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MULTI-DECOMPOSITION OF PRODUCT GRAPHS INTO PATHS AND Y-TREES OF ORDER FIVE

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Abstract. Let K_n, $\overline{K_n}$, P_n, and Y_n respectively denote a complete graph, the complement of a complete graph, a path, and a Y-tree on n vertices. Let $G_1 \times G_2$ and $G_1 \otimes G_2$ denote the tensor product and the wreath product of the graphs G_1 and G_2, respectively. Let G be a given graph. Decomposition of the graph G is defined as the partition of the edge set of G into its subgraphs. We say that the graph G has a (H_1, H_2) multi-decomposition, if it can be decomposed into $\alpha \geq 0$ copies of H_1 and $\beta \geq 0$ copies of H_2, where H_1, H_2 are subgraphs of G. In this paper, we obtain necessary and sufficient conditions for the existence of (P_5, Y_5) - multi-decomposition of $K_m \times K_n$ and $K_m \otimes \overline{K_n}$.

1 Introduction

Every graph considered here is finite, loopless, and undirected. Let K_n denote a complete graph on n vertices, $K_{m,n}$ be a complete bipartite graph with partite sets of cardinality m and n, and P_k be a path on k vertices. A *Y-tree* on 5 vertices, denoted by Y_5, is a tree obtained from P_4 by attaching a pendant vertex to a 2 degree vertex. A 1-factor of a graph G is a spanning 1-regular subgraph of G and is denoted by I. We use $K_{m,m} - I$ to denote $K_{m,m}$ with a 1-factor removed. A *crown graph* $C_{n,l}$, $l \leq n$ is a bipartite graph on $2n$ vertices with partite sets $\{u_0, u_1, ..., u_{n-1}\}$ and $\{v_0, v_1, ..., v_{n-1}\}$ and edge set $\{u_i v_j : 1 \leq j - i \leq l \quad \text{with arithmetic modulo } n\}$. Note that $K_{m,m} - I$ is isomorphic to $C_{m,m-1}$.

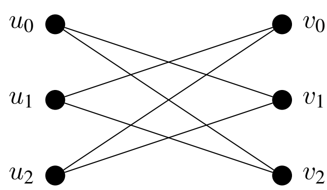

Figure 1: Crown graph $C_{3,2}$.

For two graphs G_1 and G_2, their *tensor product* $G_1 \times G_2$ has the set of vertices $V(G_1) \times V(G_2)$ and set of edges $E(G_1 \times G_2) = \{(g, h)(g', h') | gg' \in E(G_1) \text{ and } hh' \in E(G_2)\}$. The tensor product is commutative, associative, and distributive over the edge-disjoint union of graphs.

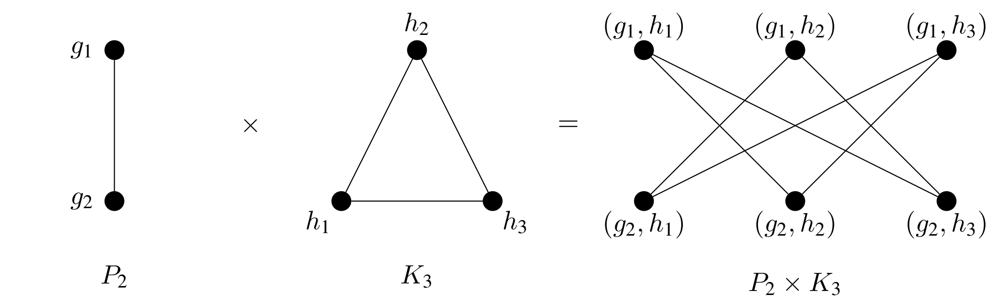

Figure 2: Tensor Product of P_2 and K_3.

Also, their *wreath product* $G_1 \otimes G_2$ has the vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(g,h)(g^{'},h^{'}) | \, gg^{'} \in E(G_1)$ or $g = g^{'}, \; hh^{'} \in E(G_2)\}$.

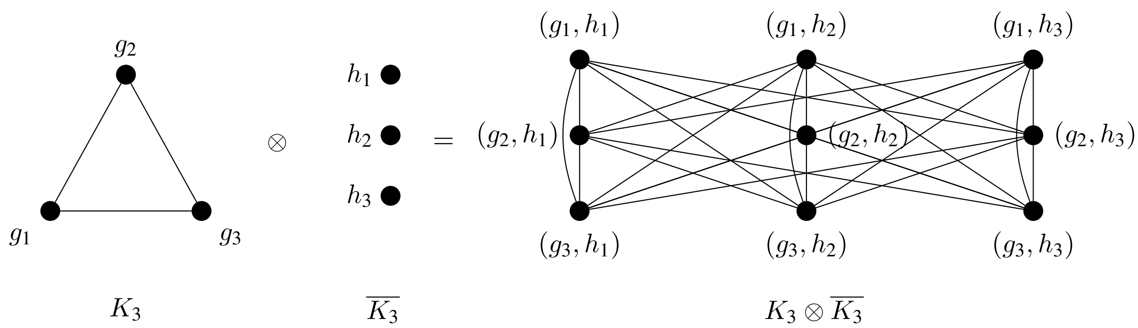

Figure 3: Wreath Product of K_3 and $\overline{K_3}$.

A *decomposition* of a graph G is the partition of G into an edge-disjoint set of subgraphs $H_1, H_2, ..., H_r$ of G such that every edge of G belongs to exactly one H_i, $1 \leq i \leq r$. If all subgraphs in the decomposition of G are isomorphic to a graph H, then we say that G is H-decomposable. If G can be decomposed into α copies of H_1 and β copies of H_2, then we say that G has a (H_1, H_2) - *multi-decomposition* or $\{H_1^{\alpha}, H_2^{\beta}\}$ - decomposition. We call the pair (α, β) *admissible*, if it satisfies the necessary condition $\alpha|E(H_1)| + \beta|E(H_2)| = |E(G)|$ with α, $\beta \geq 0$, for the existence of (H_1, H_2) - multi-decomposition. If there exists a (H_1, H_2) - multi-decomposition of G for all admissible pairs (α, β), we say G has a $(H_1, H_2)_{\{\alpha,\beta\}}$ - decomposition.

