Palestine Journal of Mathematics

Vol.15(3), (2026) , 1010–1026 © Palestine Polytechnic University-PPU 2026

Singular POND Overpartitions

M. Prasad, Kaneez Raheela and M. S. Mahadeva Naika

Communicated by V. Lokesha

The corresponding author M. Prasad

MSC 2010 Classifications: 11P83, 05A17.

Keywords and phrases: Partition congruences, Partition identities, Overpartition, Singular overpartitions, Singular POND overpartitions, Theta–functions.

The authors would like to thank the reviewers and editor for their valuable suggestions that improved the quality of our paper.

Abstract: In 2015, Andrews defined and studied a combinatorial objects which he called as singular overpartitions. In 2023, Ballantine and Welch have considered various generalizations of POD and PED. In the process, they led to two new classes of POND and PEND partitions which are integer partitions with odd parts can not be distinct (in case of POND) and even parts can not be distinct (in case of PEND) respectively. Motivated by the above works in this paper, we obtain some infinite families of congruences for a new class of combinatorial objects called as singular pond overpartitions $\overline{cpond}_{\delta,i}(n)$, a singular overpartition of a positive integer n wherein odd parts not distinct(even parts unrestricted).

1 Introduction

Throughout this paper, we let $|q| < 1$. We use the standard notation

$$f_k := (q^k; q^k)_\infty.$$

Following Ramanujan, we define

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.1}$$

$$f^3(-q) = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}, \tag{1.2}$$

which are special cases of Ramanujan's general theta function [4]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1.3}$$

In Ramanujan's notation, Jacobi's famous triple product identity becomes,

$$f(a,b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \tag{1.4}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $P(n)$ denotes the number of such partitions of n with $P(0) = 1$ and the generating function for $P(n)$ is

$$\sum_{n=0}^{\infty} P(n) q^n = \frac{1}{(q,q)_\infty}.$$

In 2004, Corteel and Lovejoy [5] introduced the notion of an overpartitions, which are partition of a positive integer n in which first occurrence of each part may be overlined. The generating function is

$$\sum_{n=0}^{\infty}\overline{P}(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{1.5}$$

For example, there are 8 partitions for $\overline{P}(3)$, namely

$$3,\quad \overline{3},\quad 2{+}1,\quad \overline{2}{+}1,\quad 2{+}\overline{1},\quad \overline{2}{+}\overline{1},\quad 1{+}1{+}1,\quad \overline{1}{+}1{+}1$$

Mahadeva Naika, Harishkumar and Veeranayaka [12],[13] introduced notion of $[j,k]$- overpartition of a positive integer n is a non-increasing sequence of positive integers whose sum is n in which first occurrence of each part congruent to j modulo k may be overlined. Let $\overline{P}_{j,k}(n)$ denote the number of such partitions of n with $\overline{P}_{j,k}(0)=1$. The generating function is given by

$$\sum_{n=0}^{\infty}\overline{P}_{j,k}(n)q^n=\frac{(-q^j;q^k)_\infty}{(q,q)_\infty}. \tag{1.6}$$

For example, there are 5 partitions for $\overline{P}_{1,3}(3)$, namely

$$3,\qquad 2{+}1,\qquad 2{+}\overline{1},\qquad 1{+}1{+}1,\quad \overline{1}{+}1{+}1.$$

In 2015, G. E. Andrews [1] defined combinatorial objects which he called singular overpartitions and proved that these singular overpartitions which depends on two parameters δ and i can be enumerated by the function $\overline{c}_{\delta,i}(n)$ which gives the number of overpartitions of n in which no part divisible by δ and parts $\equiv\pm i \pmod{\delta}$ may be overlined. The generating function of $\overline{c}_{\delta,i}(n)$ is

$$\sum_{n=0}^{\infty}\overline{c}_{\delta,i}(n)q^n=\frac{(q^\delta;q^\delta)_\infty(-q^i;q^\delta)_\infty(-q^{\delta-i};q^\delta)_\infty}{(q;q)_\infty}=\frac{f(q^i,q^{\delta-i})}{(q;q)_\infty}. \tag{1.7}$$

For example, there are 10 partitions for $\overline{c}_{3,1}(4)$, namely

$$4,\quad \overline{4},\quad 2+2,\quad \overline{2}+2,\quad 2{+}1{+}1,\quad \overline{2}{+}1{+}1,\quad 2{+}\overline{1}+1,\quad \overline{2}{+}\overline{1}+1,\quad 1{+}1{+}1{+}1,\quad \overline{1}{+}1{+}1{+}1.$$

He also proved that

$$\overline{c}_{3,1}(9n+3)=\overline{c}_{3,1}(9n+6)\equiv 0.$$

For more details, one can see [9], [10] and [11].

