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Totally umbilical $GCR-$lightlike warped product submainfolds of an indefinite nearly Cosymplectic manifold

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Abstract. In an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$, we prove that there are no totally umbilical $GCR-$lightlike warped product submanifolds of the type $\Sigma_T \times_f \Sigma_\perp$ such that $\Sigma_\perp$ is tangent to the characteristic vector field ξ, where Σ_T and $\Sigma_\perp$ are invariant and anti-invariant submanifolds in $\overline{\Sigma}$, respectively. The existence of totally umbilical $GCR-$lightlike warped product submanifolds of the type $\Sigma_T \times_f \Sigma_\perp$ is also established by presenting a characterization theorem involving the extrinsic invariants in $\overline{\Sigma}$ when Σ_T is tangent to the characteristic vector field ξ.

1 Introduction

The concept of warped product manifolds is among the present fascinating study topics in semi-Riemannian geometry. Bishop-O'Neill [1] initiated the study by considering the warped product of non-degenerate type. The research intensity in this topic increased significantly after O'Neill [2] presented its numerous applications to mathematical physics, especially in the theory of general relativity and black holes. Since then, the study gained momentum, and several researchers studied the warped product manifolds and their submanifolds in different settings (see [3–6]). On the other side, a lightlike (or null) counterpart of non-degenerate warped theory, however, had to wait for Duggal [7–9] and Sahin [10]. Meanwhile, Calin, in [11], proved that the structure vector field ξ belongs to the screen distribution when it is tangent to the submanifold. Following this notion, researchers in [12,13] and [14] derived fundamental properties of generalized Cauchy Riemann $GCR-$lightlike submanifolds along with warped aspects in an indefinite almost contact or complex setting by focusing on the structure vector field ξ. The widespread belief about this notion has led to its use in numerous recent studies concerning lightlike warped product (sub-)manifolds (for example, [15–21]). Motivated by the above literature and applications, we here present the study of GCR-lightlike warped product submanifolds along with totally umbilical aspects in indefinite nearly Cosymplectic manifolds.
The article starts with Section 2, which provides an overview of the fundamentals of indefinite nearly Cosymplectic manifolds and lightlike submanifolds. Researchers can find additional information in [7]. In Section 3, we review basics concerning $GCR-$lightlike submanifolds and provide several derived results through lemmas and theorems for future reference. In Section 4, the present research proves that totally umbilical $GCR-$lightlike warped product submanifolds of the form $\Sigma_T \times_f \Sigma_\perp$ cannot exist when $\Sigma_\perp$ is a tangent to the characteristic vector field ξ with invariant Σ_T and anti-invariant $\Sigma_\perp$ submanifolds in $\overline{\Sigma}$. Finally, we give a characterization theorem that shows the existence of totally umbilical $GCR-$lightlike warped product submanifolds of the type $\Sigma_T \times_f \Sigma_\perp$ under the condition that Σ_T is tangent to the characteristic vector field ξ concerning the extrinsic invariant, i.e., the shape operator and warping function in $\overline{\Sigma}$.

2 Preliminaries

Let $\overline{\Sigma}$ be a $(2m+1)-$dimensional semi-Riemannian manifold; then $\overline{\Sigma}$ is said to be an indefinite almost contact structure [22] if there exists a triplet (ϕ, ξ, η) satisfying

$$\phi^2(E) = -E + \eta(E)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

for any $E \in \mathfrak{X}(\overline{\Sigma})$. Whereas ϕ indicates a tensor field of type $(1,1)$, I is the identity map, ξ is a vector field, η is a $1-$form, and $\mathfrak{X}(\overline{\Sigma})$ is the Lie algebra of vector fields on $\overline{\Sigma}$. As a direct consequence of (2.1), we deduce that $rank(\phi) = 2m, \quad \phi\xi = 0, \quad \eta \circ \phi = 0$. If we equip a semi-Riemannian metric $\overline{g}$ with (ϕ, ξ, η) such that

$$\overline{g}(\phi E_1, \phi E_2) = \overline{g}(E_1, E_2) - \eta(E_1)\eta(E_2), \tag{2.2}$$

then for any $E_1, E_2 \in \mathfrak{X}(\overline{\Sigma})$, $(\phi, \xi, \eta, \overline{g})$ is termed as an indefinite almost contact metric structure, and $\overline{\Sigma}$ endowed with $(\phi, \xi, \eta, \overline{g})$ is called an indefinite almost contact metric manifold symbolized as $\overline{\Sigma}(\phi, \xi, \eta, \overline{g})$ (see [22]). From (2.1) and (2.2), it is obvious that the η is metrically equivalent to the vector field ξ, i.e., $\eta(E) = \overline{g}(E, \xi)$ and $\overline{g}(\xi, \xi) = 1$. In addition,

$$\overline{g}(\phi E_1, E_2) = \overline{g}(E_1, \phi E_2). \tag{2.3}$$

Definition 2.1. [23] $\overline{\Sigma}(\phi, \xi, \eta, \overline{g})$ is called *indefinite nearly Cosymplectic* if $(\overline{\nabla}_E \phi)E = 0$, for any $E \in \mathfrak{X}(\overline{\Sigma})$.