Graph decomposition is a fundamental area of research in combinatorics and graph theory, providing structural insights and applications across various domains, such as network design, parallel computing, biology, chemistry, and communication systems. The concept of multi-decomposition, introduced by Abueida and Daven [1], has gained increasing attention in recent years. Decomposition of product graphs into paths and cycles on five vertices was obtained by Jeevadoss and Muthusamy [7]. Ilayaraja et al. [6] established necessary and sufficient conditions for the existence of the decomposition of $K_m \times K_n$ and $K_m \otimes \overline{K_n}$ into paths and stars of order five. The same problem with four vertices was studied by Ezhilarasi et al. [4]. Multi-decomposition of the tensor product of complete graphs into various structures, such as cycles and stars, was considered in [3] and [9]. Some related studies are [5] and [10]. The results presented in [2] motivate the authors to the present investigation on (P_5, Y_5) - multi-decomposition of $K_m \times K_n$ and $K_m \otimes \overline{K_n}$. To prove our results, we recall the following.

Theorem 1.1. *([11]) Let k, l, and n be positive integers. $C_{n,l}$ is P_{k+1}-decomposable if and only if $nl \equiv 0 \; (mod \;\; k)$ and*

$$k \leq \begin{cases} 2, & if \;\, n = l = 2 \\ 2n - 3, & if \;\, l \; is \; even \; and \; l \geq 3 \\ l, & if \;\, l \; is \; odd. \end{cases}$$

Theorem 1.2. *([2]) There exists a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_{m,n}$ if and only if any one of the following holds:*
1. $m = 2k, k$ is even, $n = 2$ and α is even
2. $m = 2k, k \geq 3$ is odd, $n = 2$ and α is odd
3. $m = 4k$ and $n \geq 3$
4. $m = 2k_1$ and $n = 2k_2$; where $k_1, k_2 \geq 3$ are odd

Theorem 1.3. *([2]) For non-negative integers α, β, and $n \geq 8$, $K_n = \alpha P_5 \oplus \beta Y_5$ if and only if $4(\alpha + \beta) = \binom{n}{2}$.*

Definition 1.4. ([8]) A *Steiner triple system* is an ordered pair (S, T), where S is a finite set of points (or symbols) and T is a collection of 3-element subsets of S, called *triples*, such that each pair of distinct elements of S occurs together in exactly one triple of T. The *order* of a Steiner triple system (S, T) is the cardinality of the set S.

Theorem 1.5. *([8]) A Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$. Note that $|T| = \frac{v(v-1)}{6}$ if $|S| = v$.*

Notations:

- rG denotes r disjoint copies of graph G.
- We write $G = H_1 \oplus H_2$ if G can be decomposed into H_1 and H_2.
- We denote the vertices of the first partite set with m vertices of $K_{m,m} - I$ by 1_i, and the vertices of the second partite set with m vertices by 2_i, where $1 \leq i \leq m$. Note that $K_{m,m} - I \cong K_2 \times K_m$.
- Let the m-partite graphs $P_m \times K_n$, $K_m \times K_n$, and $K_m \otimes \overline{K_n}$ have the vertex set $\{i_j : 1 \leq i \leq m$ and $1 \leq j \leq n\}$.
- $G - v$ is the graph obtained by deleting the vertex v from G. We use $\mathcal{K}$ to denote the graph $G - 1_5$ if $G = K_{5,5} - I$.
- P_5 and Y_5 denote a path and a Y-tree on 5 vertices, respectively.

2 Preliminaries

We recall the graph structure T on 8 edges, defined in [2] and shown in Figure 4(a), which can be decomposed into two P_5, two Y_5, and one P_5 & Y_5. Here we introduce some graphs T_i, $1 \leq i \leq 11$, on 8 edges, which are useful in proving our results and have the same properties as T.

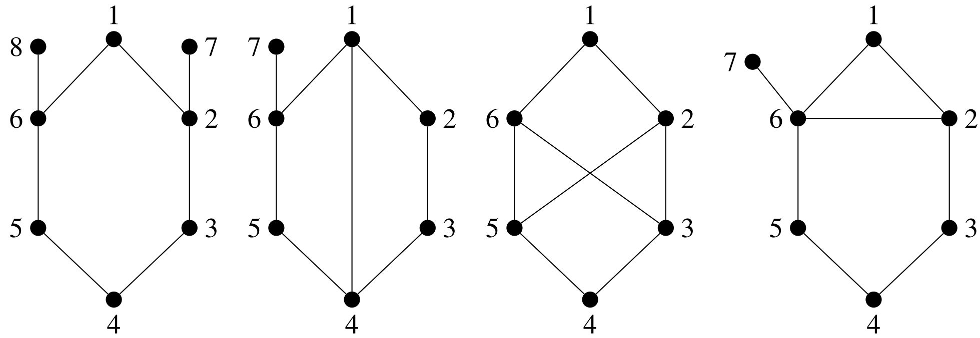

Fig. 4(a): Graph T Fig. 4(b): Graph T_1 Fig. 4(c): Graph T_2 Fig. 4(d): Graph T_3

