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Quasi-Conformal flat imperfect fluid GRW spacetime with certain modified gravity

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Abstract. In this paper, we analyze modified gravity and spacetime with C^*-curvature. At first, we illustrate that a C^*-flat imperfect fluid GRW spacetime, if it satisfies Einstein's field equations, then it implies a dark energy era. Additionally, we see that such a spacetime is either Minkowski spacetime or locally isometric to a de-Sitter spacetime. Finally, we have explored that a C^*-flat imperfect fluid GRW spacetime is a solution of $f(R,T)$, $f(R,G)$, and $f(R,L_m)$-gravity. Several energy conditions are investigated in terms of Ricci scalar with the model $f(R,G) = exp(-\frac{R}{\alpha}) + \alpha\ln(6G)$ and $f(R,L_m) = \frac{R^3}{2} + L_m + \beta$. For this model the weak, dominant, and null energy conditions are fulfilled, but the strong energy condition is violated, which implies that the universe is now accelerating.

1 Introduction

In 1915, Albert Einstein introduced his theory of "General Relativity" (briefly, GR). In GR, a spacetime is defined as a Lorentzian manifold M^n equipped with a (Lorentzian) metric g of signature $(-,+,+,\ldots,+)$ which admits a velocity vector. Numerous authors have investigated spacetimes in several contexts, such as [3, 11].

A Lorentzian manifold M^n is said to be a generalized Robertson-Walker (briefly, GRW) [1, 5] spacetime, if it's metric can be expressed as

$$ds^2 = -(dt)^2 + \phi^2(t) g^*_{u_1 u_2} dx^{u_1} dx^{u_2}$$

in which ϕ being a function dependent on t and $g^*_{u_1u_2} = g^*_{u_1u_2}(x^{u_3})$ are only function of x^{u_3} $(u_1, u_2, u_3 = 2, 3, \ldots., n)$ and also ϕ is the warping function or scale factor of t only. Also, a GRW-spacetime is an warped product $-I \times_{\phi^2} \tilde{M}$, where $I \subset \mathbb{R}$ is a open interval and $\tilde{M}$ is an $(n-1)$-dimensional Riemannian manifold. If the dimension of the Riemannian manifold M^n is 3 with constant curvature, then the GRW spacetime becomes a Robertson-Walker (briefly, RW) spacetime.

A Lorentzian manifold of dimension 4 is called a perfect fluid spacetime (briefly, PFS) if the Ricci tensor R_{ab} can be expressed as

$$R_{ab} = \gamma_1 g_{ab} + \gamma_2 \lambda_a \lambda_b,$$

where γ_1, γ_2 are scalars and λ_a, λ_b are velocity vectors (that is, $\lambda_a\lambda^a = -1, \lambda^b = g^{ab}\lambda_a$). According to GR theory, a symmetric tensor field gives the matter field, which is denoted by T_{ab}. For a PFS, the energy momentum tensor (briefly, EMT) [16] is of the form

$$T_{ab} = pg_{ab} + (p + \sigma)\lambda_a\lambda_b. \tag{1.1}$$

In an imperfect fluid GRW spacetime (briefly, $IFGRWS$), the stress-energy-momentum tensor T is essential for investigating the result of the imperfect fluid, as cited in [15]. The expression of this tensor is given by

$$T_{ab} = pg_{ab} + (p + \sigma)\lambda_a\lambda_b + P_{ab}, \tag{1.2}$$

where P denotes a symmetric traceless tensor such that $P_{ab}\lambda^a = 0$ and $P_{ab}g^{ab} = 0$.

In (1.2), p and σ denote isotropic pressure and energy density, respectively. Additionally, the PFS is called isentropic if $p = \sigma$. Moreover, if $p + \sigma = 0$, the PFS is named the dark energy era. The PFS can be referred to as stiff matter if $p = \sigma$, dust matter fluid if $p = 0$, and radiation era if $p = \frac{\sigma}{3}$[6] . According to Einstein's field equations (briefly, $EFEs$), without a cosmological constant,

$$R_{ab} - \frac{R}{2}g_{ab} = kT_{ab}, \tag{1.3}$$

where R is the Ricci scalar and k is Einstein's constant. It is related to the Newtonian gravitational constant G by $k = \frac{8\pi G}{c^4}$, where c is the speed of light.
The quasi-conformal curvature tensor, a generalization of the conformal curvature tensor, finds applications across various domains in mathematics and physics. To study an infinitesimal non-homothetic conformal transformation in a compact orientable manifold of dimension $n > 2$ with constant Ricci scalar. Yano and Sawaki [44] created a new tensor denoted by C^* and defined by

$$C^{*\,d}_{abc} = \gamma R^d_{abc} + \delta\left(R^d_c g_{ab} - R^d_b g_{ac} + R_{ab}\delta^d_c - R_{ac}\delta^d_b\right) - \frac{R}{4}\left(\frac{\gamma}{3} + 2\delta\right)\left(\delta^d_c g_{ab} - \delta^d_b g_{ac}\right), \tag{1.4}$$

