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A study on $*-$reversible modules

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Abstract In this paper, we introduce the concept of $*-$reversible modules as an extension of $*-$reversible rings to modules. We study $*-$reversible modules as modules whose endomorphism rings are $*-$reversible rings. Moreover, it demonstrates that while reduced modules are reversible, but not necessarily be $*-$reversible modules. Additionally, the definition of a $*-$reversible module is generalized and extended to various combinations of f and g. Properties associated with $*-$reversible rings are also generalized to the context of $*-$reversible modules.

1 Introduction

Throughout the discussion, let R denote an associative ring with unity, and all modules are unitary. The sets of idempotent and nilpotent elements of the endomorphism ring of a module M are denoted as $I(M)$ and $N(M)$ respectively. For an $R-module$ M, let $X = End(M)$, which denotes the endomorphism ring of M. We also define the right annihilator $r_M(I) = \{m \in M : Im = 0\}$, $r_X(I) = \{f \in X : If = 0\}$, the left annihilator $l_X(I) = \{\phi \in X : \phi I = 0\}$ for a nonempty subset I of X. We denote the set of $n \times n$ full matrices over a module M as $Mat_n(M)$, the $n \times n$ upper triangular matrices as $U_n(M)$, and the subset of upper triangular matrices with equal diagonal entries as $D_n(M) = \{(a_{ij}) \in U_n(M) : a_{ii} = a_{11}, \forall\, i\}$.

A ring R (potentially devoid of an identity element) is categorized as reversible if the product $ab = 0$, for $a, b \in R$, necessitates the reverse product $ba = 0$. This notion of a reversible ring was first introduced by P. M. Cohn in 1999 [5]. Since then, many authors have studied various concepts related to the reversible ring.

A ring R is termed $*-$reversible if, for any elements x and y in R, the product xy equals zero necessarily implies that the product yx^* equals zero. This notion of a $*-$reversible ring was introduced by Fakieh et al. [9]. Motivated by this work, we try to introduce the concept of a $*-$reversible module. Also, we establish the relations among $*-$reversible modules, symmetric modules, IFP modules, reversible modules, and reduced modules. We further discuss some properties of $*-$reversible modules.

In 1999, Anderson and Comillo [3] conducted an investigation into ring structures, specifically those wherein the product of zero elements commutes. They adopted the nomenclature ZC_2 to describe this property, subsequently demonstrating that semigroups lacking non-zero nilpotent elements conform to the ZC_2 condition. Furthermore, Anderson and Comillo examined the characteristics of rings that satisfy the ZC_2 property. Concurrently, in 1977, Krempa-Niewieczerzal [14] employed the term C_0 to refer to this characteristic.

In 2004, Lee and Zhou generalized the concept of reduced rings to the realm of module theory, introducing reduced modules as a theoretical analogue [15]. A module M over a ring R is categorised as reduced if, given any m in M and a in R, the condition $ma^2 = 0$ necessarily yields $mRa = 0$. Later, in 2024, Kimuli-Ssevviiri [12] extended this concept to endo-reduced modules by defining a module M as endo-reduced if its ring of endomorphisms forms a reduced ring.

2 Preliminaries

In this section, we present some basic definitions and results needed for our work.

A ring R is completely semiprime if, and only if, every non-zero element a complies with the condition that $a = 0$ whenever $a^2 = 0$. Consequently, this condition precludes the existence of non-trivial nilpotent elements within R. Such a ring is also referred to as a reduced ring.

An $R-$module M is called reduced if the condition $ma^2 = 0$, wherein $a \in R$ and $m \in M$, necessitates $mRa = 0$. This concept can be considered a generalization of reduced rings; specifically, a ring is characterized as reduced if it encompasses no non-zero nilpotent elements. Thus, an $R-$module M is a reduced module if $End(M)$ is a reduced ring.

A ring R that satisfies the condition ab equal to zero necessitates aRb equal to zero, for a, b in R, is denoted as an IFP ring. It is easy to verify that reversible rings are IFP rings. Furthermore, when each idempotent is central, the ring is commonly denoted as Abelian. Nonetheless, the converse of the assertion that IFP rings are Abelian is not true in general. [10]

A module, denoted by M over R, is said to exhibit the Insertion of Factors Property (IFP module) if for any element a in the ring R and any element m in M, whenever $ma = 0$, then $mra = 0$ for every r in R [13]. A module M over R is called an IFP module when its endomorphism ring satisfies the IFP ring condition.