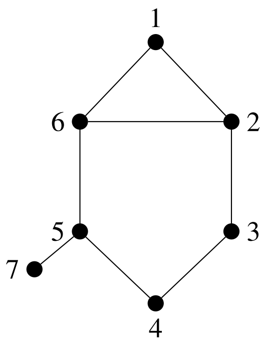

Fig. 4(e): Graph T_4

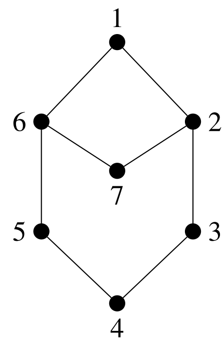

Fig. 4(f): Graph T_5

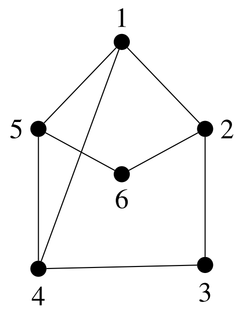

Fig. 4(g): Graph T_6

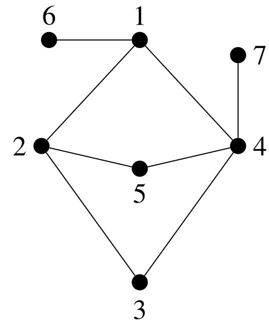

Fig. 4(h): Graph T_7

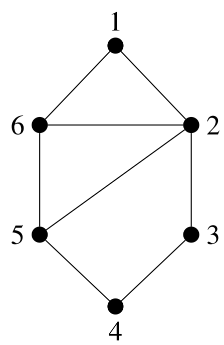

Fig. 4(i): Graph T_8

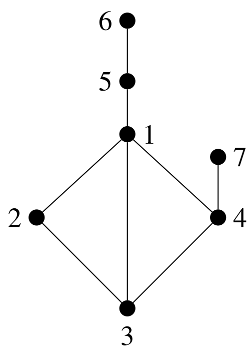

Fig. 4(j): Graph T_9

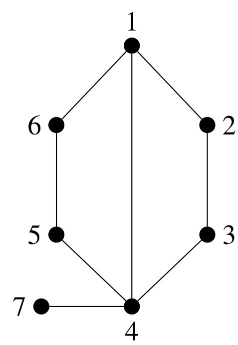

Fig. 4(k): Graph T_{10}

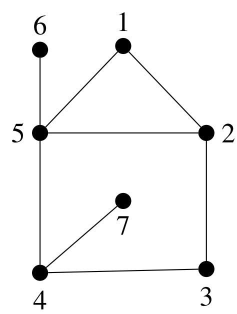

Fig. 4(l): Graph T_{11}

Figure 4: Graphs T to T_{11}

Figure No.	$2P_5$	$2Y_5$	$1P_5 \oplus 1Y_5$
4(a)	$\{86,61,12,27\}$, $\{65,54,43,32\}$	$\{61,68,65,54\}$, $\{21,27,23,34\}$	$\{72,23,34,45\}$, $\{65,68,61,12\}$
4(b)	$\{14,45,56,67\}$, $\{61,12,23,34\}$	$\{67,65,61,12\}$, $\{41,45,43,32\}$	$\{12,23,34,45\}$, $\{67,65,61,14\}$
4(c)	$\{56,61,12,23\}$, $\{63,34,45,52\}$	$\{61,65,63,34\}$, $\{21,23,25,54\}$	$\{61,12,25,54\}$, $\{32,34,36,65\}$
4(d)	$\{21,16,65,54\}$, $\{76,62,23,34\}$	$\{67,61,65,54\}$, $\{21,26,23,34\}$	$\{62,23,34,45\}$, $\{65,67,61,12\}$
4(e)	$\{21,16,65,57\}$, $\{62,23,34,45\}$	$\{21,26,23,34\}$, $\{54,57,56,61\}$	$\{61,12,23,34\}$, $\{54,57,56,62\}$
4(f)	$\{21,16,65,54\}$, $\{67,72,23,34\}$	$\{21,27,23,34\}$, $\{61,67,65,54\}$	$\{12,23,34,45\}$, $\{61,65,67,72\}$
4(g)	$\{51,14,43,32\}$, $\{12,26,65,54\}$	$\{41,43,45,56\}$, $\{23,26,21,15\}$	$\{15,56,62,23\}$, $\{43,45,41,12\}$
4(h)	$\{61,14,43,32\}$, $\{12,25,54,47\}$	$\{21,23,25,54\}$, $\{43,47,41,16\}$	$\{61,12,25,54\}$, $\{41,47,43,32\}$
4(i)	$\{61,12,25,54\}$, $\{56,62,23,34\}$	$\{61,62,65,54\}$, $\{21,25,23,34\}$	$\{12,25,54,43\}$, $\{65,61,62,23\}$
4(j)	$\{65,51,13,34\}$, $\{74,41,12,23\}$	$\{12,14,15,56\}$, $\{31,32,34,47\}$	$\{23,31,15,56\}$, $\{43,47,41,12\}$
4(k)	$\{56,61,14,47\}$, $\{12,23,34,45\}$	$\{12,14,16,65\}$, $\{45,47,43,32\}$	$\{23,34,45,56\}$, $\{12,16,14,47\}$
4(l)	$\{74,45,51,12\}$, $\{65,52,23,34\}$	$\{56,51,54,47\}$, $\{21,25,23,34\}$	$\{65,51,12,23\}$, $\{43,47,45,52\}$

Remark 2.1. If two graphs G_1 and G_2 have (H_1, H_2) multi-decomposition, then $G_1 \oplus G_2$ has such a decomposition.

3 (P_5, Y_5) - multi-decomposition of $K_{m,m} - I$

In this section, we obtain necessary and sufficient conditions for the existence of (P_5, Y_5) - multi-decomposition of $K_{m,m} - I$, $m \geq 4$, as follows.

