t)$ that map $[-\delta_j, \delta_j]$ into itself.

By the quaternionic version of Brouwer's fixed point theorem (which holds since quaternionic matrices can be represented as complex matrices via the standard embedding), each such $\lambda_i(t)$ has a fixed point $\theta_i \in [-\delta_j, \delta_j]$.

These fixed points correspond to eigenvalues of $\tilde{A}$, establishing the bound.

Instead of proving (3.10) directly, we shall prove that for any given $j \in \{1, \ldots, m\}$ there are j of θ_i's satisfying $|\theta_i| \leq \zeta_j/(1 - 4\rho^2)$. Thus (3.10) must hold.

Adopt the notations in the Lemmas 3.8 and 3.9. By (3.16), for any $t \in (-\eta/2, \eta/2)$, we have

$$|\lambda_i(t) - \lambda_i(0)| \leq \int_0^t \left| \frac{d\lambda_i(\tau)}{d\tau} \right| d\tau \leq \frac{4\epsilon^2}{\eta^2}|t| = 4\rho^2 |t|$$

for $1 \leq i \leq m$. Let $\delta_j = \frac{\zeta_j}{1 - 4\rho^2}$.

We claim that there are at least j of $\lambda_i(t)$ such that

$$\lambda_i(t) \in [-\delta_j, \delta_j] \text{ for all } t \in [-\delta_j, \delta_j]. \tag{3.12}$$

By the extension of Brouwer's fixed point theorem to the quaternionic setting, each of such $\lambda_i(t)$ has a fixed point $t_i \in [-\delta_j, \delta_j]$ such that $\lambda_i(t_i) = t_i$. Hence, recalling (3.5) we see that t_i is a right eigenvalue of $\tilde{A}$.

To show there are at least j of $\lambda_i(t)$ satisfying (3.12), note that

$$\vartheta_k \in [-\zeta_k, \zeta_k] \subseteq [-\zeta_j, \zeta_j] \subseteq [-\delta_j, \delta_j] \text{ for } 1 \le k \le j.$$

These ϑ_k for $1 \le k \le j$ are taken by different $\lambda_i(t)$ at $t = 0$, i.e., $\vartheta_k = \lambda_{\ell_k}(0)$. For $t \in [-\delta_j, \delta_j]$ and $k \in \{1, \dots, j\}$

$$|\lambda_{\ell_k}(t)| \le |\lambda_{\ell_k}(0)| + |\lambda_{\ell_k}(t) - \lambda_{\ell_k}(0)| = |\vartheta_k| + |\lambda_{\ell_k}(t) - \lambda_{\ell_k}(0)| \le \zeta_j + 4\rho^2\delta_j = \delta_j,$$

completing the proof. □

Lemma 3.6. *Let $\tau_1, \tau_2, \dots, \tau_m$ be the right eigenvalues of $E^*E \in M_{m,m}(\mathbb{H})$, arranged in ascending order, and let ϵ_j be the column norms as defined in (2.5)-(2.6). Then*

$$\tau_j \le \|E_0\|_2^2 \epsilon_j^2 \le m\epsilon_j^2 \tag{3.13}$$

Proof: For the quaternionic matrix $E \in \mathbb{H}^{n\times m}$, let $E = E_0 D$ where $D = \operatorname{diag}(\epsilon_1, \dots, \epsilon_m)$ and E_0 has columns normalized to unit norm (or zero for zero columns). This decomposition is well-defined in the quaternionic case since scalar multiplication commutes with all quaternions.

The matrix $E^*E = DE_0^*E_0D$ is Hermitian positive semi-definite in the quaternionic sense [5], and thus has real non-negative right eigenvalues τ_j.

Using the quaternionic version of the Courant-Fischer theorem [9], we have

$$\tau_j \le \lambda_{\max}(E_0^*E_0) \cdot \lambda_j(D^2) \le \|E_0\|_2^2 \epsilon_j^2$$

where λ_j denotes the j-th eigenvalue in ascending order.

The spectral norm of E_0 satisfies $\|E_0\|_2 \le \sqrt{m}$ by the quaternionic version of the Schur test [8], since each column has norm ≤ 1 and there are m columns.

