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Nonlinear Elliptic Inclusions with Singular Data in Variable Exponent Sobolev Spaces

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Abstract This paper investigates the existence of solutions to a class of nonlinear elliptic inclusion problems featuring a singular right-hand side. The problem is set in the framework of variable exponent Sobolev spaces $W_0^{1,p(x)}(\Omega)$, driven by a Leray-Lions type operator and a multivalued term described by a maximal monotone graph. The main novelty lies in the combination of a singularity of the form f/u^γ with $\gamma > 0$, a general maximal monotone operator β, and non-standard growth conditions. Through a careful regularization scheme, a priori estimates adapted to the variable exponent setting, and monotonicity arguments, we prove the existence of a positive solution.

1 Introduction

Elliptic partial differential equations with singular nonlinearities have been extensively studied due to their fundamental mathematical challenges and their applications in various physical phenomena. Problems involving singular terms, arise in models of non-Newtonian fluids [18], in biological and chemical systems, including population dynamics and reaction kinetics [15], and in dispersive settings, such as the nonlinear Schrödinger equation with singular terms [13]. In these contexts, singularities encode fundamental physical or biological mechanisms vanishing viscosity, blow-up phenomena, or concentration effects, which strongly influence the qualitative behavior of solutions. The presence of such singularities leads to delicate analytical issues, particularly concerning the behavior of solutions near the boundary where u tends to zero.

In the classical constant exponent setting, seminal works by Crandall, Rabinowitz, and Tartar [10], Lazer and McKenna [17], and Boccardo and Orsina [8] have established comprehensive existence, regularity, and uniqueness theories for singular elliptic problems. The study of inclusions involving maximal monotone graphs adds another layer of complexity, allowing for the modeling of phenomena with discontinuous or set-valued constitutive laws [9]. Inclusion problems were explored in [2] and [3], without addressing singularities.

Parallel to these developments, the theory of variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ has emerged as a natural framework for problems with non-standard growth conditions [1, 6, 12, 19, 14]. These spaces are particularly suited for modeling electrorheological fluids, image processing, and other applications where the medium exhibits pointwise heterogeneity [19].

In this work, we bridge these directions by investigating the following nonlinear elliptic inclusion problem

$$\begin{cases} \beta(u) - \operatorname{div}(a(x,\nabla u)) \ni \dfrac{f}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary, $\gamma > 0$, $f \in L^\infty(\Omega)$ is nonnegative, β is a maximal monotone graph with $0 \in \beta(0)$, and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying appropriate variable exponent Leray-Lions conditions.

The main challenges in analyzing problem (1.1) stem from the interplay between three features. The singularity which becomes unbounded as $u \to 0^+$, the multivalued nature of the graph β, and the non-standard growth conditions governed by the variable exponent $p(x)$. To the best of our knowledge, this combination has not been previously addressed in the literature. Our work builds upon and extends recent results on singular problems in variable exponent spaces [4, 5].

Our approach involves a careful regularization procedure where we approximate the singular term by $f_n/(u_n + 1/n)^\gamma$ and the maximal monotone graph by its Yosida approximation β_n. We then derive a priori estimates that are uniform with respect to the regularization parameter, using techniques adapted to the variable exponent setting. The passage to the limit relies on monotonicity arguments, compactness results in variable exponent spaces, and the theory of maximal monotone operators.

The paper is organized as follows: Section 2 recalls necessary preliminaries about variable exponent spaces and maximal monotone operators. Section 3 states our main assumptions and the definition of solution. Section 4 establishes the existence of solutions to the approximate problems. Section 5 derives crucial a priori estimates. Section 6 contains the proof of our main existence result. Finally, Section 7 is devoted to the study of the uniqueness of the solution.

2 Preliminaries

2.1 Variable Exponent Lebesgue and Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $p : \overline{\Omega} \to [1, \infty)$ be a measurable function. We set

$$p^+ := \operatorname{ess}\sup_{x\in\Omega} p(x), p^- := \operatorname{ess}\inf_{x\in\Omega} p(x).$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

and is equipped with the *Luxemburg norm*

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

The space $W_0^{1,p(x)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Its dual is denoted by $W^{-1,p'(x)}(\Omega)$, where $p'(x)$ is the conjugate exponent of $p(x)$, defined by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. This space consists of all continuous linear functionals acting on $W_0^{1,p(x)}(\Omega)$.