where γ, δ are constants and R^e_{abc} is the curvature tensor. Throughout this paper, we consider that $(\gamma + \delta) \neq 0$. According to [20], a Lorentzian manifold has constant sectional curvature if and only if C^* is flat.
On the contrary, the Weyl curvature tensor plays a significant role in both relativity theory and geometry. Many researchers have investigated spacetime with the Weyl tensor. A Weyl tensor is denoted by C and defined by

$$C_{eabc} = R_{eabc} + \frac{R}{(n-1)(n-2)}\{g_{ec}g_{ab} - g_{eb}g_{ac}\}$$
$$- \frac{1}{n-2}\left(g_{ab}R_{ec} - g_{ac}R_{eb} + g_{ec}R_{ab} - g_{eb}R_{ac}\right),$$

where R_{eabc} is the Riemann curvature tensor.
Furthermore, we know that

$$\nabla_c C^c_{dab} = \frac{1}{2}\left[\{\nabla_b R_{da} - \nabla_a R_{db}\} - \frac{1}{2(n-1)}\{g_{da}\nabla_b R - g_{db}\nabla_a R\}\right].$$

If $\nabla_c C^c_{dab} = 0$, then the Weyl tensor is called harmonic.

In GR, the energy conditions (briefly, ECs) are important assets for studying black holes in modified gravity. The ECs are precisely derived from the Raychaudhuri equation, which is contingent on the non-negativity condition $R_{ab}v^a v^b \geq 0$, where v^a is a null vector. This is compared with the null energy condition (briefly, NEC) $T_{ab}v^a v^b \geq 0$. In particular, $T_{ab}\lambda^a\lambda^b \geq 0$, for every time-like vector λ^a states that the weak energy condition (briefly, WEC).

In mathematical physics, $f(R)$-theory is known as a type of modified gravity theory that generalizes Einstein's GR theory. H. A Buchdahl [25] introduced the idea of $f(R)$-theory. This theory has been extensively investigated by several authors [32, 38]. It is well known that Einstein's field equation were derived using the Einstein-Hilbert action. In $f(R)$-theory, the Ricci scalar R is replaced by an analytic function f(R).

In [13, 22], $f(R,T)$ gravity was first presented by Harko et al. Ordines et al [17] used the $f(R,T)$-gravity while mentioning the modified Earth's atmosphere. Numerous authors have analyzed $f(R,T)$-gravity from different angles [18]. $f(R,T)$-gravity generalizes the $f(R)$-gravity by adding the trace of the energy-momentum tensor T into the gravitational action [26, 29, 34].

Moreover, $f(R,G)$-gravity theory [10, 23] is another modification of $f(R)$ gravity theory that is formed by replacing the Ricci scalar R by R and G. In $f(R,G)$, R and G are the Ricci scalar curvature and Gauss-Bonnet invariant, respectively. For more details, see [31, 35].

Harko and Lobo introduced a new type of modified theory, which is known as $f(R,L_m)$-gravity theory [12, 24]. It is an extended theory of $f(R)$-gravity[4] that exactly links any random function of R with the Lagrangian density L_m. A model suggested by Harko and Lobo, for example, is $f(R,L_m) = \frac{R^3}{2} + L_m + \beta$ ($\beta > 0$ is an arbitrary constant) [12]. For more details, see [33, 36].

In [15], the authors have investigated the $f(R)$-gravity in a projectively flat spacetime and analyzed their outcomes utilizing two common models of $f(R)$-gravity. In 2022, De and Hazra [7] investigated the impact of the quasi-conformal curvature tensor in spacetimes with $f(R,G)$-gravity, where $f(R,G) = R + \alpha G^{\beta}$. In 2023, De and Hazra [8] have investigated the impact of the m-projective curvature tenor in $f(R,G)$-gravity and $f(R,L_m)$-gravity with the model $f(R,G) = exp(R) + \alpha \ln(6G)$. In 2024, the authors of [19] investigated the impact of the projective curvature tensor in $f(R,G)$, $f(R,T)$, and $f(R,L_m)$-gravity with the model $f(R,G) = exp(R) + \alpha(6G)^{\beta}$. These findings served as an inspiration for the present paper, which is designed to examine EC_s in term of Ricci scalar R in a C^* flat $IFGRWS$ equipped with $f(R,T)$, $f(R,G)$, and $f(R,L_m)$-gravity, respectively, and we set new model $f(R,G) = exp(-\frac{R}{a}) + \ln(6G)$ (a is scalar) and $f(R,L_m) = \frac{R^3}{2} + L_m + \beta$ for explain EC_s.

After preliminaries in Section 3, the properties of $IFGRWS$ admitting C^* are explored. Finally, we provide C^*-flat $IFGRWS$ equipped with $f(R,T)$, $f(R,G)$, and $f(R,L_m)$-gravity.