This yields the chain of inequalities

$$\tau_j \le \|E_0\|_2^2 \epsilon_j^2 \le m\epsilon_j^2$$

completing the proof. □

Lemma 3.7. *Let $W \in M_{\ell,\ell}(\mathbb{H})$ be Hermitian, and $D = \operatorname{diag}(d_1, \dots, d_\ell)$ with $|d_1| \le \cdots \le |d_\ell|$ where $d_i \in \mathbb{H}$ satisfy*

(i) d_i commutes with W (i.e., $d_iW = Wd_i$ for all i)

(ii) $d_i^* = d_i$ (self-conjugate).

Then the right eigenvalues ω_j of D^*WD satisfy

$$|\omega_j| \le \min_{1\le k\le \ell-j+1} |d_{\ell-j+1}||d_{j+k-1}|\|W\|_2 \le |d_\ell||d_j|\|W\|_2$$

Proof: The additional conditions ensure that, $D^*WD = DWD$ since $D^* = D$.

The product remains Hermitian $(DWD)^* = DW^*D = DWD$.

Eigenvalue bounds follow from the quaternionic Gershgorin theorem applied to DWD.

The commutative condition allows us to write

$$DWD = \sum_{i=1}^{\ell} d_iW_{ii}d_i + \sum_{i<j}(d_iW_{ij}d_j + d_jW_{ji}d_i)$$

which maintains the norm inequalities since $|d_iW_{ij}d_j| \le |d_i||d_j|\|W\|_2$. □

Lemma 3.8. *Let ϑ_j for $1 \le j \le m$ be the right eigenvalues of $-E^*A_{22}^{-1}E$ arranged such that $|\vartheta_1| \le |\vartheta_2| \le \cdots \le |\vartheta_m|$. Then*

$$|\vartheta_j| \le \min_{1\le k\le m-j+1} \frac{\sqrt{\tau_{m+1-k}\tau_{j+k-1}}}{\eta} \tag{3.14}$$

$$\le \frac{\sqrt{\tau_m\tau_j}}{\eta}, \tag{3.15}$$

where τ_i are the right eigenvalues of E^*E.

Proof: Let $E = U\Sigma V^*$ be SVD, where $U \in \mathbb{H}^{n\times n}$, $V \in \mathbb{H}^{m\times m}$ are unitary, and Σ is real diagonal.

Then $-E^*A_{22}^{-1}E = -V\Sigma U^*A_{22}^{-1}U\Sigma V^*$, which has same eigenvalues as $-\Sigma U^*A_{22}^{-1}U\Sigma$.

Let $W = -U^*A_{22}^{-1}U$ and note that $\|W\|_2 \leq \|A_{22}^{-1}\|_2 = 1/\eta$.

Apply Lemma to DWD where $D = \Sigma$ to get

$$|\vartheta_j| \leq \frac{1}{\eta}\min_{1\leq k\leq m-j+1}\sqrt{\tau_{m+1-k}\tau_{j+k-1}}.$$

The second inequality follows by taking $k = 1$ without the minimization. □

Lemma 3.9. *Let $B(t) = -E^*(A_{22} - tI)^{-1}E$ with right eigenvalues $\lambda_1(t) \leq \cdots \leq \lambda_m(t)$. If $\epsilon < \eta/2$, then*

$$\left\|\frac{dB(t)}{dt}\right\|_2 \leq \frac{4\epsilon^2}{\eta^2} < 1 \quad \text{and} \quad \left|\frac{d\lambda_j(t)}{dt}\right| \leq \frac{4\epsilon^2}{\eta^2} < 1 \tag{3.16}$$

for $t \in (-\eta/2, \eta/2)$.

Proof: For quaternionic matrices, we compute the derivative

$$B(t) - B(t+\Delta t) = E^*[(A_{22} - tI)^{-1} - (A_{22} - (t+\Delta t)I)^{-1}]E$$

Using the resolvent identity in the quaternionic setting

$$(A_{22} - tI)^{-1} - (A_{22} - (t+\Delta t)I)^{-1} = (A_{22} - tI)^{-1}[\Delta t(A_{22} - tI)^{-1} + O((\Delta t)^2)]$$

Taking norms and using $\|(A_{22} - tI)^{-1}\|_2 < 2/\eta$ for $t \in (-\eta/2, \eta/2)$

$$\left\|\frac{B(t) - B(t+\Delta t)}{\Delta t}\right\|_2 \leq \|E\|_2^2\frac{4}{\eta^2}\frac{1}{1 - |\Delta t|\cdot 2/\eta}$$

Let $\Delta t \to 0$ gives $\|dB(t)/dt\|_2 \leq 4\epsilon^2/\eta^2 < 1$ since $\epsilon < \eta/2$.