Lemma 3.1. *For $(\alpha, \beta) \neq (3, 0)$, $K_{4,4} - I$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 3$.*

Proof. We can write $K_{4,4} - I = 1T_1 \oplus 1Y_5$. The edge sets of T_1 and Y_5 are: $\{2_41_1, 1_12_3, 2_31_2, 1_22_1, 2_11_4, 1_42_2, 1_12_2, 2_31_4\}$ and $\{1_22_4, 2_41_3, 1_32_2, 1_32_1\}$ respectively. By Remark 2.1, we have a (P_5, Y_5) -multi-decomposition for the admissible pairs (α, β) when $\alpha + \beta = 3$ except $(3, 0)$, since by Theorem 1.1, $K_{4,4} - I$ cannot be decomposed into $3P_5$. □

Lemma 3.2. *The graph $K_{5,5} - I$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 5$.*

Proof. We can write $K_{5,5} - I = 1T \oplus 3P_5$. The edge sets of the required copies of T and P_5 are: $\{2_41_2, 1_22_3, 2_31_1, 1_12_5, 2_51_3, 1_32_2, 1_22_1, 1_32_1\}$,$\{2_21_4, 1_42_3, 2_31_5, 1_52_4\}$,$\{1_32_4, 2_41_1, 1_12_2, 2_21_5\}$, $\{1_52_1, 2_11_4, 1_42_5, 2_51_2\}$. By Remark 2.1, we have a (P_5, Y_5) - multi- decomposition for the admissible pairs $(\alpha, \beta) \in \{(5, 0), (4, 1), (3, 2)\}$. Also, we can write $K_{5,5} - I = 1T \oplus 3Y_5$ and the corresponding edge sets of the requied copies of T and Y_5 are as follows: $\{2_21_1, 1_12_3, 2_31_4, 1_42_5, 2_5 1_3, 1_32_1, 1_32_4, 1_12_4\}$,$\{1_12_5, 2_51_2, 1_22_4, 1_22_3\}$,$\{2_31_5, 1_52_2, 2_21_4, 2_21_3\}$,$\{2_41_5, 1_52_1, 2_11_4, 2_11_2\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(0, 5), (1, 4), (2, 3)\}$. Hence, $K_{5,5} - I$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 5$. □

Lemma 3.3. *The graph $\mathcal{K}$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 4$.*

Proof. We can write $\mathcal{K} = 1T_1 \oplus 2P_5$. The corresponding edge sets are: $\{1_12_3, 2_31_4, 1_42_2, 2_21_3, 1_3 2_1, 2_11_2, 2_31_2, 1_42_1\}$,$\{2_21_1, 1_12_5, 2_51_2, 1_22_4\}$,$\{1_12_4, 2_41_3, 1_32_5, 2_51_4\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(4, 0), (3, 1), (2, 2)\}$. Also, we can write $\mathcal{K} = 1T_2 \oplus 2Y_5$. The edge sets of T_2 and $2Y_5$ are: $\{1_12_2, 2_21_3, 1_32_5, 2_51_4, 1_42_3, 2_31_1, 1_12_5, 2_21_4\}$,$\{2_51_2, 1_22_4, 2_41_3, 2_41_1\}$,$\{2_31_2, 1_22_1, 2_11_4, 2_11_3\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(0, 4), (1, 3), (2, 2)\}$. Hence, $\mathcal{K}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 4$. □

Lemma 3.4. *For $m \equiv 0$ or $1 (mod\ 4)$ and $m \neq 4$, $K_{m,m} - I$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition.*

Proof. **Case 1**: $m \equiv 0\ (mod\ 4)$. Let $m = 4p$ for $p \in \mathbb{N}$ and $p \neq 1$
$K_{4p,4p} - I = p\mathcal{K} \oplus pK_{4,4p-5}$
Case 2: $m \equiv 1\ (mod\ 4)$. Let $m = 4p + 1$ for $p \in \mathbb{N}$
$K_{4p+1,4p+1} - I = K_{5,5} - I \oplus (p-1)K_{5,4} \oplus (p-1)\mathcal{K} \oplus (p-1)^2K_{4,4}$
By Lemmas 3.2 and 3.3, Theorem 1.2 and Remark 2.1, we have a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_{m,m} - I$, except $m = 4$. □

Theorem 3.5. *For non-negative integers α, β and $m \geq 4$, $K_{m,m} - I$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition if and only if $m \equiv 0$ or $1 (mod\ 4)$ and $\beta \neq 0$ when $m = 4$.*

Proof. Necessity follows from the edge divisibility condition and sufficiency from the Theorem 1.1, Lemmas 3.1 and 3.4. □

4 (P_5, Y_5) - multi-decomposition of $K_m \times K_n$

In this section, we obtain necessary and sufficient conditions for the existence of (P_5, Y_5) - multi-decomposition of $K_m \times K_n$, $m \geq 4$ and $n \geq 2$, as follows.

Lemma 4.1. *The graph $K_4 \times K_3$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 9$.*