2 Preliminaries

A spacetime is said to be C^*-flat spacetime if C^* vanishes throughout the spacetime. Consider a C^*-flat spacetime, then(1.4) implies

$$\gamma R^d_{abc} = \delta\left(-R^d_c g_{ab} + R^d_b g_{ac} - R_{ab}\delta^d_c + R_{ac}\delta^d_b\right) + \frac{R}{4}\left(\frac{\gamma}{3} + 2\delta\right)\left(\delta^d_c g_{ab} - \delta^d_b g_{ac}\right). \tag{2.1}$$

Multiplying (2.1) with g_{de}, we obtain

$$\gamma R_{eabc} = \delta\left(-R_{ec} g_{ab} + R_{eb} g_{ac} - R_{ab} g_{ec} + R_{ac} g_{eb}\right) + \frac{R}{4}\left(\frac{\gamma}{3} + 2\delta\right)\left(g_{ec} g_{ab} - g_{eb} g_{ac}\right). \tag{2.2}$$

Transvecting (2.2) by g^{ec}, we infer

$$R_{ab} = \frac{R}{4} g_{ab}. \tag{2.3}$$

Thus, we state:

Proposition 2.1. *C^*-flat spacetime represents an Einstein spacetime.*

Utilizing (2.3) in (2.2), we have

$$R_{eabc} = \frac{R}{12}\left(g_{ab} g_{ec} - g_{ac} g_{eb}\right). \tag{2.4}$$

This shows that the curvature is constant. The converse is trivial, so we have

Proposition 2.2. *A spacetime is of constant curvature if and only if it is C^*-flat.*

Taking covariant derivative of (1.4), we obtain

$$\nabla_d C^{*\,d}_{abc} = \gamma \nabla_d R^d_{abc} + \delta\left(\nabla_c R_{ab} - \nabla_b R_{ac} + g_{ab}\nabla_c R - g_{ac}\nabla_b R\right) - \frac{1}{4}\left(\frac{\gamma}{3} + \delta\right)\left(g_{ab}\nabla_c R - g_{ac}\nabla_b R\right). \tag{2.5}$$

It is well-known that

$$\nabla_d R^d_{abc} = \nabla_c R_{ab} - \nabla_b R_{ac} \tag{2.6}$$

and

$$\nabla_d R^d_c = \frac{3}{2}\nabla_c R. \tag{2.7}$$

Applying (2.6) in (2.5) we get

$$\nabla_d C^{*\,d}_{abc} = (\gamma + \delta)\left(\nabla_c R_{ab} - \nabla_b R_{ac}\right) - \frac{1}{4}\left(2\delta - \gamma\right)\left(g_{ab}\nabla_c R - g_{ac}\nabla_b R\right). \tag{2.8}$$

If $\nabla_d C^{*\,d}_{abc} = 0$, then (2.8) gives

$$(\gamma + \delta)\left(\nabla_c R_{ab} - \nabla_b R_{ac}\right) = \frac{1}{4}\left(2\delta - \gamma\right)\left(g_{ab}\nabla_c R - g_{ac}\nabla_b R\right). \tag{2.9}$$

Multiplying (2.9) with g^{ab} we get

$$(\gamma + \delta)\left(\nabla_c R - \nabla_b R^b_c\right) = \frac{3}{4}\left(2\delta - \gamma\right)\nabla_c R. \tag{2.10}$$

Since $\nabla_d R^d_c = \frac{3}{2}\nabla_c R$, therefore from (2.10)

$$\frac{1}{4}\left(8\delta - \gamma\right)\nabla_c R = 0.$$

Hence

$$R = \text{constant}, \tag{2.11}$$

provided $(8\delta - \gamma) \neq 0$.
Hence (2.9) becomes

$$\left(\nabla_c R_{ab} - \nabla_b R_{ac}\right) = 0. \tag{2.12}$$

That provides $(\gamma + \delta) \neq 0$, which says that R_{ab} is a Codazzi tensor.

Conversely, if the equation (2.12) holds, then transvecting (2.12) with g^{ab}, we have

$$\nabla_k R = 0. \tag{2.13}$$

Now, from (2.8), (2.12), and (2.13), one obtains

$$\nabla_d C^{*\,d}_{abc} = 0.$$

That implies that C^*- curvature is harmonic. Thus

Proposition 2.3. *In a semi-Riemannian spacetime, the Ricci tensor R_{ab} is Codazzi type if and only if C^* is harmonic with a certain restriction.*

3 $IFGRWS$ admitting C^*

If C^*-flat $IFGRWS$ satisfies EFE, then using (1.2), (1.3) and (2.3), we get

$$\left(k\rho + \frac{R}{4}\right) g_{ab} + k\left(\rho + \sigma\right)\lambda_a\lambda_b + P_{ab} = 0. \tag{3.1}$$

Transvecting (3.1) with g^{ab} entails that

$$3k\rho - k\sigma + R = 0. \tag{3.2}$$

Also, by substituting λ^a into the equation (3.1), we have

$$R = 4k\sigma. \tag{3.3}$$

Using (3.3) into (3.2) we get

$$\rho + \sigma = 0. \tag{3.4}$$

If C^*-flat $IFGRWS$ satisfies EFE, then it represents the dark energy era[6].