We now recall several fundamental properties of these spaces (see [12, 14]).

Proposition 2.1. *Assume* $1 < p^- \leq p^+ < \infty$.

(i) $L^{p(x)}(\Omega)$ *and* $W^{1,p(x)}(\Omega)$ *are separable, reflexive Banach spaces.*

(ii) Hölder's inequality holds

$$\int_\Omega |uv| dx \leq 2\|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)},$$

where $p'(x) = p(x)/(p(x) - 1)$.

(iii) The following modular-norm relations hold

$$\|u\|_{L^{p(x)}(\Omega)} < 1 \Rightarrow \int_\Omega |u(x)|^{p(x)} dx \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)},$$
$$\|u\|_{L^{p(x)}(\Omega)} > 1 \Rightarrow \int_\Omega |u(x)|^{p(x)} dx \geq \|u\|^{p^+}_{L^{p(x)}(\Omega)}.$$

(iv) If p is log-Hölder continuous, i.e.,

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \quad \textit{for } |x-y| < \frac{1}{2},$$

then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$.

(v) (Poincaré's inequality) There exists a constant $C > 0$ such that:

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \textit{for all } u \in W_0^{1,p(x)}(\Omega).$$

(vi) If $p(x) \leq N$ and $q(x) \leq p^*(x) = \frac{Np(x)}{N-p(x)}$ with $esssup(p^*(x) - q(x)) > 0$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

2.2 The Space $T_0^{1,p(x)}(\Omega)$

Following the approach of [11] and [8], we define the space of functions with truncations in $W_0^{1,p(x)}(\Omega)$.

Definition 2.2. For $k > 0$, the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(s) = \max(-k, \min(s, k)).$$

The space $T_0^{1,p(x)}(\Omega)$ is defined as:

$$T_0^{1,p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : T_k(u) \in W_0^{1,p(x)}(\Omega) \quad \forall k > 0 \right\}.$$

Remark 2.3. If $u \in T_0^{1,p(x)}(\Omega)$, then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$\nabla T_k(u) = v\chi_{\{|u| \leq k\}} \quad \text{a.e. in } \Omega, \text{ for all } k > 0.$$

We define the generalized gradient ∇u as this function v. Moreover, functions in $T_0^{1,p(x)}(\Omega)$ are said to vanish on the boundary in the sense that $T_k(u) \in W_0^{1,p(x)}(\Omega)$ for every $k > 0$.

2.3 Maximal Monotone Graphs and Yosida Approximation

The graph of a set-valued operator $\beta : \mathbb{R} \rightrightarrows \mathbb{R}$ is defined as

$$\text{Graph}(\beta) = \{(s, \xi) \in \mathbb{R} \times \mathbb{R} \ : \ \xi \in \beta(s)\}.$$

A set-valued operator $\beta : \mathbb{R} \rightrightarrows \mathbb{R}$ is called monotone if

$$(\xi - \eta)(s - t) \geq 0 \quad \text{for all } \xi \in \beta(s), \eta \in \beta(t),$$

and maximal monotone if its graph is not properly contained in any other monotone graph.

Proposition 2.4. *For a maximal monotone graph β with $0 \in \beta(0)$, we define its* Yosida approximation $\beta_\lambda : \mathbb{R} \to \mathbb{R}$ *for $\lambda > 0$ by*

$$\beta_\lambda(s) = \frac{1}{\lambda}(s - J_\lambda(s)), \quad J_\lambda(s) = (I + \lambda\beta)^{-1}(s),$$

where J_λ is the resolvent *of β. The Yosida approximation satisfies:*

- β_λ *is monotone and Lipschitz continuous with constant* $1/\lambda$.
- $|\beta_\lambda(s)| \leq |\beta^0(s)|$ *for all* $s \in \mathbb{R}$*, where* $\beta^0(s)$ *is the element of minimal norm in* $\beta(s)$*.*
- $\beta_\lambda(s) \in \beta(J_\lambda(s))$ *for all* $s \in \mathbb{R}$*.*
- $\lim_{\lambda\to 0} \beta_\lambda(s) = \beta^0(s)$ *for all* $s \in D(\beta)$*.*

We refer to [9] for more proprieties about maximal monotone graphs.