Proof. We can write $K_4 \times K_3 = 3T_2 \oplus 3P_5$. The edge sets of the required copies of T_2 and P_5 are: $\{3_11_3, 1_32_1, 2_11_2, 1_24_1, 4_12_2, 2_23_1, 3_11_2, 1_34_1\}$,$\{3_21_3, 1_32_2, 2_21_1, 1_14_2, 4_22_1, 2_13_2, 3_21_1, 1_34_2\}$,$\{3_31_2, 1_22_3, 2_31_1, 1_14_3, 4_32_1, 2_13_3, 3_31_1, 1_24_3\}$,$\{3_12_3, 2_34_1, 4_13_2, 3_24_3\}$,$\{4_23_1, 3_14_3, 4_32_2, 2_23_3\}$,$\{3_22_3, 2_34_2, 4_23_3, 3_34_1\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(3,6), (4,5), (5,4), (6,3), (7,2), (8,1), (9,0)\}$. Also, we can write $K_4 \times K_3 = 1T \oplus 2T_3 \oplus 3Y_5$ and the edge sets of the required copies of T, T_3 and Y_5 are: $\{3_11_3, 1_34_2, 4_23_3, 3_32_2, 2_24_3, 4_32_1, 4_33_2, 1_33_2\}$,$\{4_11_2, 1_22_1, 2_13_2, 3_21_1, 1_14_3, 4_33_1, 3_11_2, 1_24_3\}$,$\{1_12_2, 2_23_1, 3_14_2, 4_22_1, 2_11_3, 1_34_1, 4_12_2, 2_21_3\}$,$\{2_13_3, 3_31_1, 1_12_3, 1_14_2\}$,$\{3_12_3, 2_34_1, 4_13_3, 4_13_2\}$,$\{3_31_2, 1_22_3, 2_33_2, 2_34_2\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(0,9), (1,8), (2,7), (3,6), (4,5), (5,4), (6,3)\}$. Hence, $K_4 \times K_3$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 9$. □

Lemma 4.2. *The graph $K_4 \times K_4$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 18$.*

Proof. We can write $K_4 \times K_4 = 2T \oplus 2T_1 \oplus 1T_2 \oplus 1T_4 \oplus 1T_5 \oplus 1T_6 \oplus 1T_7$. $\{4_21_4, 1_44_3, 4_32_2, 2_23_3, 3_32_4, 2_43_2, 1_44_1, 2_44_1\}$,$\{4_42_1, 2_14_2, 4_21_1, 1_14_3, 4_31_2, 1_23_4, 2_13_3, 1_23_3\}$,$\{2_33_4, 3_44_3, 4_33_2, 3_21_4, 1_42_1, 2_11_3, 3_41_3, 4_32_1\}$,$\{4_11_3, 1_33_1, 3_14_2, 4_22_3, 2_31_4, 1_42_2, 1_32_2, 3_11_4\}$,$\{3_12_2, 2_24_4, 4_42_3, 2_34_1, 4_11_2, 1_23_1, 3_12_3, 2_24_1\}$,$\{3_34_4, 4_41_3, 1_34_2, 4_22_4, 2_44_3, 4_33_1, 4_43_1, 3_12_4\}$, $\{3_24_4, 4_41_1, 1_12_2, 2_23_4, 3_42_1, 2_11_2, 1_24_4, 2_13_2\}$,$\{1_13_2, 3_22_3, 2_31_2, 1_22_4, 2_41_3, 2_31_1, 1_12_4, 3_21_3\}$,$\{3_24_1, 4_13_4, 3_41_1, 1_13_3, 3_31_4, 4_13_3, 3_44_2, 3_34_2\}$ are the respective edge sets of required copies of T and T_i, $1 \leq i \leq 7$ and $i \neq 3$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for all the admissible pairs whenever $\alpha + \beta = 18$. □

Lemma 4.3. *The graph $P_3 \times K_6$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 15$.*

Proof. We can write $P_3 \times K_6 = 6T \oplus 3P_5$. The edge sets of $6T$ and $3P_5$ are: $\{1_32_1, 2_11_2, 1_22_3, 2_31_1, 1_12_2, 2_21_4, 1_52_1, 1_52_2\}$,$\{3_22_1, 2_13_4, 3_42_2, 2_23_1, 3_12_3, 2_31_5, 2_11_4, 2_31_4\}$,$\{1_32_2, 2_23_3, 3_32_1, 2_13_6, 3_62_3, 2_31_6, 2_23_5, 2_33_5\}$,$\{1_32_4, 2_41_5, 1_52_6, 2_61_4, 1_42_5, 2_51_6, 2_43_1, 2_53_1\}$,$\{1_62_4, 2_43_3, 3_32_5, 2_53_4, 3_42_6, 2_61_2, 2_43_5, 2_63_5\}$,$\{1_22_5, 2_53_6, 3_62_4, 2_41_1, 1_12_6, 2_63_1, 2_53_2, 2_63_2\}$,$\{3_52_1, 2_11_6, 1_62_2, 2_23_6\}$,$\{1_22_4, 2_43_2, 3_22_3, 2_33_4\}$,$\{1_12_5, 2_51_3, 1_32_6, 2_63_3\}$. By Remark 2.1, we have a (P_5, Y_5) - multi- decomposition for the admissible pairs $(\alpha, \beta) \in \{(3,12), (4,11), (5,10), (6,9), (7,8), (8,7), (9,6), (10,5), (11,4), (12,3), (13,2), (14,1), (15,0)\}$. Also, we can write $P_3 \times K_6 = 6T \oplus 3Y_5$. The edge sets of $6T$ and $3Y_5$ are: $\{1_32_1, 2_11_6, 1_62_3, 2_31_4, 1_42_6, 2_63_1, 2_13_3, 2_63_3\}$,$\{1_42_1, 2_11_5, 1_52_3, 2_33_1, 3_12_2, 2_21_3, 2_13_6, 2_23_6\}$,$\{1_62_2, 2_23_4, 3_42_1, 2_11_2, 1_22_4, 2_43_3, 2_21_5, 2_41_5\}$,$\{1_12_4, 2_41_3, 1_32_6, 2_63_4, 3_42_5, 2_51_4, 2_41_6, 2_51_6\}$,$\{1_12_5, 2_53_1, 3_12_4, 2_43_5, 3_52_6, 2_61_5, 2_53_2, 2_63_2\}$,$\{1_42_2, 2_21_1, 1_12_6, 2_61_2, 1_22_5, 2_51_3, 2_23_3, 2_53_3\}$,$\{3_42_3, 2_33_5, 3_52_2, 3_52_1\}$,$\{3_22_4, 2_43_6, 3_62_5, 3_62_3\}$,$\{2_13_2, 3_22_3, 2_31_2, 2_31_1\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(0,15), (1,14), (2,13), (3,12), (4,11), (5,10), (6,9), (7,8), (8,7), (9,6), (10,5), (11,4), (12,3)\}$. Hence, $P_3 \times K_6$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 15$. □