Theorem 3.1. *If C^*-flat $IFGRWS$ satisfying EFE, then it represents the dark energy era.*

The equations (1.2) and (1.3) simultaneously imply

$$R_{ab} = \left(k\rho + \frac{R}{2}\right) g_{ab} + k\left(\rho + \sigma\right)\lambda_a\lambda_b + P_{ab}. \tag{3.5}$$

Transvecting (3.5) by $\lambda^a\lambda^b$, we get

$$R_{ab}\lambda^a\lambda^b = -\frac{R}{2} + k\sigma. \tag{3.6}$$

Therefore from (3.3) and (3.6), we obtain

$$R_{ab}\lambda^a\lambda^b = -k\sigma. \tag{3.7}$$

If for any timelike vector λ, $R_{ab}\lambda^a\lambda^b \geq 0$, then the spacetime satisfies the strong energy condition (briefly, SEC).
Now, let a spacetime satisfies SEC. Then from (3.7) we get

$$k\sigma \leq 0. \tag{3.8}$$

Since $k \geq 0$ and σ is non negative, from (3.3) and (3.8), we get

$$R = 0. \tag{3.9}$$

In that case, (2.4) implies $R_{eabc} = 0$, which implies that the spacetime has zero sectional curvature. Therefore, a C^*- flat Minkowski spacetime and an imperfect fluid spacetime are locally isometric[9]. Thus, we write:

Theorem 3.2. *A C^*-flat $IFGRWS$ satisfying the SEC is locally isometric to Minkowski spacetime.*

From (3.8), we get either $\sigma = 0$ or $\sigma > 0$, since σ cannot be negative.

Remark 3.3. If $\sigma = 0$, then from equation (3.4) we get $p = 0$, and the spacetime is said to be Minkowski spacetime.

Remark 3.4. If $\sigma > 0$, then equation (3.4) and $\sigma > 0$ together imply de-Sitter spacetime.

Thus, we get:

Theorem 3.5. *A C^*-flat $IFGRWS$ is either de-Sitter spacetime or Minkowski spacetime.*

4 P^*-flat $IFGRWS$ solutions satisfying $f(R,T)$-gravity

Here, we consider the form of $f(R,T)$ as

$$f(R,T) = R + 2f(T). \tag{4.1}$$

The Einstein-Hilbert action is defined by

$$E = \int \left[\frac{f(R,T) + 16\pi L_m}{16\pi} \right] (-g)^{\frac{1}{2}} \, d^4x. \tag{4.2}$$

The stress energy tensor is denoted by T_{ab} and define by

$$T_{ab} = \frac{-2L_m \delta\left(\sqrt{-g}\right)}{\sqrt{-g}\delta^{ab}},$$

where L_m depends on g.
From equation (4.2), we have obtained the subsequent field equations:

$$\begin{aligned} &f_R(R,T)R_{ba} - \frac{1}{2}f(R,T)g_{ba} - \left[\nabla_b\nabla_a - g_{ba}\Box\right] f_R(R,T) \\ &= 8\pi T_{ba} - [T_{ba} + \ominus_{ba}]f_R(R,T), \end{aligned} \tag{4.3}$$

Here $\Box$ is the d'Alembert operator and

$$\ominus_{ba} = -2T_{ba} + G_{ba}L_m - 2g^{cd}\frac{\delta^2 L_m}{\delta g^{ab}\delta g^{cd}}.$$

We assume $L_m = -p$ and utilizing (1.2), we get

$$T_{ab} = -pg_{ab} + (p+\sigma)\lambda_a\lambda_b + P_{ab}. \tag{4.4}$$

Using (4.4), we get the changes of stress energy is

$$\ominus_{ab} = -2T_{ab} - pg_{ab}. \tag{4.5}$$

Equations (4.1) and (4.3) together produce

$$R_{ab} = \frac{R}{2}g_{ab} + 8\pi T_{ab} + f(T)g_{ab} - 2[T_{ab} + \ominus_{ab}]f^{'}(T). \tag{4.6}$$

We consider the case where the EMT remains conserved for an imperfect fluid solution within the framework of $f(R,T)$-gravity. Using (4.4), (4.5), and (4.6), we obtain the Ricci tensor as

$$R_{ab} = \left[\frac{R}{2} + f(T) - 8p\pi\right] g_{ab} + (p+\sigma)\left\{8\pi + 2f^{'}(T)\right\}\lambda_a\lambda_b + \left\{2f^{'}(T) + 8\pi\right\}P_{ab}. \tag{4.7}$$

From equation (2.3) and (4.7) we get,

$$\left[\frac{R}{4} + f(T) - 8p\pi\right] g_{ab} + (p+\sigma)\left\{8\pi + 2f^{'}(T)\right\}\lambda_a\lambda_b + \left\{2f^{'}(T) + 8\pi\right\}P_{ab} = 0 \tag{4.8}$$

Contracting the foregoing equation we have

$$R + 4f(T) - 32p\pi - (p+\sigma)\left\{2f^{'}(T) + 8\pi\right\} = 0. \tag{4.9}$$

Transvecting (4.8) by g^{ab}, we get

$$-R - 4f(T) + 32p\pi + 4(p+\sigma)\left\{8\pi + 2f^{'}(T)\right\} = 0. \tag{4.10}$$

Adding (4.9) and (4.10), we acquire

$$\left\{f^{'}(T)+4\pi\right\}(p+\sigma)=0.$$

Now we consider either $p+\sigma=0$ or, $p+\sigma\neq 0$. Then the condition represents

Remark 4.1. If $p+\sigma=0$, which is a condition of the dark energy era.