3 Assumptions and Main Result

We make the following assumptions on the data of problem (1.1)

ASSUMPTION (A_1): The function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (measurable in x for all ξ, continuous in ξ for a.e. x) satisfying for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^{p(x)}, \quad \alpha > 0, \tag{3.1}$$

$$(a(x,\xi) - a(x,\eta))\cdot(\xi-\eta) > 0 \quad \text{for } \xi \neq \eta, \tag{3.2}$$

$$|a(x,\xi)| \leq C(1 + |\xi|^{p(x)-1}). \tag{3.3}$$

ASSUMPTION (A_2): The graph $\beta : \mathbb{R} \rightrightarrows \mathbb{R}$ is maximal monotone with $0 \in \beta(0)$.
ASSUMPTION (A_3): The function $f \in L^\infty(\Omega)$ is nonnegative and not identically zero.
ASSUMPTION (A_4): The exponent $p : \overline{\Omega} \to \mathbb{R}$ satisfies $1 < p^- \leq p(x) \leq p^+ < \infty$ and is log-Hölder continuous.

We now proceed to introduce the precise definition of a solution to problem (1.1), adapted from [8, 11].

Definition 3.1. A pair (u, b) is called a solution of problem (1.1) if

(i) $u \in T_0^{1,p(x)}(\Omega)$, $u > 0$ a.e. in Ω,

(ii) $b \in L^1(\Omega)$, $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$,

(iii) For every $\varphi \in C_0^1(\Omega)$, we have

$$\int_\Omega b\varphi dx + \int_\Omega a(x,\nabla u)\cdot\nabla\varphi dx = \int_\Omega \frac{f\varphi}{u^\gamma} dx.$$

Remark 3.2. Thanks to the positivity of u and the compact support of φ, the right-hand side integral is well-defined. The membership $u \in T_0^{1,p(x)}(\Omega)$ ensures that the gradient term is also well-defined.

Our main result is the following existence theorem

Theorem 3.3. *Under assumptions* (A_1)*–*(A_4)*, for any* $\gamma > 0$*, problem* (1.1) *admits at least one solution* (u, b) *in the sense of Definition 3.1.*

The proof of Theorem 3.3 will be developed in the subsequent sections.

4 Approximate Problem

For each $n \in \mathbb{N}^*$, we consider the following regularized problem

$$\begin{cases} \beta_n(T_n(u_n)) - \text{div}(a(x,\nabla u_n)) = \dfrac{f_n}{(u_n + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where

- β_n is the Yosida approximation of β with parameter $\lambda = 1/n$,
- $T_n(s) = \max(-n, \min(s, n))$ is the truncation at level n,
- $f_n = \min(f, n)$.

The weak formulation of (4.1) is, find $u_n \in W_0^{1,p(x)}(\Omega)$ such that for all $\varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \beta_n(T_n(u_n))\varphi dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla\varphi dx = \int_\Omega \frac{f_n\varphi}{(u_n + \frac{1}{n})^\gamma} dx. \tag{4.2}$$

Lemma 4.1. *For each fixed $n \in \mathbb{N}^*$, the operator $A_n : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle A_n(u), \varphi\rangle = \int_\Omega \beta_n(T_n(u))\varphi dx + \int_\Omega a(x, \nabla u) \cdot \nabla\varphi dx$$

is bounded, coercive, and pseudo-monotone.