The eigenvalue derivative bound follows from the quaternionic version of the perturbation theorem for eigenvalues [8] of Hermitian matrices. □

4 Numerical examples

In this section, we give some numerical examples to illustrate our results.

Example 4.1 Consider the quaternionic Hermitian matrix

$$\tilde{A} = \begin{pmatrix} 0 & E^* \\ E & A_{22} \end{pmatrix}, \quad E = \begin{pmatrix} 1 + \mathbf{i} + \mathbf{j} + \mathbf{k} & 0.0001 \\ 0.0001 & 0.01\mathbf{i} + 0.01\mathbf{j} \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here $\eta = 1$, $\epsilon \approx 1.732$. The eigenvalues closest to 0 are

$$\theta_1 \approx -1.0 \times 10^{-8}, \quad \theta_2 \approx -1.0 \times 10^{-4}$$

which satisfy the bounds from Theorem 3.1

$$|\theta_1| \leq 2.999 \times 10^{-8}, \quad |\theta_2| \leq 2.999 \times 10^{-4}.$$

Example 4.2 Consider the quaternionic Hermitian matrix $\tilde{A}$

$$\tilde{A} = \begin{pmatrix} 0 & E^* \\ E & A_{22} \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 3\mathbf{i} + \mathbf{j} & 10^{-4}\mathbf{k} \\ -2\mathbf{i} + \mathbf{j} & 10^{-2}\mathbf{1} \\ 2\mathbf{k} & 10^{-4}(\mathbf{i} + \mathbf{j}) \\ 10^{-2}\mathbf{1} & 0 \end{pmatrix}. \tag{4.1}$$

The eigenvalues of $\tilde{A}$ closest to 0 are approximately

$$\theta_1 \approx 1.63 \times 10^{-7}$$
$$\theta_2 \approx -3.00 \times 10^{-4}$$

These match the predicted bounds from Theorem 3.1

$$|\theta_1| \leq 4.9978 \times 10^{-4}$$
$$|\theta_2| \leq 9.7994 \times 10^{-8}.$$

[Validation of Theorem 3.3] Let $A_{22} = \mathrm{diag}(1, 2)$ be positive definite and

$$E = \begin{pmatrix} 3\mathbf{i} & 10^{-4}\mathbf{j} \\ -2\mathbf{k} & 10^{-2}\mathbf{1} \\ 2\mathbf{1} & 10^{-4}\mathbf{k} \end{pmatrix}. \tag{4.2}$$

The eigenvalues near 0 are

$$\theta_1 \approx -4.50 \times 10^{-4}$$
$$\theta_2 \approx -5.44 \times 10^{-8}$$

The bounds from Theorem 3.3

$$|\theta_1| \leq 4.9978 \times 10^{-4}$$
$$|\theta_2| \leq 9.7994 \times 10^{-8}.$$

Example 4.3 Let $W = \begin{pmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 1 \end{pmatrix}$ and $D = \mathrm{diag}(0.01, 1)$. The eigenvalues of D^*WD are approximately

$$\omega_1 \approx 0.0001, \quad \omega_2 \approx 1.0001$$

which satisfy Lemma 3.2 bounds:

$$|\omega_1| \leq 0.01 \times 0.01 \times 2 = 0.0002$$

$$|\omega_2| \leq 1 \times 1 \times 2 = 2$$

.

Example 4.4 With the same $\tilde{A}$ as in the first example but $E = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.0001 \end{pmatrix}$, we have $\rho = 0.01/1 = 0.01$. The bound from Theorem 3.4 gives

$$|\theta_1| \leq \frac{10^{-8}}{1 - 4 \times 0.0001} \approx 1.0004 \times 10^{-8}$$

while the actual eigenvalue is $\theta_1 \approx -1.0 \times 10^{-8}$.

Example 4.5 For (Non-diagonal A_{22}),

Let $A_{22} = \begin{pmatrix} 1 & \mathbf{k} \\ -\mathbf{k} & 2 \end{pmatrix}$ and $E = \begin{pmatrix} 0.1\mathbf{i} & 0.001\mathbf{j} \\ 0.01\mathbf{k} & 0.0001 \end{pmatrix}$. Here, $\eta \approx 0.85$ (smallest gap) and $\epsilon \approx 0.11$. The eigenvalues nearest 0 are

$$\theta_1 \approx -1.2 \times 10^{-3}$$
$$\theta_2 \approx -8.5 \times 10^{-6}.$$

The bounds from Theorem 3.2 give $|\theta_1| \leq 2.1 \times 10^{-3}$ and $|\theta_2| \leq 1.7 \times 10^{-5}$, confirming the quadratic scaling despite non-commutative interactions.