Remark 4.2. If $p+\sigma\neq 0$, then $f^{'}(T)+4\pi=0$. Hence, (4.7) implies that the spacetime is Einstein spacetime.

Therefore, we get

Theorem 4.3. *A P^*- flat spacetime satisfying $f(R,T)$-gravity describes either the dark energy era or the Einstein spacetime.*

5 C^* flat $IFGRWS$ solutions satisfying $f(R,G)$-gravity

Here, we consider an extended theory of gravity known as $f(R,G)$-gravity. In that case, the gravitational force is defined by

$$S=\frac{1}{2k}\int(-g)^{\frac{1}{2}}f(R,G)d^4x+S_{mat}, \tag{5.1}$$

where S_{mat} denotes the matter action. G is the Gauss-Bonnet invariant defined by

$$G=R^2+R_{eabc}R^{eabc}-4R_{ab}R^{ab}. \tag{5.2}$$

We get the field equation from (5.1) as

$$R_{ab}-\frac{R}{2}g_{ab}=kT_{ab}+\omega_{ab}=kT^{eff}_{ab}, \tag{5.3}$$

where

$$\begin{aligned}\omega_{ab}=&\nabla_a\nabla_b f_R-g_{ab}\Box f_R+2R\nabla_a\nabla_b f_G-2g_{ab}R\Box f_G-4R^c_a\nabla_c\nabla_b f_G\\&-4R^c_b\nabla_c\nabla_a f_G+4R_{ab}\Box f_G+4g_{ab}R^{cd}\nabla_c\nabla_d f_G+4R_{acdb}\nabla^c\nabla^d f_G\\&-\frac{1}{2}g_{ab}\left(Rf_R+Gf_G-f\right)+(1-f_R)\left(R_{ab}-\frac{1}{2}g_{ab}R\right)\end{aligned} \tag{5.4}$$

and the effective EMT is denoted by T^{eff}_{ab}. f_R is the partial derivative of f with respect to R. We know that different energy conditions can be expressed in terms of σ and p as follows [7]

$$\begin{aligned}&\sigma+p\geq 0\Longleftrightarrow \text{NEC},\\&\sigma\geq 0\quad\text{and}\quad\sigma+p\geq 0\Longleftrightarrow\text{WEC},\\&\sigma\geq 0\quad\text{and}\quad\sigma\pm p\geq 0\Longleftrightarrow\text{DEC},\\&\sigma+3p\geq 0\quad\text{and}\quad\sigma+p\geq 0\Longleftrightarrow\text{SEC}.\end{aligned}$$

where DEC and NEC are said to be dominant energy conditions and null energy conditions, respectively.

From (2.3), we have

$$R^{ab}=\frac{R}{4}g^{ab}. \tag{5.5}$$

Equation (2.3) and (5.5) together imply

$$R_{ab}R^{ab}=\frac{R^2}{4}. \tag{5.6}$$

From (2.4), it follows that

$$R^{eabc}=\frac{R}{12}\left\{g^{ab}g^{ec}-g^{ac}g^{eb}\right\}. \tag{5.7}$$

Multiplying (2.4) and (5.7), we get

$$R_{eabc}R^{eabc} = \frac{R^2}{6}. \tag{5.8}$$

Equations (5.2), (5.6), and (5.8) infer that

$$G = \frac{R^2}{6}. \tag{5.9}$$

Since for a C^*-flat spacetime R is constant, the equation (5.4) implies that

$$\omega_{ab} = R_{ab} + \left(\frac{f}{2} - \frac{R}{2}\right) g_{ab}. \tag{5.10}$$

Using (5.10) and (1.2) in (5.3), we get

$$\left(kp + \frac{f}{2}\right) g_{ab} + k(p+\sigma)\lambda_a\lambda_b + kP_{ab} = 0. \tag{5.11}$$

Contracting (5.11) with λ^a, we get

$$\sigma = \frac{f}{2k}. \tag{5.12}$$

Multiplying (5.11) by g^{ab} and using (5.12), we infer

$$p = -\frac{f}{2k}. \tag{5.13}$$

Hence, we conclude the following:

Theorem 5.1. *In a C^*-flat $IFGRWS$ satisfying $f(R,G)$-gravity σ and p are represented by (5.12) and (5.13).*

By adding (5.12) and (5.13), we get

$$p + \sigma = 0,$$

which implies that NEC is verified.
Now, we focus on the energy condition of $f(R,G)$-gravity model A.