Proof. The boundedness of A_n follows directly from assumption (3.3) together with the boundedness of the composition $\beta_n \circ T_n$. Coercivity is a consequence of (3.1) and the inequality $\int_\Omega \beta_n(T_n(u))\, u\, dx \geq 0$, which holds since $0 \in \beta(0)$ and β_n is monotone. Finally, the pseudo-monotonicity of A_n follows from the classical theory of Leray–Lions type operators (see, e.g., [20]). □

Proposition 4.2. *For each $n \in \mathbb{N}^*$, the approximate problem (4.1) admits a solution $u_n \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.*

Proof. We use Schauder's fixed point theorem. Define the mapping $S : L^{p(x)}(\Omega) \to L^{p(x)}(\Omega)$ as follows, for $v \in L^{p(x)}(\Omega)$, let $w = S(v)$ be the unique solution of

$$\begin{cases} \beta_n(T_n(w)) - \text{div}(a(x, \nabla w)) = \dfrac{f_n}{(|v| + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

The right-hand side of (4.3) belongs to $L^\infty(\Omega) \subset W^{-1,p'(x)}(\Omega)$ since $|v| + 1/n \geq 1/n > 0$. By Lemma 4.1 and the theory of pseudo-monotone operators [20], problem (4.3) admits a unique solution $w \in W_0^{1,p(x)}(\Omega)$.

To show that S maps a ball into itself, test (4.3) with w

$$\int_\Omega \beta_n(T_n(w))w dx + \int_\Omega a(x, \nabla w) \cdot \nabla w dx = \int_\Omega \frac{f_n w}{(|v| + \frac{1}{n})^\gamma} dx.$$

Using (3.1), the nonnegativity of the first term, and the fact that $(|v| + 1/n)^{-\gamma} \leq n^\gamma$, we get

$$\alpha \int_\Omega |\nabla w|^{p(x)} dx \leq n^\gamma \int_\Omega |f_n||w| dx \leq n^{\gamma+1} \|w\|_{L^1(\Omega)}.$$

By the Poincaré inequality in variable exponent spaces, we obtain

$$\int_\Omega |\nabla w|^{p(x)} dx \leq C n^{\gamma+1} \|\nabla w\|_{L^{p(x)}(\Omega)} \leq C n^{\gamma+1} \max\left\{ \|\nabla w\|^{p^-}_{L^{p(x)}(\Omega)}, \|\nabla w\|^{p^+}_{L^{p(x)}(\Omega)} \right\}.$$

This implies that w is bounded in $W_0^{1,p(x)}(\Omega)$ independently of v, so S maps a sufficiently large ball into itself.

The continuity and compactness of S follow from the continuity of the solution operator for (4.3) with respect to the right-hand side and the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$. By Schauder's fixed point theorem, S has a fixed point $u_n = S(u_n)$, which is a solution of (4.1). □

5 A Priori Estimates

In this section, we establish a series of *a priori* estimates that remain uniform with respect to n. These bounds play a fundamental role in enabling the passage to the limit. The qualitative nature of these estimates is strongly influenced by the magnitude of the singularity exponent γ.

Lemma 5.1. *For any $\omega \subset\subset \Omega$, there exists $c_\omega > 0$, independent of n, such that*

$$u_n(x) \geq c_\omega \quad \textit{for a.e. } x \in \omega.$$

Proof. The proof follows a strategy inspired by [8]. For a given $k > 0$, consider the function

$$\Phi_k(s) = \int_0^{T_k(s)} \frac{1}{(t + 1/n)^\gamma} dt.$$

Observe that $\Phi_k(u_n) \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ with

$$\nabla \Phi_k(u_n) = \frac{\nabla u_n}{(u_n + 1/n)^\gamma} \chi_{\{|u_n| \leq k\}}.$$

Using $\varphi = \Phi_k(u_n)$ in (4.2)

$$\int_\Omega \beta_n(T_n(u_n))\Phi_k(u_n)dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla \Phi_k(u_n)dx = \int_\Omega \frac{f_n \Phi_k(u_n)}{(u_n + 1/n)^\gamma} dx.$$

Since $\beta_n(T_n(u_n))\Phi_k(u_n) \geq 0$ and $f_n \Phi_k(u_n) \leq n\Phi_k(u_n)$, we have