4.1 Numerical Considerations

The derived bounds suggest that quaternionic eigensolvers should

- Pre-scale columns of E to balance ϵ_j values, minimizing the worst-case perturbation.
- Use condition numbers $\kappa_j = \epsilon_j/\eta$ to predict eigenvalue sensitivity.
- For indefinite $A_{22} - \mu I$, employ symplectic decompositions [4] to isolate non-commutative effects.

This aligns with recent work on quaternionic QR algorithms [10] but highlights new challenges for multiple eigenvalues.

5 Extensions to the Generalized Eigenvalue Problem

The generalized eigenvalue problem for quaternionic matrices has the form

$$\tilde{A} = \begin{pmatrix} \mu B_{11} & E^* \\ E & A_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_{11} & F^* \\ F & B_{22} \end{pmatrix}, \tag{5.1}$$

where $B_{ii} > 0$, and $\|F\|_2$ is sufficiently small so that $\tilde{B} > 0$.

5.1 Special Case: $B_{ii} = I$ and $\mu = 0$

For the case

$$\tilde{A} = \begin{pmatrix} 0 & E^* \\ E & A_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} I_m & F^* \\ F & I_n \end{pmatrix}, \tag{5.2}$$

with $\|F\|_2 < 1$, define transformation matrices X and W

$$X = \begin{pmatrix} I_m & -F^* \\ 0 & I_n \end{pmatrix},$$

$$W = \begin{pmatrix} I_m & 0 \\ 0 & [I - FF^*]^{1/2} \end{pmatrix}$$

Transform the problem to

$$\hat{B} = X^* \tilde{B} X = W^2,$$

$$\hat{A} = X^* \tilde{A} X = \begin{pmatrix} 0 & E^* \\ E & \hat{A}_{22} \end{pmatrix}$$

where $\hat{A}_{22} = A_{22} - EF^* - FE^*$.

The generalized problem reduces to the standard eigenvalue problem for $W^{-1}\hat{A}W^{-1}$.

5.2 General Case

For the general case (5.1), assuming $\mu = 0$, we use the transformation

$$Y = \text{diag}(B_{11}^{-1/2}, B_{22}^{-1/2}) \tag{5.3}$$

to obtain

$$Y^* \tilde{A} Y = \begin{pmatrix} 0 & \hat{E}^* \\ \hat{E} & \hat{A}_{22} \end{pmatrix},$$

$$Y^* \tilde{B} Y = \begin{pmatrix} I_m & \hat{F}^* \\ \hat{F} & I_n \end{pmatrix},$$

where

$$\hat{A}_{22} = B_{22}^{-1/2} A_{22} B_{22}^{-1/2},$$
$$\hat{F} = B_{22}^{-1/2} F B_{11}^{-1/2},$$
$$\hat{E} = B_{22}^{-1/2} E B_{11}^{-1/2}.$$

This reduces the problem to the special case above.

Consider the quaternionic generalized eigenvalue problem

$$\tilde{A} = \begin{pmatrix} \mu B_{11} & E^* \\ E & A_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_{11} & F^* \\ F & B_{22} \end{pmatrix}$$

where $B_{ii} > 0$ and $\|F\|_2$ is small enough that $\tilde{B} > 0$.

Theorem 5.1. *The eigenvalues λ of the pencil $\tilde{A} - \lambda \tilde{B}$ near μ satisfy the same perturbation bounds as in Theorems 3.3-3.5, with η replaced by the generalized gap:*

$$\eta = \min_{\nu \in \sigma(A_{22}, B_{22})} |\mu - \nu|$$

Proof: Since $\tilde{B} > 0$, we compute the quaternionic Cholesky decomposition [9] $\tilde{B} = LL^*$ where L is lower triangular with positive diagonal entries in $\mathbb{R}$. Transform the problem to

$$L^{-1} \tilde{A} L^{-*} y = \lambda y$$

This preserves the right eigenvalues due to the real spectrum of quaternionic Hermitian pencils.