$$f(R,G) = exp(-\frac{R}{a}) + ln(6G)$$

$$\sigma = \frac{exp(-\frac{R}{a}) + ln(6G)}{2k}. \tag{5.14}$$

$$p = -\frac{exp(-\frac{R}{a}) + ln(6G)}{2k}. \tag{5.15}$$

Using (5.9), the equation (5.14) and (5.15) reduce to

$$\sigma = \frac{exp(-\frac{R}{a}) + ln(R^2)}{2k}. \tag{5.16}$$

$$p = -\frac{exp(-\frac{R}{a}) + ln(R^2)}{2k}. \tag{5.17}$$

Adding equations (5.16) and (5.17), we see that the energy condition for this model, NEC, is verified which implies that WEC is also verified. We investigate the character of the density parameter, DEC, and SEC. One can see from Figure 1 that the energy density is positive for the parameter ranges $R \in [-2.25, -2]$ and $a \in [1, 1.25]$. Figure 2 gives an overview of $\sigma - p$, which generates a non-negative range. Using Figure 1, Figure 2, and $\sigma + p = 0$, we see that DEC is satisfied. From Figure 3, it is observed that the SEC is not satisfied. Therefore, this indicates the presence of late-time acceleration in the Universe[7].

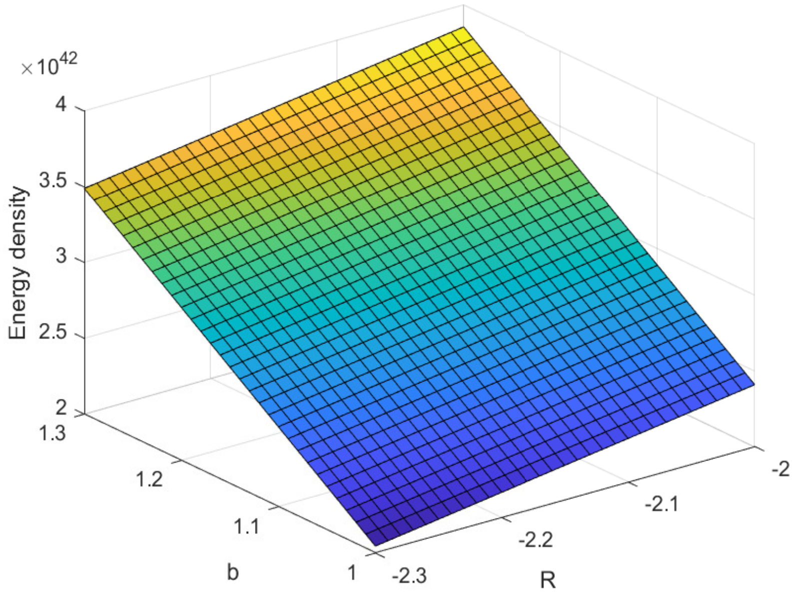

Figure 1. Evolution of σ with reference to $R \in [-2.3, -2]$, $b \in$[1, 1.3] and k = 2.077 $\times$ 10^{-43} .

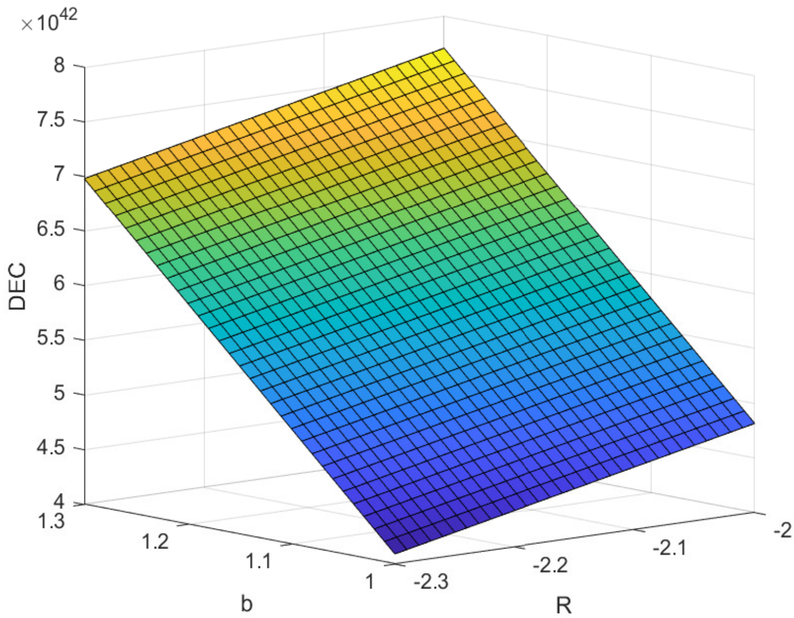

Figure 2. Evolution of DEC with reference to $R \in [-2.3, -2]$,
$b \in$[1, 1.3] and k = 2.077 $\times$ 10^{-43} .

6 C^* flat $IFGRWS$ solutions satisfying $f(R, L_m)$-gravity

In this section, we investigate C^*-flat $IFGRWS$ satisfying $f(R, L_m)$-gravity. Our hypothesis states that the action term has the following form.