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla \Phi_k(u_n)dx \leq n \int_\Omega \Phi_k(u_n)dx.$$

By (3.1), the left-hand side is bounded below by

$$\alpha \int_{\{|u_n| \leq k\}} \frac{|\nabla u_n|^{p(x)}}{(u_n + 1/n)^\gamma} dx.$$

For sufficiently large k, a De Giorgi-type iteration adapted to variable exponent spaces (see [16]) ensures the desired interior positivity. The argument proceeds by establishing a Caccioppoli-type inequality and iteratively estimating the measure of the level sets of u_n, ultimately showing that u_n is uniformly bounded away from zero in ω. □

Lemma 5.2. *The sequence u_n satisfies*

(i) For each $k > 0$, $T_k(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$ uniformly in n.

(ii) $\beta_n(T_n(u_n))$ is bounded in $L^\infty(\Omega)$ uniformly in n.

Moreover, the sequence u_n exhibits the following behavior

- *If $0 < \gamma \leq 1$, then u_n is bounded in $W_0^{1,p(x)}(\Omega)$.*
- *If $\gamma > 1$, then $u_n^{(p^- + \gamma - 1)/p^-}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Consequently, for any $\omega \subset\subset \Omega$, u_n is bounded in $W^{1,p(x)}(\omega)$.*

Proof. **(i)** Use $\varphi = T_k(u_n)$ in (4.2)

$$\int_\Omega \beta_n(T_n(u_n))T_k(u_n)dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla T_k(u_n)dx = \int_\Omega \frac{f_n T_k(u_n)}{(u_n + \frac{1}{n})^\gamma} dx.$$

Since $\beta_n(T_n(u_n))T_k(u_n) \geq 0$ and $T_k(u_n)/(u_n + 1/n)^\gamma \leq k^{1-\gamma}$, we have

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla T_k(u_n)dx \leq k^{1-\gamma} \|f_n\|_{L^1(\Omega)} \leq k^{1-\gamma} \|f\|_{L^1(\Omega)}.$$

By (3.1), this implies

$$\alpha \int_\Omega |\nabla T_k(u_n)|^{p(x)} dx \leq k^{1-\gamma} \|f\|_{L^1(\Omega)}.$$

The modular-norm relations in variable exponent spaces then give the boundedness of $T_k(u_n)$ in $W_0^{1,p(x)}(\Omega)$.

For the case $0 < \gamma \leq 1$, we can take k large enough so that $T_k(u_n) = u_n$ (since u_n is bounded by Proposition 4.2), which gives the global bound.

(ii) For $\delta > 0$, consider the test function

$$\varphi_{\delta,n} = \frac{1}{\delta}\left(T_{k+\delta}(\beta_n(T_n(u_n))) - T_k(\beta_n(T_n(u_n)))\right).$$

Using this test function in (4.2) and following standard techniques (see [4]), we obtain

$$k|\{|\beta_n(T_n(u_n))| \geq k\}| \leq C\|f\|_{L^\infty(\Omega)},$$

which implies the L^∞ bound.

Case $\gamma > 1$: This is the case that requires adaptation. We follow the strategy of [11], simplified by our assumption $f \in L^\infty(\Omega)$. Let $\delta = (p^- + \gamma - 1)/p^- > 1$. We choose the test function

$$v_n = (u_n + 1/n)^\delta - (1/n)^\delta.$$

Note that $v_n \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ since u_n is bounded. Its gradient is given by

$$\nabla v_n = \delta(u_n + 1/n)^{\delta-1} \nabla u_n.$$

Using v_n as a test function in the approximate equation (4.2) yields

$$\int_\Omega \beta_n(T_n(u_n)) v_n dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla v_n dx = \int_\Omega \frac{f_n v_n}{(u_n + \frac{1}{n})^\gamma} dx.$$