In a given ring R, if for every element a, there exists an element b, both in R, such that the product aba equals a, then R is categorized as a (von Neumann) regular ring. Furthermore, if the endomorphism ring that is $End(M)$ is a (von Neumann) regular ring, M is referred to as a regular module.

A module M over a ring R is considered symmetric when the endomorphism ring of M satisfies the criterion of symmetry. This condition is equivalently expressed by the statement: for every endomorphisms f, g, h of M, if $fgh = 0$, then the reverse permutation gfh also equals zero.

A $2-$degree anti-isomorphism $a \longmapsto a^*$ in a ring R is characterized by the properties, that is, $(a^*)^* = a$; $(a+b)^* = a^* + b^*$; and $(ab)^* = b^*a^*$. A ring R endowed with such an involution $*$ is denoted as a $*-$ring [6, 7]. An $R-$module M is a $*-$module if $End(M)$ is a $*-$ring.

An element x within a $*-$ring R is called a $*-$cancellable if $x^*x = 0$ necessarily implies that $x = 0$. If every element $x \in R$ possesses this characteristics, then R is classified as a $*-$cancellable ring. An $R-$module M is called a $*-$cancellable module if $End(M)$ is a $*-$cancellable ring.[6]

A ring, denoted by R, possesses an involution $*$, and is called a $*-$proper if the condition $x^*x = 0$ necessitates $x = 0$ for all elements x in R [6]. A module M over R equipped with an involution $*$ is called a $*-$proper module if $End(M)$ is a $*-$proper ring.

A module M is abelian if it satisfies the condition $mae = mea$ for every element m in M and idempotent e in R. It is demonstrated in [2] that reduced modules, symmetric modules, semicommutative modules, and Armendariz modules inherently possess this characteristic. An $R-$module M is termed Abelian when its endomorphism ring, that is, $End(M)$, exhibits Abelian ring properties.

A ring R is characterized as semicommutative if it fulfills the following condition: for any elements a and b in R, the product ab vanishing necessitates the vanishing of the product aRb. Correspondingly, a module M over R is termed semicommutative if for any module element m and any ring element a, the product ma vanishing necessitates the vanishing of the product mRa. Buhphang and Rege in [4] conducted an investigation into the intrinsic characteristics of semicommutative modules, whereas Agayev and Harmanci broadened the scope of their analysis to comprehend the properties of semicommutative rings and modules, [1] with particular emphasis on the inherent semicommutativity of subrings within matrix rings.

3 Main Results

In this section, we present our main results.

Definition 3.1. A $*-$module M over a $*-$ring R is called a $*-$reversible if the endomorphism ring of M, that is, $End(M)$ is a $*-$reversible ring. Equivalently, for all $f, g \in End(M)$, $fg = 0$ implies $gf^* = 0$.

Example 3.2. (i) The $\mathbb{Z}-$module $\mathbb{Z}_6 \bigoplus \mathbb{Z}_6$, which is a reduced module. Let $t = (x, y) \in \mathbb{Z}_6 \bigoplus \mathbb{Z}_6$. Taking $t^* = (y, x)$. Consider $\alpha = (3, 0), \beta = (0, 1) \in \mathbb{Z}_6 \bigoplus \mathbb{Z}_6$. Then $\alpha\beta = (3, 0)(0, 1) = (0, 0) = 0$ while $\beta\alpha^* = (0, 1)(3, 0)^* = (0, 1)(0, 3) = (0, 3) \neq 0$. Hence, $\mathbb{Z}_6 \bigoplus \mathbb{Z}_6$ is not a $*-$reversible module.

(ii) Let M be a module whose endomorphism ring X is strongly regular. Then M is a reversible module, but it is not a $*-$reversible module. Let $\mathbb{Z}-$module $\mathbb{Z}_{10} \bigoplus \mathbb{Z}_{10}$, which is a strongly regular. Define an involution $*$ on $\mathbb{Z}_{10} \bigoplus \mathbb{Z}_{10}$ as $(x, y)^* = (y, x)$, for $(x, y) \in \mathbb{Z}_{10} \bigoplus \mathbb{Z}_{10}$. Taking $a = (0, 5)$, $b = (3, 0) \in \mathbb{Z}_{10} \bigoplus \mathbb{Z}_{10}$. Then $ab = 0$ implies $ba = 0$, while $ba^* = (5, 0) \neq 0$.