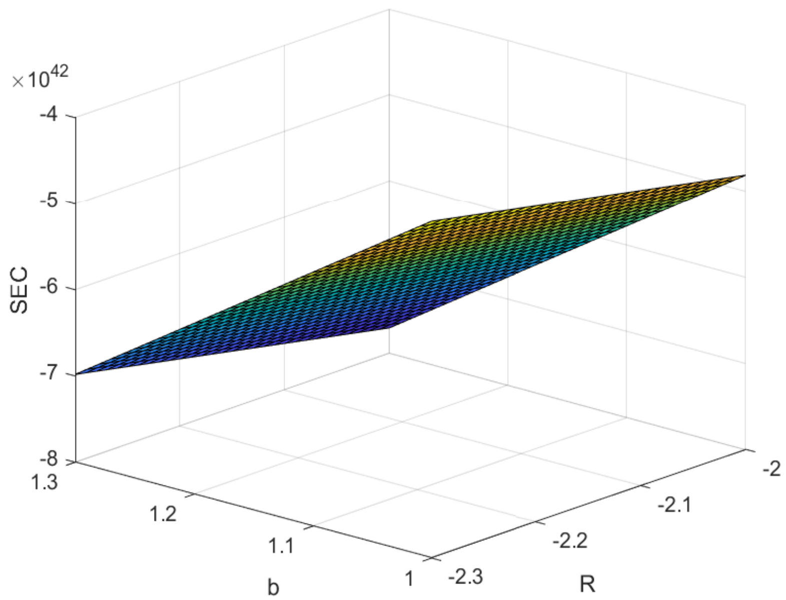

Figure 3. Evolution of SEC with reference to $R \in [-2.3, -2]$, $b \in[1, 1.3]$ and $k = 2.077 \times 10^{-43}$.

$$S = \int \sqrt{-g} f(R, L_m) d^4x. \tag{6.1}$$

EMT of the matter is as follows:

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{ab}}.$$

The field equation of $f(R, L_m)$-gravity can be expressed in their modified form [12], with respect to g from modified action (6.1) as

$$\begin{aligned} &f_R(R, L_m) R_{ab} + \left(g_{ab} \nabla_c \nabla^d - \nabla_a \nabla^b\right) f_R(R, L_m) - \frac{3}{4} \left[f(R, L_m) - f_{L_m}(R, L_m) L_m\right] g_{lk} \\ &= \frac{3}{4} f_{L_m}(R, L_m) T_{ab}. \end{aligned} \tag{6.2}$$

For our research, we take the new model define below:

$$f(R, L_m) = \frac{R^3}{2} + L_m + \beta \tag{6.3}$$

where $\beta > 0$ is arbitrary constant.
From (6.2) and (6.3), we have

$$\frac{3}{2} R^2 R_{ab} - \frac{3}{4} \left(\beta + \frac{R^3}{2}\right) g_{ab} = \frac{3}{4} T_{ab}. \tag{6.4}$$

Equation (1.2), (2.3) and (6.4) reflect that

$$(p + \beta)\, g_{ab} + (p + \sigma)\, \lambda_a \lambda_b = 0. \tag{6.5}$$

Multiplying (6.5) with λ^a and g^{ab} separately, we get

$$\sigma = \beta, \tag{6.6}$$

and

$$p = -\beta. \tag{6.7}$$

The combination of (6.6) and (6.7) gives $p + \sigma = 0$. Thus, we conclude:

Theorem 6.1. *A C^*-flat $IFGRWS$ solutions satisfying $f(R, L_m) = \frac{R^3}{2} + L_m + \beta$ becomes a dark energy era.*

Now, we investigate the EC_s for the model (6.3). Adding equations (6.5) and (6.6), we see that the energy condition for this model, NEC is verified which implies that WEC is also verified. We investigate the character of density parameter, DEC and SEC. One can see from figure 4 that the energy density is positive for the parameter ranges $R \in [-2.23, -2]$ and $a \in [1, 1.13]$. Figure 5, gives an overview of $\sigma - p$, which generates a non-negative range. Using Figure 4, Figure 5, and $\sigma + p = 0$, we see that DEC is satisfied. From Figure 6, it is observed that the SEC is not satisfied. Therefore, this indicates the presence of late-time acceleration in the Universe [7].

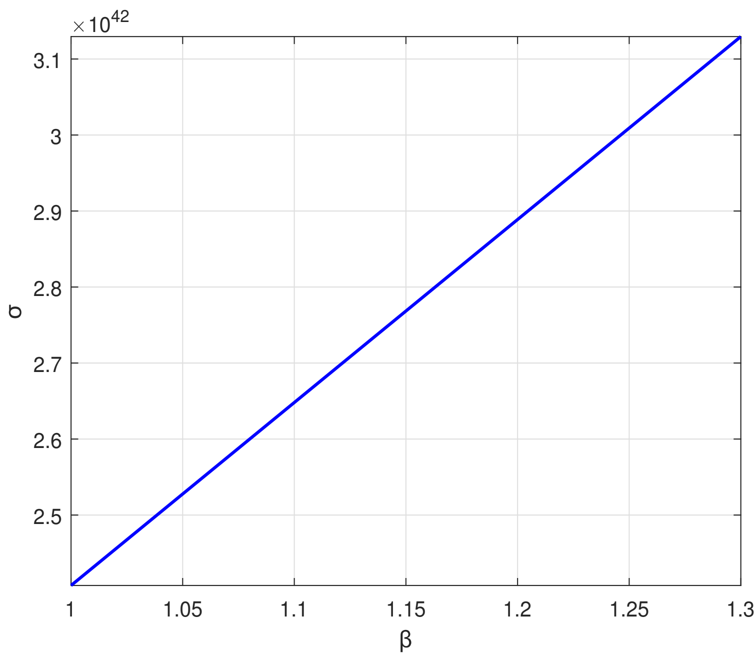

Figure 4. Evolution of σ with reference to $R \in [-2.25, -2]$, $a \in$[1, 1.25] and k = 2.077 × 10^{-43} .