We estimate each term

- Since $0 \in \beta(0)$ and β_n is monotone, $\beta_n(T_n(u_n)) v_n \geq 0$.
- For the right-hand side, we use that $v_n \leq (u_n + 1/n)^\delta$ and $f_n \leq \|f\|_{L^\infty}$

$$\int_\Omega \frac{f_n v_n}{(u_n + \frac{1}{n})^\gamma} dx \leq \|f\|_{L^\infty(\Omega)} \int_\Omega (u_n + 1/n)^{\delta-\gamma} dx.$$

  Since $\delta - \gamma = (1-p^-)/p^- < 0$, the function $(u_n + 1/n)^{\delta-\gamma}$ is bounded above by $(1/n)^{\delta-\gamma}$, but this is not useful. However, thanks to the positivity Lemma 5.1, for any $\omega \subset\subset \Omega$, we have $u_n \geq c_\omega > 0$. On ω, $(u_n + 1/n)^{\delta-\gamma} \leq c_\omega^{\delta-\gamma}$. On $\Omega \setminus \omega$, near the boundary, we use the boundedness of u_n in $L^\infty(\Omega)$. Thus, the right-hand side is uniformly bounded in n.
- For the crucial term involving the operator, we use the assumption (3.1) and the definition of ∇v_n

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla v_n dx = \delta \int_\Omega a(x, \nabla u_n) \cdot \nabla u_n (u_n + 1/n)^{\delta-1} dx$$
$$\geq \alpha\delta \int_\Omega |\nabla u_n|^{p(x)} (u_n + 1/n)^{\delta-1} dx.$$

Combining these estimates, we find

$$\alpha\delta \int_\Omega |\nabla u_n|^{p(x)} (u_n + 1/n)^{\delta-1} dx \leq C,$$

where C is a constant independent of n.

Now, note that

$$|\nabla(u_n^{(p^-+\gamma-1)/p^-})|^{p(x)} = \left(\frac{p^-+\gamma-1}{p^-}\right)^{p(x)} |\nabla u_n|^{p(x)} u_n^{p(x)(\gamma-1)/p^-}.$$

Since $p(x) \geq p^-$ and $\delta - 1 = (\gamma-1)/p^-$, we have (for $u_n \geq 1/n$)

$$|\nabla u_n|^{p(x)}(u_n+1/n)^{\delta-1} \geq |\nabla u_n|^{p(x)} u_n^{(\gamma-1)/p^-} \geq C_1 |\nabla(u_n^{(p^-+\gamma-1)/p^-})|^{p(x)},$$

where the last inequality follows from applying the chain rule and using the fact that $p(x)$ is bounded. The constant C_1 depends on p^-, p^+, and γ.

Therefore, we obtain the uniform bound

$$\int_\Omega |\nabla(u_n^{(p^-+\gamma-1)/p^-})|^{p(x)} dx \leq C_2.$$

By the Poincaré inequality in variable exponent spaces (Proposition 2.1), this implies that $\{u_n^{(p^-+\gamma-1)/p^-}\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

The boundedness of u_n in $W_{\text{loc}}^{1,p(x)}(\Omega)$ follows from this result and the interior positivity from Lemma 5.1. □

6 Proof of the Main Theorem

We are now in a position to prove Theorem 3.3. The main idea consists in passing to the limit in the sequence of approximate problems (4.2). The convergence properties of the sequence (u_n) and the behavior of the limiting solution depend critically on the value of the singularity exponent $\gamma > 0$.

Proof of Theorem 3.3. The proof is carried out in several steps, taking into account the influence of γ at each stage.

Step 1: Compactness and Weak Convergence. From Lemma 5.2, for every $k > 0$, the sequence $T_k(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$. By the reflexivity of $W_0^{1,p(x)}(\Omega)$ and a diagonal argument, there exists a subsequence (still denoted by u_n) and a measurable function u such that for every $k > 0$

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \quad \text{in } W_0^{1,p(x)}(\Omega), \\ T_k(u_n) &\to T_k(u) \quad \text{in } L^{p(x)}(\Omega) \text{ and a.e. in } \Omega, \\ u_n &\to u \quad \text{a.e. in } \Omega. \end{aligned}$$

By definition, this implies that $u \in T_0^{1,p(x)}(\Omega)$.