(iii) Let $M = U_2(\mathbb{Z}_2)$. For any $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$, taking $A^* = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$. Choose $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M$. Then $ab = 0$ implies $ba = 0$, while $ba^* \neq 0$. Thus, M is a reversible module, but it is not a $*-$reversible module.

Theorem 3.3. *The following are equivalent for a module M equipped with an involution $*$:*

(i) M is $*-$reversible;

(ii) $fg = 0$ implies $g^*f = 0$, for all f and g in X;

(iii) $fg = 0$ implies $f^*g = 0$, for all f and g in X;

(iv) $fg = 0$ implies $fg^* = 0$, for all f and g in X.

Proof. $(i) \implies (ii)$ For every f and g in X, we consider $fg = 0$. Taking an involution on $fg = 0$, we have $g^*f^* = 0$. Since M is a $*-$reversible module, then from $g^*f^* = 0$ we have $f^*g = 0$. Again, we have taking an involution on $f^*g = 0$, then $g^*f = 0$. Therefore, $fg = 0$ implies $g^*f = 0$.

$(ii) \implies (i)$ Take $fg = 0$. Now, applying an involution on $fg = 0$, we have $g^*f^* = 0$. Then by hypothesis, we have $fg^* = 0$. Again, taking an involution on $fg^* = 0$ then $gf^* = 0$, and thereby, M is a $*-$reversible.

$(ii) \implies (iii)$ Consider $fg = 0$ implies $g^*f = 0$. Taking an involution on $g^*f = 0$, we have $f^*g = 0$. Thus, $fg = 0$ implies $f^*g = 0$.

$(iii) \implies (iv)$ Assume $fg = 0$. Taking an involution on $fg = 0$, we have $g^*f^* = 0$. Then, by our hypothesis, we have $gf^* = 0$. Again, we have taking an involution on $gf^* = 0$, then $fg^* = 0$. Thus, $fg = 0$ implies $fg^* = 0$. □

Corollary 3.4. *For a module M that is $*-$reversible, then the ring of endomorphisms of M, symbolized as $End(M)$, possesses the characteristic of being an integral domain.*

Proof. Since every $*-$reversible module is reversible by Theorem 3.7. Then $End(M)$ is a reversible ring. Also from [[5], Theorem 2.1], we have $End(M)$ is an integral domain. □

Theorem 3.5. *For a $*-$cancellable module M, the following conditions are equivalent:*

(i) M is $*-$reversible;

(ii) $fg \in I(M)$ implies $gf \in I(M)$, for $f, g \in X$.

Proof. "$(i) \implies (ii)$" Take $fg \in I(M)$ for f *and* g in X. Then $(fg)^2 = fg \implies fg(1 - fg) = 0$. By our hypothesis, we have $0 = \{g(1 - fg)\}f^* = f^*\{g(1 - fg)\}^*$. Taking an involution on $f^*\{g(1 - fg)\}^* = 0$, we have $g(1 - fg)f = 0 \implies gf = (gf)^2$. So $gf \in I(M)$.

"$(ii) \implies (i)$" Given the condition (ii). Take $fg = 0$. Then $gf \in I(M)$, so $gf = gfgf = 0$ by condition (ii), thereby, satisfying the reversible condition and confirming that M is reversible. Now, we have $fg = 0 \implies fgf^* = 0 \implies gf^*f = 0 \implies gf^*fg^* = 0 \implies gf^*(gf^*)^* = 0$. Since M is $*-$cancellable, so $gf^*(gf^*)^* = 0 \implies gf^* = 0$. Thus, $fg = 0$ implies $gf^* = 0$. Hence, M is a $*-$reversible. □

Lemma 3.6. *Every $*-$reversible module is symmetric.*

Proof. Consider a $*-$reversible module M and $fgh = 0$ for each $f, g, h \in X$. Then $f(gh) = 0$ implies $f(gh)^* = 0$, by Theorem 3.3. Taking an involution on $f(gh)^* = 0$, we have $ghf^* = 0$. Again, using Theorem 3.3, $(gh)f^* = 0$ induces $ghf = 0$. Thus, $fgh = 0$ implies $ghf = 0$ for any $f, g, h \in End(M)$. Hence, M is a symmetric module. □