Let (Σ, g) be a submainfold of a $(2m+1)-$dimensional semi-Riemannian manifold $(\overline{\Sigma}, \overline{g})$ of index $i \in (0, 2m+1)$, where g is the induced metric of $\overline{g}$ on Σ, i.e., $g = \overline{g}_{\Sigma}$. Then Σ is a lightlike (degenerate) submainfold of $\overline{\Sigma}$ [7] if g is of constant rank n and the normal bundle $\mathfrak{X}(\Sigma)^{\perp}$ is a subbundle of the tangent bundle of Σ that coincides with the radical distribution $Rad(\mathfrak{X}(\Sigma)) = \{\zeta \in \mathfrak{X}(\Sigma) : g(\zeta, E,) = 0, \forall\, E \in \mathfrak{X}(\Sigma)\}$. Additionally, there exists a non-degenerate complementary vector bundle called a screen distribution denoted by $S(\mathfrak{X}(\Sigma))$ of $Rad(\mathfrak{X}(\Sigma))$ in $\mathfrak{X}(\Sigma)$ of Σ such that

$$\mathfrak{X}(\Sigma) = Rad(\mathfrak{X}(\Sigma)) \perp S(\mathfrak{X}(\Sigma)).$$

Now consider a screen transversal vector bundle $S(\mathfrak{X}(\Sigma)^{\perp})$, the complementary vector bundle of $Rad(\mathfrak{X}(\Sigma))$ in $\mathfrak{X}(\Sigma)^{\perp}$. Since for any local basis $\{E_i\}$ of $Rad(\mathfrak{X}(\Sigma))$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(\mathfrak{X}(\Sigma)^{\perp})$ in Σ such that $\overline{g}(E_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$. Hence, there exists a lightlike transversal vector bundle $ltr(\mathfrak{X}(\Sigma))$ locally spanned by $\{N_i\}$. Let $tr(\mathfrak{X}(\Sigma))$ be a complementary (but not orthogonal) vector bundle to $\mathfrak{X}(\Sigma)$ in $T\overline{\Sigma}|_{\Sigma}$, then

$$tr(\mathfrak{X}(\Sigma)) = ltr(\mathfrak{X}(\Sigma)) \perp S(\mathfrak{X}(\Sigma)^{\perp}), \tag{2.4}$$

$$T\overline{\Sigma}|_{\Sigma} = \mathfrak{X}(\Sigma) \oplus tr(\mathfrak{X}(\Sigma)),$$

$$T\overline{\Sigma}|_{\Sigma} = S(\mathfrak{X}(\Sigma)) \perp S(\mathfrak{X}(\Sigma)^{\perp}) \perp \{Rad(\mathfrak{X}(\Sigma)) \oplus ltr(\mathfrak{X}(\Sigma))\}, \tag{2.5}$$

where $\oplus$ denotes the non-orthogonal direct sum.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{\Sigma}$; then the Gauss and Weingarten formulas are

$$\overline{\nabla}_{E_1} E_2 = \nabla_{E_1} E_2 + h(E_1, E_2), \tag{2.6}$$

$$\overline{\nabla}_{E_1} \zeta = -A_\zeta E_1 + \nabla^t_{E_1} \zeta, \tag{2.7}$$

for any $E_1, E_2 \in \mathfrak{X}(\Sigma)$ and $\zeta \in tr(\mathfrak{X}(\Sigma))$, where $\{\nabla_{E_1} E_2, A_\zeta E_1\}$ and $\{h(E_1, E_2), \nabla^t_{E_1}\zeta\}$ belong to $\mathfrak{X}(\Sigma)$ and $tr(\mathfrak{X}(\Sigma))$, respectively. Here ∇ is the induced linear connection on Σ, ∇^t is the linear connection on the vector bundle $tr(\mathfrak{X}(\Sigma))$, h is the second fundamental form on $\mathfrak{X}(\Sigma)$ with values in $tr(\mathfrak{X}(\Sigma))$, and A_ζ is the shape operator, which is a linear endomorphism of $\mathfrak{X}(\Sigma)$. In view of (2.4), let us consider the projection morphisms $\mathcal{L}$ and $\mathcal{S}$ of $tr(\mathfrak{X}(\Sigma))$ on $ltr(\mathfrak{X}(\Sigma))$ and $S(\mathfrak{X}(\Sigma^{\perp}))$, respectively. Then (2.6) and (2.7) can be written as

$$\overline{\nabla}_{E_1} E_2 = \nabla_{E_1} E_2 + h^l(E_1, E_2) + h^s(E_1, E_2), \tag{2.8}$$

$$\overline{\nabla}_{E_1} \zeta = -A_\zeta E_1 + \mathfrak{D}^l_{E_1}\zeta + \mathfrak{D}^s_{E_1}\zeta, \tag{2.9}$$