Lemma 4.4. *The graph $K_4 \times K_6$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 45$.*

Proof. Since $K_4 \times K_6 = 3(P_3 \times K_6)$, the proof follows from Lemma 4.3 for all admissible pairs (α, β) such that $\alpha + \beta = 45$. □

Lemma 4.5. *Let $m \equiv 0 \ (mod\ 4)$ and $n \equiv 0 \ (mod\ 2)$. Then there exists a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_m \times K_n$ except $\beta = 0$ when $m = 4$ and $n = 2$.*

Proof. Let $m = 4p$ and $n = 2q$ for $p, q \in \mathbb{N}$
Case 1: $p = 1$
When $q = 1$, $K_4 \times K_2 \cong K_{4,4} - I$. For $q \neq 1$, and q is even, $q = 2s$ for $s \in \mathbb{N}$. We can write $K_4 \times K_{2q} = K_4 \times K_{4s} = K_4 \times K_4 \oplus 4K_{4,3(4s-4)} \oplus K_4 \times K_{4(s-1)}$. By Lemma 4.2 and Theorem 1.2, the recursive relation gives a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_4 \times K_{4s}$.

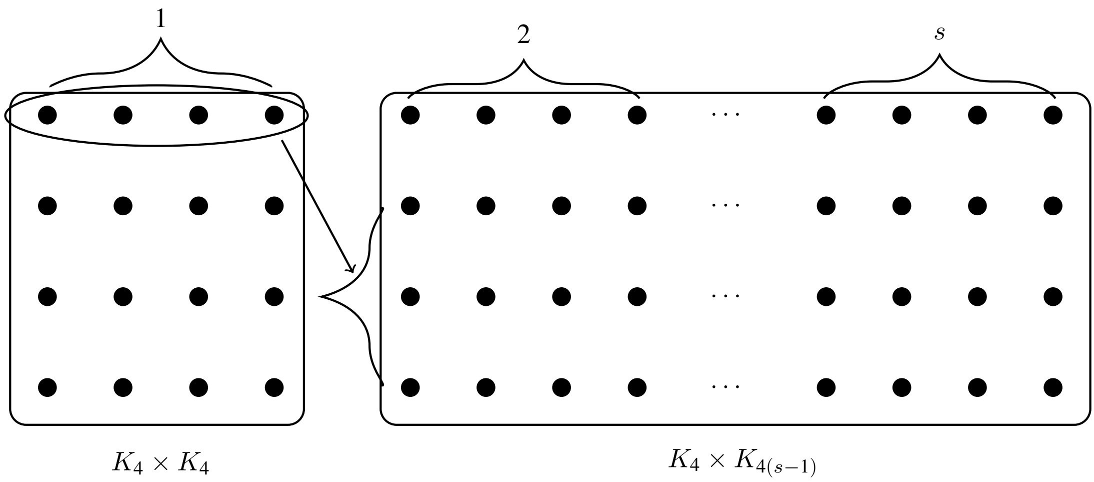

Figure 5: $K_4 \times K_{4s} = K_4 \times K_4 \oplus 4K_{4,3(4s-4)} \oplus K_4 \times K_{4(s-1)}$.

In the above Figure, $\rightarrow$ denotes a $K_{4,3(4s-4)}$. Similarly, the edges between $K_4 \times K_{4(s-1)}$ and each row of $K_4 \times K_4$ give a $K_{4,3(4s-4)}$, and hence a total of $4K_{4,3(4s-4)}$.

Suppose q is odd, $q = 2s+1$ for $s \in \mathbb{N}$. We can write $K_4 \times K_{2q} = K_4 \times K_{4s+2} = K_4 \times K_6 \oplus 4K_{6,12(s-1)} \oplus K_4 \times K_{4(s-1)}$. By Lemma 4.4, Theorem 1.2 and the above case where $n = 4s$, we have the required $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_4 \times K_{4s+2}$.

Case 2: $p \neq 1$
We can write $K_{4p} \times K_{2q} = q(2q-1)(K_{4p,4p} - I)$. By Theorem 3.5, $K_{4p,4p} - I$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition and by Remark 2.1, $K_{4p} \times K_{2q}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition. □

Lemma 4.6. *Let* $m \equiv 0 \ (mod\ 4)$ *and* $n \equiv 1 \ (mod\ 2)$. *Then there exists a* $(P_5, Y_5)_{\{\alpha,\beta\}}$ *- decomposition of* $K_m \times K_n$.