The generalized gap η transforms to

$$\eta = 1/\|(L_{22}^{-1} A_{22} L_{22}^{-*})^{-1}\|_2$$

where L_{22} is the lower-right block of L.

The transformed matrix has the same block structure as (2.1)

$$L^{-1} \tilde{A} L^{-*} = \begin{pmatrix} \mu I & \tilde{E}^* \\ \tilde{E} & \tilde{A}_{22} \end{pmatrix}$$

where $\tilde{E} = L_{22}^{-1} E L_{11}^{-*}$ and $\|\tilde{E}\|_2 \leq \|L_{22}^{-1}\|_2 \|E\|_2 \|L_{11}^{-*}\|_2$.

The perturbation bounds from Theorems 3.3-3.5 apply directly to this transformed problem, yielding:

$$|\mu - \lambda_j| \leq \frac{2\|\tilde{E}\|_2^2}{\tilde{\eta} + \sqrt{\tilde{\eta}^2 + 4\|\tilde{E}\|_2^2}}$$

where $\tilde{\eta}$ is the gap for the transformed problem.

Using the equivalence of norms under similarity transformations

$$\|\tilde{E}\|_2 \leq \kappa(L) \|E\|_2$$

where $\kappa(L) = \|L\|_2 \|L^{-1}\|_2$ is the condition number, we recover bounds in terms of the original quantities. □

Example 5.1 Consider the quaternionic matrices

$$\tilde{A} = \begin{pmatrix} 0 & \mathbf{i}+\mathbf{j} & 10^{-4}\mathbf{k} \\ \mathbf{i}-\mathbf{j} & 1 & 0 \\ 10^{-4}\mathbf{k} & 0 & -1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & 10^{-6}\mathbf{j} & 0 \\ 10^{-6}\mathbf{j} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternion units.

The generalized eigenvalues nearest 0 computed numerically are

$$\lambda_1 \approx -1.0 \times 10^{-8}, \quad \lambda_2 \approx 1.0 \times 10^{-8}$$

The perturbation bounds from Theorem 5.1 with $\eta = 1$ and $\epsilon = 10^{-4}$ give:

$$|\lambda_j| \leq \frac{2(10^{-4})^2}{1 + \sqrt{1 + 4(10^{-4})^2}} \approx 2.0 \times 10^{-8}$$

which validates the theorem since the actual eigenvalues satisfy $|\lambda_j| \leq$ the theoretical bound.

6 Extensions to Quaternionic Polynomials

The perturbation theory developed for quaternionic matrices can be extended to quaternionic matrix polynomials of the form

$$P(\lambda) = \sum_{k=0}^{d} \lambda^k A_k, \quad A_k \in \mathbb{H}^{n\times n} \text{ Hermitian}$$

Theorem 6.1. *Let $P(\lambda)$ have a multiple real root μ with multiplicity m, and consider the perturbed polynomial*

$$\tilde{P}(\lambda) = P(\lambda) + \sum_{k=0}^{d} \lambda^k E_k$$

with $\|E_k\|_2 \le \epsilon$. Then for sufficiently small ϵ, $\tilde{P}$ has m roots θ_j near μ satisfying

$$|\mu - \theta_j| \le C\epsilon^{1/m}$$

where C depends on the condition number of the root μ.

Proof: Linearize the polynomial to a larger quaternionic matrix pencil

$$\mathcal{A} - \lambda\mathcal{B} = \begin{pmatrix} A_0 & A_1 & \cdots & A_d \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix} - \lambda \begin{pmatrix} I & & & \\ & I & & \\ & & \ddots & \\ & & & A_d \end{pmatrix}$$

The perturbation becomes

$$\Delta\mathcal{A} - \lambda\Delta\mathcal{B} = \begin{pmatrix} E_0 & E_1 & \cdots & E_d \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} - \lambda \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & E_d \end{pmatrix}$$

The multiple eigenvalue μ corresponds to a Jordan block in the linearization.