where, $h^l(E_1, E_2) = \mathcal{L}(h(E_1, E_2))$, $h^s(E_1, E_2) = \mathcal{S}(h(E_1, E_2))$, $\mathfrak{D}^l_{E_1}\zeta = \mathcal{L}(\nabla^{\perp}_{E_1}\zeta)$, and $\mathfrak{D}^s_{E_1}\zeta = \mathcal{S}(\nabla^{\perp}_{E_1}\zeta)$. Since, h^l and h^s are $ltr(\mathfrak{X}(\Sigma))$ and $S(\mathfrak{X}(\Sigma)^{\perp}))-$valued, therefore h^l and h^s are called the lightlike second fundamental form and the screen second fundamental form on Σ, respectively. Moreover, we have

$$\overline{\nabla}_{E_1}\zeta_1 = -A_{\zeta_1}E_1 + \nabla^l_{E_1}\zeta_1 + \mathfrak{D}^s(E_1, \zeta_1), \tag{2.10}$$

$$\overline{\nabla}_{E_1}\zeta_2 = -A_{\zeta_2}E_1 + \nabla^s_{E_1}\zeta_2 + \mathfrak{D}^l(E_1, \zeta_2), \tag{2.11}$$

for any $E_1 \in \mathfrak{X}(\Sigma)$, $\zeta_1 \in ltr(\mathfrak{X}(\Sigma))$, and $\zeta_2 \in S(\mathfrak{X}(\Sigma^{\perp}))$. Further, using (2.4), (2.5), and (2.8)-(2.11), we obtain

$$\begin{aligned} \overline{g}(h^s(E_1, E_2), \zeta_2) + \overline{g}(E_2, \mathfrak{D}^l(E_1, \zeta_2)) &= g(A_{\zeta_2}E_1, E_2), \\ \overline{g}(h^l(E_1, E_2), E) + \overline{g}(E_2, h^l(E_1, E)) &= -g(E_2, \nabla_{E_1}E), \\ \overline{g}(A_{\zeta_1}E_1, \zeta_1') + \overline{g}(\zeta_1, A_{\zeta_1'}E_1) &= 0, \end{aligned} \tag{2.12}$$

for any $E_1, E_2 \in \mathfrak{X}(\Sigma)$, $\zeta_2 \in S(\mathfrak{X}(\Sigma^{\perp}))$, $E \in Rad(\mathfrak{X}(\Sigma))$ and $\zeta_1, \zeta_1' \in ltr(\mathfrak{X}(\Sigma))$.

3 Totally Umbilical $GCR-$Lightlike Submanifold

Definition 3.1. [13] Let $(\Sigma, g, S(\mathfrak{X}(\Sigma)))$ be a real lightlike submanifold of an almost contact metric manifold $(\overline{\Sigma}, \overline{g})$ such that the characteristic vector field ξ is tangent to Σ. Then Σ is called a $GCR-$lightlike submanifold of $\overline{\Sigma}$ if it satisfies the following conditions:

(i) There exist two subbundles $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of $Rad(\mathfrak{X}(\Sigma))$ such that

$$Rad(\mathfrak{X}(\Sigma)) = \mathfrak{D}_1 \oplus \mathfrak{D}_2, \text{ with } \phi(\mathfrak{D}_1) = \mathfrak{D}_1,\ \phi(\mathfrak{D}_2) \subset S(\mathfrak{X}(\Sigma)).$$

(ii) There exist two subbundles $\mathfrak{D}_0$ and $\overline{\mathfrak{D}}$ of $S(\mathfrak{X}(\Sigma))$ such that

$$S(\mathfrak{X}(\Sigma)) = \{\phi(\mathfrak{D}_2) \oplus \overline{\mathfrak{D}}\} \perp \mathfrak{D}_0 \perp \{\xi\}, \text{ with } \phi(\overline{\mathfrak{D}}) = \mathcal{L} \perp \mathcal{S},$$

where $\mathfrak{D}_0$ is an invariant non-degenerate distribution on Σ, $\{\xi\}$ is a one-dimensional distribution spanned by ξ, and $\mathcal{L}$, $\mathcal{S}$ are the vector subbundles of $ltr(\mathfrak{X}(\Sigma))$, $S(\mathfrak{X}(\Sigma)^{\perp})$, respectively.

Hence, the tangent bundle $\mathfrak{X}(\Sigma)$ of Σ is decomposed as

$$\mathfrak{X}(\Sigma) = \mathfrak{D} \oplus \overline{\mathfrak{D}} \oplus \{\xi\}, \text{where} \quad \mathfrak{D} = Rad(\mathfrak{X}(\Sigma)) \oplus \mathfrak{D}_0 \oplus \phi(\mathfrak{D}_2). \tag{3.1}$$