Proof. Let $m = 4p$ and $n = 2q+1$ for $p, q \in \mathbb{N}$
Case 1: $p = 1$
If $q \equiv 0 \ (mod\ 3)$, then $n = 6s+1$ for $s \in \mathbb{N}$. We can write $K_4 \times K_{6s+1} = s(6s+1)(K_4 \times K_3)$. Similarly, if $q \equiv 1 \ (mod\ 3)$, then $n = 6s+3$ for $s \in \mathbb{N} \cup \{0\}$. We can write $K_4 \times K_{6s+3} = (2s+1)(3s+1)(K_4 \times K_3)$. By Lemma 4.1 and Remark 2.1, we have the required decomposition. The proof of the above two decompositions follows from Theorem 1.5, where we have taken $S = \{\{1_1, 2_1, 3_1, 4_1\}, \{1_2, 2_2, 3_2, 4_2\}, ..., \{1_{6s+1}, 2_{6s+1}, 3_{6s+1}, 4_{6s+1}\}\}$ and $S = \{\{1_1, 2_1, 3_1, 4_1\}, \{1_2, 2_2, 3_2, 4_2\}, ..., \{1_{6s+3}, 2_{6s+3}, 3_{6s+3}, 4_{6s+3}\}\}$ respectively. Suppose $q \equiv 2 \ (mod\ 3)$, then $n = 6s+5$ for $s \in \mathbb{N} \cup \{0\}$. We can write $K_4 \times K_{6s+5} = K_4 \times K_4 \oplus K_4 \times K_{6s+1} \oplus 4K_{4,3(6s+1)}$. By Lemma 4.2, Theorem 1.2 and the above case where $n = 6s+1$, $K_4 \times K_{6s+5}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition.
Case 2: $p \neq 1$
We can write $K_{4p} \times K_{2q+1} = q(2q+1)(K_{4p,4p} - I)$. By Theorem 3.5, $K_{4p,4p} - I$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition and by Remark 2.1, $K_m \times K_n$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition. □

Lemma 4.7. *Let* $m \equiv 1 \ (mod\ 4)$ *and* $n \equiv 0 \ (mod\ 2)$. *Then there exists a* $(P_5, Y_5)_{\{\alpha,\beta\}}$ *- decomposition of* $K_m \times K_n$.

Proof. Let $m = 4p+1$ and $n = 2q$ for $p, q \in \mathbb{N}$. We can write $K_{4p+1} \times K_{2q} = q(2q-1)(K_{4p+1,4p+1} - I)$. By Theorem 3.5, $K_{4p+1,4p+1} - I$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition and by Remark 2.1, $K_m \times K_n$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition. □

Lemma 4.8. *Let* $m \equiv 1 \ (mod\ 4)$ *and* $n \equiv 1 \ (mod\ 2)$. *Then there exists a* $(P_5, Y_5)_{\{\alpha,\beta\}}$ *- decomposition of* $K_m \times K_n$.

Proof. Let $m = 4p+1$ and $n = 2q+1$ for $p, q \in \mathbb{N}$. We can write $K_{4p+1} \times K_{2q+1} = q(2q+1)(K_{4p+1,4p+1} - I)$. By Theorem 3.5, $K_{4p+1,4p+1} - I$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition and by Remark 2.1, $K_m \times K_n$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition. □

Theorem 4.9. *For $m \geq 4$ and $n \geq 2$, $K_m \times K_n$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition if and only if one of the following holds.*

(1). $m \equiv 0 \ (mod\ 4)$ *and* $n \equiv 0 \ (mod\ 2)$, *except* $\beta = 0$ *when* $m = 4$ *and* $n = 2$

(2). $m \equiv 0 \ (mod\ 4)$ *and* $n \equiv 1 \ (mod\ 2)$

(3). $m \equiv 1 \ (mod\ 4)$ *and* $n \equiv 0 \ (mod\ 2)$

(4). $m \equiv 1 \ (mod\ 4)$ *and* $n \equiv 1 \ (mod\ 2)$

Proof. Necessity follows from edge divisibility condition, $mn(m-1)(n-1) \equiv 0 \ (mod\ 8)$. Sufficiency follows from Lemmas 4.5 to 4.8. □

5 (P_5, Y_5) - multi-decomposition of $K_m \otimes \overline{K_n}$

In this section, we obtain necessary and sufficient conditions for the existence of (P_5, Y_5) - multi-decomposition of $K_m \otimes \overline{K_n}$, $m, n \geq 2$, as follows.

Lemma 5.1. *The graph $K_3 \otimes \overline{K_2}$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 3$.*

Proof. We can write $K_3 \otimes \overline{K_2} = 1T_8 \oplus 1P_5$. The edge sets of T_8 and P_5 are: $\{3_1 1_2, 1_2 3_2, 3_2 2_2, 2_2 1_1, 1_1 2_1, 3_1 2_2, 3_1 1_1, 3_1 2_1\}$,$\{1_1 3_2, 3_2 2_1, 2_1 1_2, 1_2 2_2\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(1,2), (2,1), (3,0)\}$. Also, We can write $K_3 \otimes \overline{K_2} = 3Y_5$ and the edge sets of $3Y_5$ are: $\{1_1 3_2, 3_2 1_2, 1_2 2_1, 1_2 2_2\}$,$\{3_2 2_2, 2_2 3_1, 3_1 1_2, 3_1 1_1\}$,$\{2_2 1_1, 1_1 2_1, 2_1 3_2, 2_1 3_1\}$. Thus, we have a (P_5, Y_5) - multi-decomposition for the admissible pair $(0,3)$. Hence, $K_3 \otimes \overline{K_2}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 3$. □

Lemma 5.2. *The graph $K_4 \otimes \overline{K_2}$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 6$.*

Proof. We can write $K_4 \otimes \overline{K_2} = 1T \oplus 1T_9 \oplus 1T_{11}$. The edge sets of the required copies of T, T_9 and T_{11} are: $\{3_1 4_2, 4_2 2_2, 2_2 1_2, 1_2 2_1, 2_1 4_1, 4_1 1_1, 3_2 4_2, 3_2 4_1\}$,$\{3_1 2_1, 2_1 4_2, 1_2 4_1, 4_1 2_2, 2_2 1_1, 3_1 4_1, 3_1 2_2, 3_1 1_1\}$,$\{2_2 3_2, 3_2 2_1, 2_1 1_1, 1_1 4_2, 4_2 1_2, 1_2 3_1, 3_2 1_1, 3_2 1_2\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for all admissible pairs. Hence, $K_4 \otimes \overline{K_2}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 6$. □

Lemma 5.3. *The graph $K_5 \otimes \overline{K_2}$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 10$.*