Corollary 3.7. *Every $*-$reversible module is reversible.*

Proof. Since every symmetric module is a reversible module, by Lemma 3.6, we have every $*-$reversible module is reversible. □

Lemma 3.8. *Every $*-$reversible module is IFP.*

Proof. Since every symmetric module is an IFP module, by Lemma 3.6, we have every $*-$reversible module is IFP. □

Remark 3.9. The converse of Lemma 3.8 fails to hold in general. We consider the following example,

Assume M is a reduced module. Then,

$$D_3(M) = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \middle| \ \alpha, \beta, \gamma, \delta \in End(M) \right\}$$

is an IFP module. [11]

Define an involution $*$ on $D_3(M)$ as $\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix}^* = \begin{pmatrix} \alpha & \delta & -\gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix}$.

Assume $x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ be two elements of $D_3(\mathbb{Z})$.

Then,

$$xy = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

and

$$\begin{aligned} yx^* &= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* \\ &= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\neq 0. \end{aligned}$$

Thus, $D_3(\mathbb{Z})$ is not a $*-$reversible module.

By Example 3.2(i), a reduced module is not necessarily $*-$reversible in general. In the following, we show that the notion of reduced module and $*-$reversible module are identical over the class of $*-$cancellable module.

Theorem 3.10. *The following are equivalent for a $*-$cancellable module M:*

(i) M *is reduced;*

(ii) M *is symmetric;*

(iii) M *is reversible;*

(iv) M *is $*-$reversible.*

Proof. $(i) \implies (ii) \implies (iii)$ are clear.

$(iii) \implies (iv)$ We assume M to be a reversible module and $fg = 0$, for any $f, g \in End(M)$. Then $fg = 0 \implies fgf^* = 0 \implies gf^*f = 0 \implies gf^*fg^* = 0 \implies gf^*(gf^*)^* = 0$, by the condition that M is a reversible module. Given that M is a $*-$cancellable module, then from $gf^*(gf^*)^* = 0$, we have $gf^* = 0$. Thus, $fg = 0$ implies $gf^* = 0$, for any $f, g \in End(M)$. Hence, M is a $*-$reversible.

$(iv) \implies (i)$ We consider M is a $*-$reversible module and $f^2 = 0$, for all $f \in End(M)$. Then $ff = 0 \implies ff^* = 0$, by Theorem 3.3. Since M is $*-$cancellable, from $ff^* = 0$ we have $f = 0$. Thus, $f^2 = 0$ implies $f = 0$, thereby validating M as a reduced module. □

Lemma 3.11. *For a reduced module M, $*-$reversibility implies $*-$cancellability.*

Proof. Assume M is a $*-$reversible and $ff^* = 0$ for all $f \in End(M)$. Then $ff = 0$ by Theorem 3.3. Since M is reduced, from $ff = 0$ we get $f = 0$. Thus, $ff^* = 0$ implies $f = 0$, thereby validating M as a $*-$cancellable module. □

Theorem 3.12. *For a $*-$reversible module M, the following are equivalent;*

(i) M *is $*-$cancellable;*

(ii) $f^2 \in I(M)$ *implies* $f = f^3$, *for each* $f \in End(M)$.

Proof. "⇒" Suppose that M is a $*-$cancellable module and let $f \in End(M)$ such that $f^2 \in I(M)$. Then

$$\begin{aligned} f^2 &= f^4 \\ \implies f^2(1-f^2) &= 0 \\ \implies (f(1-f^2))^2 &= 0 \\ \implies (f(1-f^2))(f(1-f^2))^* &= 0. \end{aligned}$$

Since M is $*-$cancellable, so $(f(1-f^2))(f(1-f^2))^* = 0$ implies $f(1-f^2) = 0 \implies f = f^3$. Hence, $f^2 \in I(M)$ implies $f = f^3$, for $f \in End(M)$.

"⇐" Suppose that for any $f \in End(M)$, $f^2 \in I(M)$ implies that $f = f^3$. Let $f \in End(M)$ such that $f^*f = 0$. Then by Theorem 3.3, $(f^*)f = 0$, which implies that $f^2 = 0$. Now since $f^2 = 0 \in I(M)$, so by assumption $f = f^3 = f(f^2) = 0$. □

Theorem 3.13. *If M is a $*-$cancellable module, then the following are equivalent:*

(i) M *is $*-$reversible;*

(ii) $f^2 \in I(M) \setminus \{1\}$ *implies* $f \in I(M)$, *for* $f \in X$.