Let $\mathcal{Q}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ be the projection morphisms on $\mathfrak{D}$, $\phi\mathcal{S}$ and $\phi\mathcal{L}$, respectively, then for any $E_1 \in \mathfrak{X}(\Sigma)$, we can write $E_1 = \mathcal{Q}E_1 + \xi + \mathcal{P}_1E_1 + \mathcal{P}_2E_1$, or

$$E_1 = \mathcal{Q}E_1 + \xi + \mathcal{P}E_1, \tag{3.2}$$

where $\mathcal{P}$ is the projection morphism on $\overline{\mathfrak{D}}$. On applying ϕ to (3.2), we obtain

$$\phi E_1 = \mu E_1 + \nu\mathcal{P}_1E_1 + \nu\mathcal{P}_2E_1, \tag{3.3}$$

where $\mu E_1 \in \mathfrak{X}(\mathfrak{D})$, $\nu\mathcal{P}_1E_1 \in \mathfrak{X}(\mathcal{S}) \subset S(\mathfrak{X}(\Sigma^{\perp}))$ and $\nu\mathcal{P}_2E_1 \in \mathfrak{X}(\mathcal{L}) \subset ltr(\mathfrak{X}(\Sigma))$, therefore, we can write (3.3), as

$$\phi E_1 = \mu E_1 + \nu E_1, \tag{3.4}$$

where μE_1 and νE_1 are the tangential and the transversal components of ϕE_1, respectively. Similarly, for any $\zeta \in tr(\mathfrak{X}(\Sigma))$, we have

$$\phi\zeta = B\zeta + C\zeta, \tag{3.5}$$

where $B\zeta$ and $C\zeta$ are the sections of $\mathfrak{X}(\Sigma)$ and $tr(\mathfrak{X}(\Sigma))$, respectively.
Further, consider $\overline{\Sigma}$ be an indefinite nearly Cosymplectic manifold; then, on differentiating (3.3) and using (2.8)-(2.11) and (3.5), we derive

$$\mathfrak{D}^l(E_1, \nu\mathcal{P}_1E_2) = -\nabla^l_{E_1}\nu\mathcal{P}_2E_2 + \nu\mathcal{P}_2\nabla_{E_1}E_2 - h^l(E_1, \mu E_2) + Ch^l(E_1, E_2), \tag{3.6}$$

$$\mathfrak{D}^s(E_1, \nu\mathcal{P}_2E_2) = -\nabla^s_{E_1}\nu\mathcal{P}_1E_2 + \nu\mathcal{P}_1\nabla_{E_1}E_2 - h^s(E_1, \mu E_2) + Ch^s(E_1, E_2), \tag{3.7}$$

for all $E_1, E_2 \in \mathfrak{X}(\Sigma)$.
Next, we prove some basic preparatory results for later use.

Lemma 3.2. *Let Σ be a $GCR-$lightlike submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$ then for any $E_1, E_2 \in \mathfrak{X}(\Sigma)$ and $\zeta \in tr(\mathfrak{X}(\Sigma))$, we have the following identities:*

$$\begin{cases} (\nabla_{E_1}\mu)\,E_2 = A_{\nu E_2}E_1 + Bh(E_1, E_2), & \left(\nabla^t_{E_1}\nu\right)E_2 = Ch(E_1, E_2) - h(E_1, \mu E_2), \\ (\nabla_{E_1}\mu)E_2 = \nabla_{E_1}\mu E_2 - \mu\nabla_{E_1}E_2, & (\nabla^t_{E_1}\nu)E_2 = \nabla^t_{E_1}\nu E_2 - \nu\nabla_{E_1}E_2, \\ (\nabla_{E_1}B)\,\zeta = A_{C\zeta}E_1 - \mu A_\zeta E_1, & \left(\nabla^t_{E_1}C\right)\zeta = -\nu A_\zeta E_1 - h(E_1, B\zeta), \\ (\nabla_{E_1}B)\zeta = \nabla_{E_1}B\zeta - B\nabla^t_{E_1}\zeta, & (\nabla^t_{E_1}C)\zeta = \nabla^t_{E_1}C\zeta - C\nabla^t_{E_1}\zeta. \end{cases} \tag{3.8}$$

Proof. The proof follows immediately from the structure equation of the indefinite nearly Cosymplectic manifolds. □

Next, we examine the conditions for the distributions involved in the definition of a $GCR-$lightlike submanifold Σ of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$ to be integrable and totally geodesic.

Lemma 3.3. *If Σ is a $GCR-$lightlike submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$. Then the distribution $\mathfrak{D} \oplus \{\xi\}$ is integrable if and only if the second fundamental form satisfies $h(E_1, \mu E_2) = h(\mu E_1, E_2)$, for any $E_1, E_2 \in \mathfrak{D} \oplus \{\xi\}$.*

Proof. The proof of the Lemma can easily be obtained using (3.6) and (3.7). □

Lemma 3.4. *If Σ is a $GCR-$lightlike submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$. Then the distribution $\overline{\mathfrak{D}}$ is integrable if and only if the shape operator satisfies $A_{\nu E_3}E_4 = A_{\nu E_4}E_3$, for any $E_3, E_4 \in \overline{\mathfrak{D}}$.*

Proof. The proof follows directly using (3.8). □

Lemma 3.5. *If Σ is a $GCR-$lightlike submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$. Then the distribution $\mathfrak{D} \oplus \{\xi\}$ defines a totally geodesic foliation on Σ if and only if $Bh(E_1, \phi E_2) = 0$, for any $E_1, E_2 \in \mathfrak{D} \oplus \{\xi\}$.*