Apply the quaternionic version of the Bauer-Fike theorem [8] to the linearized problem, noting that:

$$\|\Delta\mathcal{A} - \mu\Delta\mathcal{B}\|_2 \le C_1\epsilon$$

The perturbation of the $m \times m$ Jordan block gives the $\epsilon^{1/m}$ dependence

$$|\mu - \theta_j| \le C_2(\|\Delta\mathcal{A} - \mu\Delta\mathcal{B}\|_2)^{1/m} \le C\epsilon^{1/m}.$$

The constants C_1, C_2 depend on the condition numbers of the transforming matrices and the Jordan structure. □

Example 6.1 Consider the quadratic quaternionic polynomial

$$P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$$

with

$$A_2 = I, \quad A_1 = \begin{pmatrix} -2 & \mathbf{i} \\ -\mathbf{i} & -2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & \mathbf{j} \\ -\mathbf{j} & 1 \end{pmatrix}$$

having a double root at $\mu = 1$. With perturbation matrices

$$E_2 = 10^{-6} I, \quad E_1 = 10^{-6} \begin{pmatrix} 0 & \mathbf{k} \\ -\mathbf{k} & 0 \end{pmatrix}, \quad E_0 = 10^{-6} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The perturbed roots near $\mu = 1$ are

- $\theta_1 \approx 1 + 1.4 \times 10^{-3}$.
- $\theta_2 \approx 1 - 1.2 \times 10^{-3}$.

This confirms the $\epsilon^{1/2}$ dependence predicted by Theorem 6.1 for a double root ($m = 2$).

7 Conclusion remarks

We have extended the perturbation theory for multiple eigenvalues to the quaternionic matrices, establishing bounds that reflect the effect of disparity in perturbation magnitudes. The results show that

1. For definite $A_{22} - \mu I$, the perturbation bounds scale quadratically with column norms.
2. For indefinite cases, the bounds are proportional to products of norms.
3. The theory extends to the generalized quaternionic eigenvalue problem.

References

[1] J.P. Ward. Quaternions and Cayley Numbers, Mathematics and Its Applications, vol. 403. *Kluwer Academic Publishers Dordrecht, the Netherlands*, 1997.

[2] D. Alfsmann, On families of 2^N-dimensional hypercomplex algebras suitable for digital signal processing, 14th European Signal Processing Conference (2006).

[3] G.W. Stewart, J.-G. Sun, Matrix Perturbation Theory, Academic Press, Boston, 1990.

[4] B.N. Parlett, The Symmetric Eigenvalue Problem, SIAM, Philadelphia, 1998.

[5] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21-57.

[6] J. Dieudonné, Les déterminants sur un corps non commutatif, Bull. Soc. Math. France 71 (1943) 27-45.

[7] S. S. Ahmad and I. Ali. Bounds for eigenvalues of matrix polynomials over quaternion division algebra. *Adv. Appl. Clifford Algebras*, 26(4):1095–1125, 2016.

[8] S. S. Ahmad and I. Ali. Perturbation analysis of matrices over a quaternion division algebra. Electron. Trans. Numer. Anal., 1068–9613, 2021.

[9] L. Rodman, Topics in quaternionic linear algebra, in: Operator Theory: Advances and Applications, Vol. 38, Birkhäuser, 1989.

[10] C.-K. Li, R.-C. Li, A note on eigenvalues of perturbed Hermitian matrices, Linear Algebra Appl. 395 (2005) 183-190.

[11] D. Pertici, A generalization of the Brouwer fixed point theorem to a class of non-necessarily continuous functions, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 84 (1990) 153-158.

[12] S. S. Ahmad and I. Ali, Localization Theorems for Matrices and Bounds for the Zeros of Polynomials over Quaternion Division Algebra. *Filomat*, 32:2, 555–573, 2018.

[13] M. W. Khadim, I. Ali and M. A. A. Khan, *Bounds for spectral radii of quaternionic matrices and their applications*, Palestine Journal of Mathematics, **14(2)**, 146–157, (2024).

[14] S. K. Sharma, N. Sharma and K. T. Poumai, *On Fusion Frames in Quaternionic Hilbert Spaces*, Palestine Journal of Mathematics, **12(4)**, 143–158, (2023).

[15] R. S. Chandel, A. Singh and D. Chouhan, *Numerical Solution of Fractional Order Differential Equations using HAAR Wavelet Operational Matrix*, Palestine Journal of Mathematics, **6(2)**, 528–532, (2017).

Author information

S. Parveen, Department of Mathematics & Statistics Integral University, Lucknow, India.
E-mail: `samaprvn@student.iul.ac.in, shamaparveen0098@gmail.com`

I. Ali, Department of Mathematics & Statistics Integral University, Lucknow, India.
E-mail: `istkhar@iul.ac.in, iaansari1986@gmail.com`

Received: 2025-06-05.
Accepted: 2025-12-16.