Proof. $(i) \implies (ii)$ Assume the condition (i). Suppose $f^2 \in I(M) \setminus \{1\}$ for $f \in X$. Then $f^2(1-f^2) = 0$. Now,

$$\begin{aligned} f^2(1-f^2) &= 0 \\ \implies f^2(1-f)(1+f) &= 0 \\ \implies f^2(1-f)(1-f) &= 0 \\ \implies (f(1-f))^2 &= 0 \\ \implies (f(1-f))(f(1-f))^* &= 0 \quad (By\ Theorem\ 3.3). \end{aligned}$$

Since M is $*-$cancellable, so $(f(1-f))(f(1-f))^* = 0$ implies $f(1-f) = 0 \implies f = f^2$. Hence, $f \in I(M)$.

$(ii) \implies (i)$ Suppose that $f^2 \in I(M) \setminus \{1\}$ implies $f \in I(M)$, for $f \in X$. Assume $fg = 0$, for $f, g \in X$. First, we claim that M is a reversible module. Let $(gf)^2 \in I(M) \setminus \{1\}$, then by the condition, we have $gf \in I(M)$. Now, from $gf \in I(M)$, we get $gf = gfgf = 0$. Hence, M is reversible. Next, by Theorem 3.10, we have M is a $*-$reversible module. □

Remark 3.14. Considering Theorem 3.13, if M is a $*-$reversible, we can hypothesize that the following are equivalent:

(i) M is $*-$cancellable;

(ii) If $f^2 \in I(M) \setminus \{1\}$, then $f \in I(M)$, for $f \in X$.

Theorem 3.15. *The following are equivalent for a module M.*

(i) M is reversible module;

(ii) $fg \in N(M)$ implies $gf \in N(M)$, for $f, g \in X$.

Proof. " $\Rightarrow$" Take $fg \in N(M)$ for $f, g \in X$. Then there exists a positive integer n such that $(fg)^n = 0$. Now,

$$\begin{aligned} (fg)^n &= 0 \\ \implies f(gf)^{n-1}g &= 0 \\ \implies gf(gf)^{n-1} &= 0 \\ \implies fgfg(fg)^{n-2} &= 0 \\ \implies (gf)^3(gf)^{n-3} &= 0. \end{aligned}$$

Continuing in this way, we have $(gf)^{n-1}gf = 0 \implies (gf)^n = 0 \implies gf \in N(M)$.

"$\Leftarrow$" Assume $fg = 0$, for f, g in X. Then $fg \in N(M)$. By condition (ii), we have $gf \in N(M)$. Now, from $fg = 0$ we have $fgf = 0 \implies f(gf) = 0 \implies f(gf)^2 = 0$. Continuing in this way, we have $f(gf)^i = 0$, for any positive integer $i \leq n$. Similarly, $(gf)^i g = 0$. Thus, $gf \in N(M)$ satisfies $(gf)^n = 0$ and $f(gf)^i = 0$ or $(gf)^i g = 0$, for all i. Hence, $ba = 0$. Therefore, M is a reversible module. □

Lemma 3.16. *For a reduced module M. If M is $*-$reversible, then*

$$D_2(M) = \left\{ \begin{pmatrix} m & n \\ 0 & m \end{pmatrix} \middle| \, m, n \in X \right\}$$

is a $*-$reversible module, where involution $*$ is defined on $D_2(M)$ as $\begin{pmatrix} m & n \\ 0 & m \end{pmatrix}^* = \begin{pmatrix} m^* & n^* \\ 0 & m^* \end{pmatrix}$.

Proof. Let $u = \begin{pmatrix} f_1 & g_1 \\ 0 & f_1 \end{pmatrix}$ and $v = \begin{pmatrix} f_2 & g_2 \\ 0 & f_2 \end{pmatrix} \in D_2(M)$ such that $uv = 0$. Then $f_1 f_2 = 0 = f_1 g_2 + g_1 f_2$. Since M is $*-$reversible, we have $f_2 f_1^* = 0$ and so $0 = f_1 g_2 f_1^* + g_1 f_2 f_1^* = f_1 g_2 f_1^* \implies f_1 g_2 = 0 \implies g_2 f_1^* = 0$. So from $f_1 g_2 + g_1 f_2 = 0$, we get $g_1 f_2 = 0 \implies f_2 g_1^* = 0$, by the condition that M is a $*-$reversible module. Hence, $g_2 f_1^* = 0$ and $f_2 g_1^* = 0$.