Proof. Given $\overline{\Sigma}$ an indefinite nearly Cosymplectic manifold, then using (2.3) and (2.8), we derive

$$g(\nabla_{E_1}E_2, \phi E) = -\overline{g}(\overline{\nabla}_{E_1}\phi E_2, E) = -\overline{g}(h^l(E_1, \phi E_2), E), \tag{3.9}$$

$$g(\nabla_{E_1}E_2, \phi E_3) = -\overline{g}(\overline{\nabla}_{E_1}\phi E_2, E_3) = -\overline{g}(h^s(E_1, \phi E_2), E_3), \tag{3.10}$$

for any $E_1, E_2 \in \mathfrak{D} \oplus \{\xi\}$, $E_3 \in \mathfrak{X}(\mathcal{S})$ and $E \in \mathfrak{D}_2$. Thus, by virtue of (3.9) and (3.10), the distribution $\mathfrak{D} \oplus \{\xi\}$ defines a totally geodesic foliation on Σ if and only if $h^l(E_1, \phi E_2)$ and $h^s(E_1, \phi E_2)$ has no components in $\mathcal{L}$ and $\mathcal{S}$, respectively. Hence, using (3.5), the proof is complete. □

Theorem 3.6. *Let the distribution $\mathfrak{D} \oplus \{\xi\}$ define a totally geodesic foliation in a $GCR-$lightlike submanifold Σ of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$. Then the distribution $\mathfrak{D} \oplus \{\xi\}$ is always integrable.*

Proof. The proof of the Theorem can be obtained from (3.8) of the Lemma 3.2, along with the fact that h is symmetric and the Lemma 3.3. □

Definition 3.7. [24] A lightlike submanifold Σ immersed in an almost contact metric manifold $(\overline{M}, \overline{g})$ is said to be totally umbilical in $\overline{\Sigma}$ if there is a smooth transversal vector field $\mathcal{H} \in tr(\mathfrak{X}(\Sigma))$ on Σ, called the transversal curvature vector field of Σ, such that

$$h(E_1, E_2) = \mathcal{H}\overline{g}(E_1, E_2), \tag{3.11}$$

for any $E_1, E_2 \in \mathfrak{X}(\Sigma)$. Moreover, it is not hard to see from (2.8), (2.12) and (3.11) that Σ is totally umbilical if and only if on each coordinate neighborhood $\mathcal{U}$, there exist smooth vector fields $\mathcal{H}^l \in ltr(\mathfrak{X}(\Sigma))$ and $\mathcal{H}^s \in S(\mathfrak{X}(\Sigma^\perp))$ such that

$$h^l(E_1, E_2) = \mathcal{H}^l\overline{g}(E_1, E_2), \quad h^s(E_1, E_2) = \mathcal{H}^s\overline{g}(E_1, E_2), \quad \mathfrak{D}^l(E_1, \zeta_2) = 0, \tag{3.12}$$

for all $\zeta_2 \in S(\mathfrak{X}(\Sigma^\perp))$.

Theorem 3.8. *If Σ is a totally umbilical $GCR-$lightlike submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$, then the distribution $\overline{\mathfrak{D}} \oplus \{\xi\}$ defines a totally geodesic foliation in Σ.*

Proof. From (3.8), it follows that

$$-\mu\nabla_{E_3}E_4 = A_{\nu E_4}E_3 + Bh(E_3, E_4),$$

for any $E_3, E_4 \in \overline{\mathfrak{D}}$. On taking inner product of above equation with $E_1 \in \mathfrak{D}_0$ and using the fact that Σ is a totally umbilical, we get

$$\begin{aligned} -g(\mu\nabla_{E_3}E_4, E_1) &= g(A_{\nu E_4}E_3, E_1) + g(Bh(E_3, E_4), E_1) \\ &= -\overline{g}(\overline{\nabla}_{E_3}\nu E_4, E_1) \\ &= \overline{g}(\nu E_4, \nabla_{E_3}E_1) + \overline{g}(\nu E_4, h^s(E_1, E_3)) \\ &= -\overline{g}(E_4, \nu\nabla_{E_3}E_1). \end{aligned} \tag{3.13}$$

Since Σ is a totally umbilical then from (3.8), we deduce $\nu\nabla_{E_3}E_1 = \mathcal{H}g(E_3, \mu E_1) - C\mathcal{H}g(E_3, E_1)$ = 0, using this fact in (3.13), we get $g(\mu\nabla_{E_3}E_4, E_1) = 0$, then the non-degeneracy of the distribution $\mathfrak{D}_0$ implies $\mu\nabla_{E_3}E_4 = 0$, that is, $\nabla_{E_3}E_4 \in \overline{\mathfrak{D}} \oplus \{\xi\}$, for any $E_3, E_4 \in \overline{\mathfrak{D}} \oplus \{\xi\}$. This completes the proof of the lemma. □