Proof. We can write $K_5 \otimes \overline{K_2} = 2T_1 \oplus 1T_2 \oplus 1T_6 \oplus 1T_{10}$. The edge sets of the required copies of T_1, T_2, T_6 and T_{10} are: $\{4_2 2_1, 2_1 3_2, 3_2 1_2, 1_2 4_1, 4_1 2_2, 2_2 3_1, 2_1 3_1, 3_2 2_2\}$,$\{4_2 2_2, 2_2 1_1, 1_1 3_2, 3_2 4_1, 4_1 3_1, 3_1 1_2, 2_2 1_2, 1_1 3_1\}$,$\{3_1 5_1, 5_1 2_2, 2_2 5_2, 5_2 3_2, 3_2 4_2, 3_1 4_2, 3_1 5_2, 5_1 3_2\}$, $\{5_2 4_2, 4_2 1_1, 1_1 4_1, 4_1 2_1, 2_1 5_1, 5_2 1_1, 5_2 2_1, 4_2 5_1\}$,$\{2_1 1_1, 1_1 5_1, 5_1 4_1, 4_1 5_2, 5_2 1_2, 1_2 4_2, 2_1 1_2, 5_1 1_2\}$. By Remark 2.1, we have a (P_5, Y_5) - multi-decomposition for all admissible pairs. Hence, $K_5 \otimes \overline{K_2}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 10$. □

Lemma 5.4. *$K_m \otimes \overline{K_2}$ admits a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition except when $m = 2$.*

Proof. **Case 1:** $m \equiv 0 \ (mod\ 2)$ i.e., $m = 2r$ for $r \in \mathbb{N}$
Suppose $m = 2$, then $K_m \otimes \overline{K_n} \cong C_4$, where C_4 is a cycle of length four.
Therefore, there doesn't exist a P_5 or Y_5 decomposition. Hence, we take $m \geq 4$
Subcase 1: r is even i.e., $m = 4p$ for $p \in \mathbb{N}$
We can write $K_m \otimes \overline{K_2} = p(K_4 \otimes \overline{K_2}) \oplus \frac{p(p-1)}{2} K_{8,8}$. By Lemma 5.2, Theorem 1.2 and Remark 2.1, $K_m \otimes \overline{K_2}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition.
Subcase 2: r is odd i.e., $m = 4p+2$ for $p \in \mathbb{N}$
We can write $K_m \otimes \overline{K_2} = (p-1)(K_4 \otimes \overline{K_2}) \oplus 2(K_3 \otimes \overline{K_2}) \oplus (p-1)K_{8,12} \oplus \frac{(p-1)(p-2)}{2} K_{8,8} \oplus K_{6,6}$. By Lemmas 5.1 and 5.2, Theorem 1.2 and Remark 2.1, $K_m \otimes \overline{K_2}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition.
Case 2: $m \equiv 1 \ (mod\ 2)$,

Subcase 1: $m \neq 5$
We can write $K_m \otimes \overline{K_2} = K_{m-3} \otimes \overline{K_2} \oplus K_3 \otimes \overline{K_2} \oplus K_{2(m-3),6}$. Since $m-3$ is even, Case 1 gives the $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_{m-3} \otimes \overline{K_2}$. By Lemma 5.1, Theorem 1.2 and Remark 2.1, $K_m \otimes \overline{K_2}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition.
Subcase 2: $m = 5$
Lemma 5.3 gives the required decomposition. □

Lemma 5.5. *There exists a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_m \otimes \overline{K_n}$ if one of the following holds.*

(i). $m = 2p + r$, $r \in \{0, 1\}$, $p \geq 1$ *and* $n \geq 4$ *is even*

(ii). $m = 8p + r$, $r \in \{0, 1\}$, $p \geq 1$ *and* $n \geq 3$ *is odd*

Proof. (i). We can write $K_m \otimes \overline{K_n} = \frac{m(m-1)}{2} K_{n,n}$. Therefore, by Theorem 1.2 and Remark 2.1, we have a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_m \otimes \overline{K_n}$.
(ii). We can write $K_m \otimes \overline{K_n} = K_m \times K_n \oplus \ nK_m$. Therefore, by Theorems 1.3, 4.9 and Remark 2.1, we have a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_m \otimes \overline{K_n}$. □

Theorem 5.6. *For $m, n \geq 2$, $K_m \otimes \overline{K_n}$ has a $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition if and only if one of the following holds.*

(1). $m \equiv 0 \ (mod\ 2)$ *and* $n \equiv 0 \ (mod\ 2)$, *except* $m = n = 2$

(2). $m \equiv 1 \ (mod\ 2)$ *and* $n \equiv 0 \ (mod\ 2)$

(3). $m \equiv 0 \ (mod\ 8)$ *and* $n \equiv 1 \ (mod\ 2)$

(4). $m \equiv 1 \ (mod\ 8)$ *and* $n \equiv 1 \ (mod\ 2)$

Proof. Necessity follows from edge divisibility condition, $mn^2(m-1) \equiv 0 \ (mod\ 8)$. Sufficiency follows from Lemmas 5.4 and 5.5. □

6 Conclusion

In this paper, it is proved that a necessary and sufficient condition for the existence of (P_5, Y_5) - multi-decomposition of $K_m \times K_n$, $m \geq 4$, and $n \geq 2$ is $mn(m-1)(n-1) \equiv 0 \ (mod\ 8)$, except $\beta = 0$ when $m = 4$ and $n = 2$. Also, it is proved that a necessary and sufficient condition for the existence of (P_5, Y_5) - multi-decomposition of $K_m \otimes \overline{K_n}$, $m, n \geq 2$, is $mn^2(m-1) \equiv 0 \ (mod\ 8)$, except when $m = n = 2$.

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