Now,

$$\begin{aligned} vu^* &= \begin{pmatrix} f_2 & g_2 \\ 0 & f_2 \end{pmatrix} \begin{pmatrix} f_1 & g_1 \\ 0 & f_1 \end{pmatrix}^* \\ &= \begin{pmatrix} f_2 & g_2 \\ 0 & f_2 \end{pmatrix} \begin{pmatrix} f_1^* & g_1^* \\ 0 & f_1^* \end{pmatrix} \\ &= \begin{pmatrix} f_2 f_1^* & f_2 g_1^* + g_2 f_1^* \\ 0 & f_2 f_1^* \end{pmatrix} \\ &= 0. \end{aligned}$$

Therefore, for any $u, v \in D_2(M)$, $uv = 0$ implies $vu^* = 0$, indicating that $D_2(M)$ is a $*-$reversible module. □

Remark 3.17. We may conjecture that

$$D_2(M) = \left\{ \begin{pmatrix} m & n \\ 0 & m \end{pmatrix} \middle| \, m, n \in X \right\}$$

is not a $*-$reversible if M is a reduced module.

Remark 3.18. In general, 2×2 upper triangular matrix that is

$$U_2(M) = \left\{ \begin{pmatrix} m & p \\ 0 & n \end{pmatrix} \middle| \, m \neq n \neq p \in X \right\}$$

is not a $*-$reversible module. For example, define an involution $*$ on $U_2(M)$ as $\begin{pmatrix} m & p \\ 0 & n \end{pmatrix}^* = \begin{pmatrix} n & p \\ 0 & m \end{pmatrix}$. Choose $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \in U_2(\mathbb{Z})$. Then $AB = 0$ but $BA^* \neq 0$. Thus, $U_2(\mathbb{Z})$ is not a $*-$reversible module.

Theorem 3.19. *For a module M and any $f \in X$, the following are equivalent:*

(i) M is $*-$reversible;

(ii) $r_X(f^*) = r_X(f)$;

(iii) $l_X(f^*) = l_X(f)$;

(iv) $l_X(f^*) = r_X(f)$;

(v) $r_X(f^*) = l_X(f)$.

Proof. $(i) \implies (ii)$ Choose, $g \in r_X(f^*)$. Then by hypothesis, $f^*g = 0 \implies g^*f^* = 0$. Taking an involution on $g^*f^* = 0$, we have $fg = 0$. Thus, $g \in r_X(f)$ and vice versa. Hence, $r_X(f^*) = r_X(f)$.

$(ii) \implies (i)$ Taking the condition (ii). Then $fg = 0 \implies f^*g = 0$ for each $g \in r_X(f^*) = r_X(f)$. Now, taking an involution on $f^*g = 0$, we have $g^*f = 0$ and vice versa. Hence, M is a $*-$reversible module.

$(i) \implies (iii)$ We assume M to be a $*-$reversible module and let $h \in l_M(f^*)$. Then $hf^* = 0 \implies f^*h^* = 0$, by the condition that M is a $*-$reversible module. Taking an involution on $f^*h^* = 0$, we have $hf = 0$. Thus, $h \in l_X(f)$ and vice versa. Hence, $l_X(f^*) = l_X(f)$.

$(i) \implies (iv)$ We assume M to be a $*-$reversible module. Then $gf^* = 0$, for $g \in l_X(f^*)$. Then, by assumption, from $gf^* = 0$ we have $fg = 0$. Thus, $g \in r_X(f)$ and vice versa. Hence, $l_X(f^*) = r_X(f)$.

$(i) \implies (v)$ We assume M to be a $*-$reversible module. Then $f^*g = 0$, for $g \in r_X(f^*)$. Then, by assumption, from $f^*g = 0$ we have $gf = 0$. Thus, $g \in l_X(f)$ and vice versa. Hence, $r_X(f^*) = l_X(f)$. □

4 Conclusion

In this study, we have defined the concept of a $*-$reversible module and we have established various properties relating to reduced module and IFP module. In future, the notion of $(t, e)-$reversible ring and $(t, e)-$reversible module can be studied.

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