Furthermore, on using $\mu\nabla_{E_3}E_4 = 0$, for any $E_3, E_4 \in (\overline{\mathfrak{D}})$ in (3.8), we get $A_{\nu E_4}E_3 = -Bh(E_3, E_4)$. Then, using the symmetric property of the second fundamental form h, we obtain the condition of the Lemma 3.4. In addition to this, we deduce the following result:

Theorem 3.9. *Let Σ be a totally umbilical $GCR-$lightlike submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$. Then the distribution $\overline{\mathfrak{D}}$ is always integrable.*

4 Totally Umbilical $GCR-$Lightlike Warped Product Submanifolds

The lightlike warped product submanifold [10] $B \times_f F$ is a degenerate submanifold such that the base B is an $r-$dimensional totally lightlike submanifold, the fiber F is a $d-$dimensional semi-Riemannian submanifold of $\overline{\Sigma}$, and the warping function f is a smooth positive function on B endowed with a degenerate warped metric

$$g(E_1, E_2) = g_B\left(\pi_*(E_1), \pi_*(E_2)\right) + (f \circ \pi)^2 g_F\left(\sigma_*(E_1), \sigma_*(E_2)\right),$$

for all $E_1, E_2 \in \mathfrak{X}(\Sigma)$, where '$*$' stands for the derivation map, π and σ are the natural projections of $B \times F$ onto B and F, respectively. The lightlike warped product submanifold Σ is said to be proper if $B \neq \{0\}$, $F \neq \{0\}$, and f is a non-constant smooth function on B.

Proposition 4.1. *Let $\Sigma = B \times_f F$ be a warped product manifold then for $E_1, E_2 \in \mathfrak{X}(B)$ and $E_3, E_4 \in \mathfrak{X}(F)$, we have*

(i) $\nabla_{E_1}E_2 \in \mathfrak{X}(TB)$,

(ii) $\nabla_{E_1}E_3 = \nabla_{E_3}E_1 = \left(\frac{E_1 f}{f}\right)E_3$,

(iii) $\nabla_{E_3} E_4 = \frac{-g(E_3, E_4)}{f} \nabla f$,

where ∇ *denotes the Levi-Civita connection on* Σ *and* ∇f *is the gradient of* f *defined by* $g(\nabla f, E_1) = E_1 f$.

From (3.1), we know that the tangent bundle $\mathfrak{X}(\Sigma)$ of a $GCR-$lightlike submanifold Σ of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$ is decomposed into an invariant distribution $\mathfrak{D}$ and an anti-invariant distribution $\overline{\mathfrak{D}}$. Thus, we say a warped product $\Sigma = \Sigma_T \times_f \Sigma_\perp$ as a $GCR-$lightlike warped product submanifold of $\overline{\Sigma}$ where Σ_T is an invariant submanifold and $\Sigma_\perp$ is an anti-invariant submanifold $\overline{\Sigma}$.

Definition 4.2. [16] Let Σ be a $GCR-$lightlike submanifold of an indefinite almost contact metric manifold $\overline{\Sigma}$ such that both the distributions $\mathfrak{D} \oplus \{\xi\}$ and $\bar{\mathfrak{D}}$ (or $\bar{\mathfrak{D}} \oplus \{\xi\}$ and $\mathfrak{D}$) define totally geodesic foliations in Σ. Then Σ is said to be a $GCR-$lightlike product submanifold in $\overline{\Sigma}$.

Next, we investigate the existence or nonexistence of a $GCR-$lightlike warped product submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$.

Theorem 4.3. *Let* Σ *be a totally umbilical* $GCR-$*lightlike submanifold of an indefinite nearly Cosymplectic manifold* $\overline{\Sigma}$ *such that the distribution* $\phi\mathfrak{D}_2$ *is parallel with respect to the induced connection* ∇. *Then there does not exist a* $GCR-$*lightlike warped product submanifold* $\Sigma = \Sigma_T \times_f \Sigma_\perp$ *of* $\overline{\Sigma}$ *other than a* $GCR-$*lightlike product submanifold in* $\overline{\Sigma}$, *when* $\Sigma_\perp$ *is tangent to the characteristic vector field* ξ.

Proof. Denote h_T be the second fundamental form of Σ_T in Σ; then using the Gauss formula, for $E_1, E_2 \in \mathfrak{X}(\Sigma_T)$ and $E_3 \in \mathfrak{X}(\phi\mathcal{S}) \subset \mathfrak{X}(\Sigma_\perp \oplus \{\xi\})$, we have

$$g(h_T(E_1, E_2), E_3) = -\overline{g}(\overline{\nabla}_{E_1} E_3, E_2) = -g(\nabla_{E_1} E_3, E_2) - \overline{g}(h^l(E_1, E_3), E_2).$$

Further, on using the fact that Σ is a totally umbilical $GCR-$lightlike submanifold in $\overline{\Sigma}$, that is, $\overline{g}(h^l(E_1, E_3), E_2) = g(E_1, E_3)\overline{g}(\mathcal{H}^l, E_2) = 0$ and also using Proposition 4.1(ii), we obtain

$$g(h_T(E_1, E_2), E_3) = -g(\nabla_{E_1} E_3, E_2) = -E_1 \ln f g(E_3, E_2) = 0. \tag{4.1}$$