7 Conclusion

In this article, we analyze a C^*-flat $IFGRWS$ and show that it is either a de-Sitter spacetime or locally isometric to Minkowski spacetime. We also investigate whether a C^*-flat $IFGRWS$ satisfies the SEC is locally isotropic to Minkowski spacetime. The main focus of this paper has been the investigation of C^*-flat $IFGRWS$ solutions in relation to the variation of gravity. In this paper, our results have been analyzed analytically and graphically. By analytic technique, we derive our formulation for determining the stability of two cosmological models, like $f(R, G) = exp(-\frac{R}{a}) + ln(6G)$ and $f(R, L_m) = \frac{R^3}{2} + L_m + \beta$. For the first model, Figures 1, 2, and 3 provide an overview of EC_s. Here we see that SEC is not satisfied, but NEC, WEC, and SEC are verified. Similar to the first model, Figures 4, 5, and 6 show all EC_s for the second

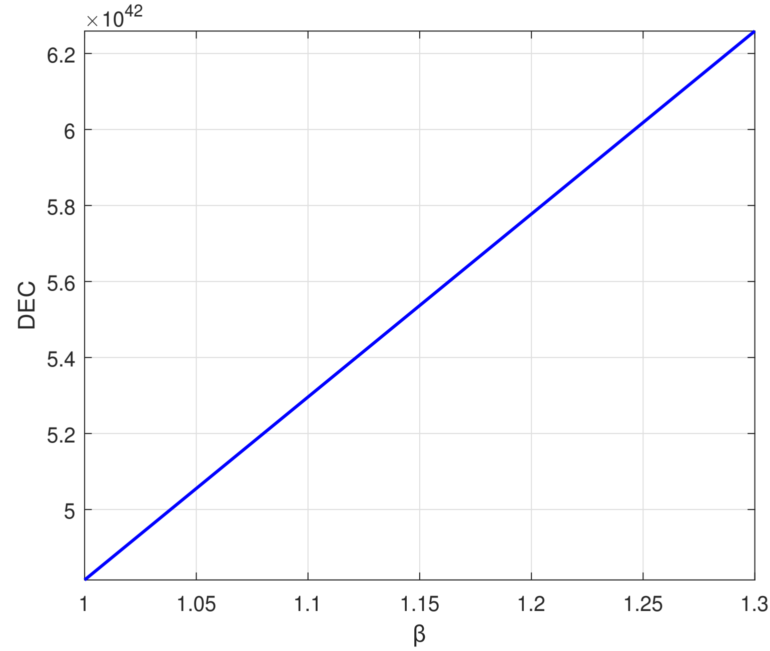

Figure 5. Evolution of DEC with reference to $R \in [-2.25, -2]$, a ∈[1, 1.25] and k = 2.077 × 10^{-43} .

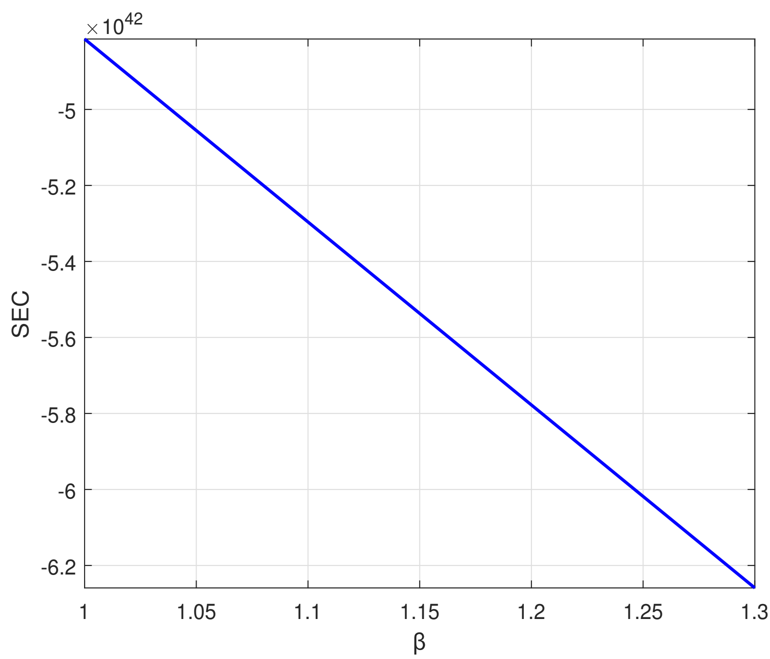

Figure 6. Evolution of SEC with reference to $R \in [-2.25, -2]$, a ∈[1, 1.25] and k = 2.077 × 10^{-43} .

model.

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