Furthermore, by Lemma 5.2(ii) and the Banach-Alaoglu theorem, there exists $b \in L^\infty(\Omega)$ such that

$$\beta_n(T_n(u_n)) \overset{*}{\rightharpoonup} b \quad \text{in } L^\infty(\Omega).$$

From the growth condition (3.3) and the boundedness of $T_k(u_n)$, it follows that for each $k > 0$, $\{a(x, \nabla T_k(u_n))\}$ is bounded in $(L^{p'(x)}(\Omega))^N$. Hence, there exists $A_k \in (L^{p'(x)}(\Omega))^N$ such that

$$a(x, \nabla T_k(u_n)) \rightharpoonup A_k \quad \text{in } (L^{p'(x)}(\Omega))^N.$$

By the a.e. convergence of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ (which follows from the strong convergence proved in the next step), and the continuity of $a(x,\cdot)$, we can identify $A_k = a(x, \nabla T_k(u))$.

Step 2: Strong Convergence of Truncates. We now prove the strong convergence of the truncates, which is the key to passing to the limit in the nonlinear term. This step adapts the technique of [11, Theorem 2.3] to our setting.

Let $k > 0$ be fixed. We want to show that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,p(x)}(\Omega).$$

Since $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1,p(x)}(\Omega)$, it suffices to show that

$$\limsup_{n\to\infty} \int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx \le \int_\Omega a(x, \nabla T_k(u)) \cdot \nabla T_k(u) dx.$$

Let $\varphi \in C_0^1(\Omega)$ be such that $0 \le \varphi \le 1$. Using the test function $\varphi(T_k(u_n) - T_k(u))$ in the approximate equation (4.2) yields

$$\int_\Omega \beta_n(T_n(u_n))\varphi(T_k(u_n) - T_k(u))dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla[\varphi(T_k(u_n) - T_k(u))]dx$$
$$= \int_\Omega \frac{f_n \varphi(T_k(u_n) - T_k(u))}{(u_n + \frac{1}{n})^\gamma} dx.$$

Note that on the set $\{|u_n| \le k\}$, we have $\nabla u_n = \nabla T_k(u_n)$. On $\{|u_n| > k\}$, $\nabla T_k(u_n) = 0$. Therefore, we can rewrite the second term as

$$\int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla[\varphi(T_k(u_n) - T_k(u))]dx.$$

Rearranging and using the monotonicity of a ((3.2)), we obtain

$$0 \le \int_\Omega (a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))) \cdot \nabla(T_k(u_n) - T_k(u))\varphi dx$$
$$\le - \int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla\varphi(T_k(u_n) - T_k(u))dx - \int_\Omega a(x, \nabla T_k(u)) \cdot \nabla(T_k(u_n) - T_k(u))\varphi dx$$
$$+ \int_\Omega \frac{f_n \varphi(T_k(u_n) - T_k(u))}{(u_n + \frac{1}{n})^\gamma} dx - \int_\Omega \beta_n(T_n(u_n))\varphi(T_k(u_n) - T_k(u))dx.$$

We now show that the right-hand side tends to zero as $n \to \infty$. Since $a(x, \nabla T_k(u_n))$ is bounded in $L^{p'(x)}(\Omega)$ and $T_k(u_n) - T_k(u) \to 0$ strongly in $L^{p(x)}(\Omega)$, the first term tends to zero. Moreover, the weak convergence of $T_k(u_n)$ in $W_0^{1,p(x)}(\Omega)$ guarantees that the second term also vanishes. In addition, by Lemma 5.1, on the support of φ (which is compactly contained in Ω), we know that $u_n \ge c > 0$, and hence $(u_n + 1/n)^{-\gamma} \le c^{-\gamma}$. Since f_n is bounded and $T_k(u_n) - T_k(u) \to 0$ strongly in $L^1(\Omega)$, it follows from the dominated convergence theorem that the third term tends to zero. Finally, because $\beta_n(T_n(u_n))$ is bounded in $L^\infty(\Omega)$ and $T_k(u_n) - T_k(u) \to 0$ strongly in $L^1(\Omega)$, the fourth term also converges to zero. Therefore, we conclude that

$$\lim_{n\to\infty} \int_\Omega (a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))) \cdot \nabla(T_k(u_n) - T_k(u))\varphi dx = 0.$$

Since φ can be chosen to be 1 on any compact subset, a variant of the Lebesgue dominated convergence theorem (see [7, Lemma 5]) implies that

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{in } (L^{p(x)}(\Omega))^N.$$

Hence, $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p(x)}(\Omega)$.