Next, denote $\overline{h}_T$ as the second fundamental form of Σ_T in $\overline{\Sigma}$, then

$$\overline{h}_T(E_1, E_2) = h_T(E_1, E_2) + h^l(E_1, E_2) + h^s(E_1, E_2). \tag{4.2}$$

Take the inner product of above expression both sides with E_3 and using (4.1), we obtain

$$\overline{g}(\overline{h}_T(E_1, E_2), E_3) = 0. \tag{4.3}$$

Now, by virtue of the fact that Σ_T is an invariant submanifold in $\overline{\Sigma}$, we have

$$\overline{h}_T(E_1, \phi E_2) = \overline{h}_T(\phi E_1, E_2) = \phi \overline{h}_T(E_1, E_2), \tag{4.4}$$

hence, from (4.2)–(4.4), we achieve

$$\begin{aligned} \overline{g}(h(E_1, \phi E_2), \phi E_3) &= \overline{g}(\overline{h}_T(E_1, \phi E_2), \phi E_3) - g(h_T(E_1, \phi E_2), \phi E_3) \\ &= -\overline{g}(\overline{h}_T(E_1, E_2), E_3) \\ &= 0, \end{aligned}$$

since $E_3 \in \mathfrak{X}(\phi\mathcal{S})$, then the above expression implies that $h^s(E_1, \phi E_2)$ has no components in $\mathcal{S}$, therefore from (3.5), we have $Bh^s(E_1, \phi E_2) = 0$, for any $E_1, E_2 \in \mathfrak{X}(\mathfrak{X}(\Sigma_T))$.

Next, let $E \in \mathfrak{X}(\mathfrak{D}_2)$ then using the definition of $\overline{\Sigma}$ and the Gauss formula, we have

$$\overline{g}(h^l(E_1, \phi E_2), E) = g(E_2, \nabla_{E_1} \phi E) + \overline{g}(E_2, h^l(E_1, \phi E)),$$

as Σ is totally umbilical and the distribution $\phi\mathfrak{D}_2$ is parallel with respect to ∇, therefore last expression gives $\overline{g}(h^l(E_1, \phi E_2), E) = 0$, since $E \in \mathfrak{X}(\mathfrak{D}_2)$, therefore $h^l(E_1, \phi E_2)$ has no components in $\mathcal{L}$, that is, $Bh^l(E_1, \phi E_2) = 0$, for any $E_1, E_2 \in \mathfrak{X}(\Sigma_T)$. Hence, from the Lemma 3.5, Σ_T defines a totally geodesic foliation in Σ. Furthermore, in light of Theorem 3.8, it is not hard to see that $\Sigma_\perp$ also defines a totally geodesic foliation in Σ. This completes the proof of the theorem. □

Henceforth, we take that the invariant submanifold Σ_T is tangent to the characteristic vector field ξ.

Proposition 4.4. *Let $\Sigma = \Sigma_T \times_f \Sigma_\perp$ be a $GCR-$lightlike warped product submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$ such that Σ_T is tangent to the characteristic vector field ξ. If Σ is a totally umbilical $GCR-$lightlike submanifold of $\overline{\Sigma}$ then $\overline{g}(h(E_1, \phi E_2), \nu E_3)$ vanishes for any $E_1, E_2 \in \mathfrak{X}(\Sigma_T \oplus \{\xi\})$ and $E_3 \in \mathfrak{X}(\phi\mathcal{S}) \subset \mathfrak{X}(\Sigma_\perp)$.*

Proof. Using the fact that $\overline{\Sigma}$ be an indefinite nearly Cosymplectic manifold, we have $\phi(\overline{\nabla}_{E_1} E_3) = \overline{\nabla}_{E_1}\phi E_3 - (\overline{\nabla}_{E_3}\phi)E_1$, for any $E_1 \in \mathfrak{X}(\Sigma_T \oplus \{\xi\})$ and $E_3 \in \mathfrak{X}(\phi\mathcal{S}) \subset \mathfrak{X}(\Sigma_\perp)$. Then, further using (2.11) and (3.12), we obtain

$$\phi(\nabla_{E_1} E_3) = -A_{\nu E_3} E_1 + \nabla^s_{E_1} \nu E_3 - (\overline{\nabla}_{E_3}\phi)E_1. \tag{4.5}$$

Take inner product of last expression both sides with ϕE_2, where $E_2 \in \mathfrak{X}(\Sigma_T \oplus \{\xi\})$ following use of (2.2) and definition 2.1 we obtain that

$$g(\nabla_{E_1} E_3, E_2) - g(\nabla_{E_1} E_3, \xi)\eta(E_2) = g(A_{\nu E_3} E_1, \phi E_2). \tag{4.6}$$

Using Proposition 4.1(ii) in (4.6), we derive $g(A_{\nu E_3} E_1, \phi E_2) = 0$. Thus by virtue of (2.12) and (3.12), we achieve $\overline{g}(h^s(E_1, \phi E_2), \nu E_3) = 0$, but $\overline{g}(h^l(E_1, \phi E_2), \nu E_3) = 0$, this completes the proof. □

Theorem 4.5. *Let Σ be a totally umbilical $GCR-$lightlike submanifold of an indefinite nearly Cosymplectic manifold $\overline{\Sigma}$ such that the distribution $\phi\mathfrak{D}_2$ is parallel with respect to the induced connection ∇. Then Σ is locally a proper $GCR-$lightlike warped product submanifold of the form $\Sigma = \Sigma_T \times_f \Sigma_\perp$, where Σ_T is an invariant submanifold of $\overline{\Sigma}$ and $\Sigma_\perp$ is an anti-invariant submanifold $\overline{\Sigma}$, if and only if the shape operator of Σ satisfies*

$$A_{\phi E_3} E_1 = -\phi E_1(\omega)E_3, \quad \forall\, E_1 \in \mathfrak{X}(\Sigma_T),\ E_3 \in \mathfrak{X}(\Sigma_\perp), \tag{4.7}$$

for some function ω on Σ such that $E_4(\omega) = 0$, $E_4 \in \mathfrak{X}(\Sigma_\perp)$.

Proof. Let Σ be a proper totally umbilical $GCR-$lightlike warped product submanifold of $\overline{\Sigma}$. Then, on comparing the tangential parts of (4.5), we derive (4.7), where $\omega = \ln f$. Since the warping function f is a smooth function on Σ_T, therefore $E_4(\omega) = 0$, for any $E_4 \in \mathfrak{X}(\Sigma_\perp)$.
Conversely, assume that Σ is a totally umbilical $GCR-$lightlike submanifold of $\overline{\Sigma}$; then, from the Proposition 4.4, we have $\overline{g}(h(E_1, \phi E_2), \nu E_3) = 0$, that is, $Bh^s(E_1, \phi E_2) = 0$, for any $E_1, E_2 \in \mathfrak{X}(\Sigma_T \oplus \{\xi\})$ and $E_3 \in \mathfrak{X}(\phi\mathcal{S}) \subset \mathfrak{X}(\Sigma_\perp)$. Let $E \in \mathfrak{X}(\mathfrak{D}_2)$ then by straightforward calculations, $\overline{g}(h^l(E_1, \phi E_2), E) = g(E_2, \nabla_{E_1}\phi E)$, since the distribution $\phi\mathfrak{D}_2$ is parallel with respect to ∇ therefore $\overline{g}(h^l(E_1, \phi E_2), E) = 0$, this implies $h^l(E_1, \phi E_2)$ has no component in $\mathcal{L}$, hence $Bh^l(E_1, \phi) = 0$. Thus, from the Lemma 3.5 and the Theorem 3.6, we conclude that the Σ_T defines totally geodesic foliation in Σ.

Next, from the Theorem 3.9, the distribution $\overline{\mathfrak{D}}$ is always integrable. On taking the inner product of (4.7) with $E_4 \in \mathfrak{X}(\phi S) \subset \mathfrak{X}(\Sigma_\perp)$, we derive

$$g(A_{\phi E_3} E_1, E_4) = -\phi E_1(\omega) g(E_3, E_4) = -g(\nabla\omega, \phi E_1) g(E_3, E_4), \tag{4.8}$$

where $\nabla\omega$ is the gradient of the function ω. Using the Gauss-Weingarten formulas and Proposition 4.1(ii), we can write

$$g(A_{\phi E_3} E_1, E_4) = g(\nabla_{E_3} E_4, \phi E_1) + \overline{g}(h^l(E_3, E_4), \phi E_1), \tag{4.9}$$

then from (4.8) and (4.9), we achieve

$$g(\nabla_{E_3} E_4, \phi E_1) = -g(\nabla\omega, \phi E_1) g(E_3, E_4) - \overline{g}(h^l(E_3, E_4), \phi E_1). \tag{4.10}$$

Denote $\nabla^\perp$ and $h_\perp$ the linear connection on $\Sigma_\perp$ and the second fundamental of $\Sigma_\perp$ on Σ, then we have

$$g(h_\perp(E_3, E_4), \phi E_1) = g(\nabla_{E_4} E_3, \phi E_1) - g(\nabla^\perp_{E_4} E_3, \phi E_1). \tag{4.11}$$

Thus, by virtue of (4.10) and (4.11), for any $E_1 \in \mathfrak{X}(\mathfrak{D}_0) \subset \mathfrak{X}(\Sigma_T)$, we derive that $h_\perp(E_3, E_4) = -\nabla\omega g(E_3, E_4)$. Hence, the integrable manifold $\Sigma_\perp$ is a totally umbilical submanifold in Σ with non-zero and parallel mean curvature such that $E_4(\omega) = 0$, for all $E_4 \in \mathfrak{X}(\overline{\mathfrak{D}})$. Thus, by Theorem 1.2 ([25], page 211), we achieve that Σ is locally a proper $GCR-$lightlike warped product submanifold of $\overline{\Sigma}$. This completes the proof of the theorem. □

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