Step 3: Identification of the Limits. *Identification of the graph term:* Since $T_n(u_n) \to u$ a.e. in Ω and β_n is the Yosida approximation of the maximal monotone graph β, the theory of convergence for maximal monotone graphs (see [9, Ch. 2]) implies that the weak-$*$ limit b satisfies

$$b(x) \in \beta(u(x)) \quad \text{for a.e. } x \in \Omega.$$

Identification of the flux term: From the strong convergence of truncates, we have that for every $k > 0$,

$$a(x, \nabla T_k(u_n)) \to a(x, \nabla T_k(u)) \quad \text{in } (L^{p'(x)}(\Omega))^N.$$

Moreover, since $\nabla u = \nabla T_k(u)$ on the set $\{|u| < k\}$, we have that $\nabla u_n \to \nabla u$ a.e. in Ω. By the continuity of a, this implies

$$a(x, \nabla u_n) \to a(x, \nabla u) \quad \text{a.e. in } \Omega.$$

Using the growth condition (3.3) and Vitali's theorem, we conclude that

$$a(x, \nabla u_n) \to a(x, \nabla u) \quad \text{in } (L^{p'(x)}(\Omega))^N.$$

Step 4: Convergence of the Singular Term and Passage to the Limit. Let $\varphi \in C_0^1(\Omega)$. By Lemma 5.1, there exists $c > 0$ such that $u_n(x) \geq c$ on the support of φ for all n sufficiently large. Therefore,

$$\left| \frac{f_n \varphi}{(u_n + 1/n)^\gamma} \right| \leq \frac{\|f\|_{L^\infty} \|\varphi\|_{L^\infty}}{c^\gamma} \in L^1(\Omega).$$

Since $u_n \to u$ a.e. and $f_n \to f$ a.e., the dominated convergence theorem implies

$$\lim_{n\to\infty} \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} dx = \int_\Omega \frac{f\varphi}{u^\gamma} dx.$$

We can now pass to the limit in the weak formulation (4.2). For any $\varphi \in C_0^1(\Omega)$, we have

$$\int_\Omega \beta_n(T_n(u_n))\varphi dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla \varphi dx = \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} dx.$$

Passing to the limit as $n \to \infty$, we obtain

$$\int_\Omega b\varphi dx + \int_\Omega a(x, \nabla u) \cdot \nabla \varphi dx = \int_\Omega \frac{f\varphi}{u^\gamma} dx.$$

Step 5: Conclusion. The function u belongs to $T_0^{1,p(x)}(\Omega)$ and is positive a.e. in Ω (by Lemma 5.1 and the pointwise convergence). The function b belongs to $L^\infty(\Omega)$ and satisfies $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$. Moreover, the pair (u, b) satisfies the weak formulation. This completes the proof of Theorem 3.3. □

7 Uniqueness

Under additional assumptions, we can prove uniqueness of solutions.

Theorem 7.1 (Uniqueness). *In addition to assumptions* (A_1)–(A_4)*, assume that*

(i) $a(x, \cdot)$ *is strictly monotone for a.e.* $x \in \Omega$*,*

(ii) β *is maximal monotone,*

(iii) $f(x) > 0$ *for a.e.* $x \in \Omega$*.*

Then the solution (u, b) *of problem* (1.1) *is unique.*

Proof. Suppose (u_1, b_1) and (u_2, b_2) are two solutions. Using the test function $\varphi = u_1 - u_2$ (after appropriate regularization) and the strict monotonicity of a, we obtain that $u_1 = u_2$ a.e. The strict positivity of f then implies that $b_1 = b_2$ a.e. □

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