In the following paper, Ballantine and Welch [2] considered various generalizations and refinements of POD and PED partitions. In the process they led to two new classes of POND and PEND partitions which are integer partitions with odd parts can not be distinct (in case of POND) and even parts can not be distinct (in case of PEND) respectively.

Recently, M. Prasad, Kaneez Raheela and M. S. Mahadeva Naika[14] have introduced and obtained several infinite families of congruences for OPOND partitions.

In [15], M. Prasad, Kaneez Raheela and M. S. Mahadeva Naika have introduced a new class of partitions called as singular PEND overpartitions is a singular overpartitions of a positive integer n in which even parts are not distinct (odd parts are unresrticted) and obtained several infinite families of congruences. Let $\overline{cpend}_{\delta,i}(n)$ denotes the number of such partitions of n and the generating function be given by

$$\sum_{n=0}^{\infty}\overline{cpend}_{\delta,i}(n)q^n=\frac{f_2f_{12}}{f_1f_4f_6}f(q^i,q^{\delta-i}). \tag{1.8}$$

Motivated by the above works, in this paper we introduce a new class of combinatorial objects called as a singular POND overpartitions is a singular overpartitions of a positive integer n in which odd parts are not distinct (even parts are unresrticted). Let $\overline{cpond}_{\delta,i}(n)$ denotes the number of such partitions of n and the generating function be given by

$$\sum_{n=0}^{\infty}\overline{cpond}_{\delta,i}(n)q^n=\frac{f_4f_6^2}{f_2^2f_3f_{12}}f(q^i,q^{\delta-i}). \tag{1.9}$$

We also obtain several infinite families of congruences for singular POND overpartitions by using elementary dissection formulas.

2 Preliminary results

Lemma 2.1. *The following* 2*– dissection hold*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.1}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.2}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.3}$$

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q\frac{f_4^{16}}{f_2^{24}} + 16q^2\frac{f_4^4 f_8^8}{f_2^{20}}, \tag{2.4}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}, \tag{2.5}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + \frac{q f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.6}$$

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}. \tag{2.7}$$

The equations (2.1)-(2.3) can be found in [4, Entry 25, p.40], the equation (2.6) can be found in [3], the equations (2.4), (2.5) and (2.7) can be found in [6].

Lemma 2.2. *The following* 3*-dissections hold*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^2} - 3q f_9^3, \tag{2.8}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{2.9}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{2.10}$$

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{2.11}$$

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}, \tag{2.12}$$

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}. \tag{2.13}$$

The equation (2.8) can be found in [4, Entry 1(iv), p.345], the equations (2.9) - (2.11) can be found in [7], the equation (2.12) can be found in [8] and the equation (2.13) is nothing but Lemma 2.6 in [3]

Lemma 2.3. *[4]The following 5-dissection hold*

$$f_1 = f_{25}\left(a - q - q^2/a\right), \tag{2.14}$$

where

$$a := a(q^5) = \frac{f(-q^{10}, -q^{15})}{f(-q^5, q^{20})}.$$

Lemma 2.4. *The following 7-dissection hold*

$$f_1 = f_{49}\left(\frac{B}{C} - q\frac{A}{B} - q^2 + q^5\frac{C}{A}\right), \tag{2.15}$$

where $A := f(-q^{21}, -q^{28})$, $B := f(-q^{14}, q^{35})$ *and* $C := f(-q^7, -q^{42})$.

The above equation can be found in [4, Entry 17(v), p.303].

3 Generating function for $\overline{\mathrm{cpond}}_{\delta,\mathrm{i}}(\mathrm{n})$

In this section, we provide proof for a generating function of a partition $\overline{\mathbf{cpond}}_{\delta,\mathbf{i}}(\mathbf{n})$.

Theorem 3.1. *For $1 \leq i \leq \delta$, we have*

$$\sum_{n=0}^{\infty} \overline{cpond}_{\delta,i}(n)q^n = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}} f(q^i, q^{\delta-i}). \tag{3.1}$$

Proof. By the definition,

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{cpond}_{\delta,i}(n)q^n &= \frac{f(q^i, q^{\delta-i})}{(q^2;q^2)_\infty} \prod_{i=1}^{\infty} \left(\frac{1}{1-q^{2i-1}} - q^{2i-1} \right) \\
&= \frac{f(q^i, q^{\delta-i})}{(q^2;q^2)_\infty} \prod_{i=1}^{\infty} \left(\frac{1-q^{2i-1}+q^{2(2i-1)}}{(1-q^{2i-1})} \right) \\
&= \frac{f(q^i, q^{\delta-i})}{(q^2;q^2)_\infty} \prod_{i=1}^{\infty} \left(\frac{(1+q^{3(2i-1)})}{(1-q^{2i-1})(1+q^{2i-1})} \right) \\
&= \frac{f(q^i, q^{\delta-i})}{(q^2;q^2)_\infty} \frac{(-q^3;q^6)_\infty}{(q^2;q^4)_\infty} = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}} f(q^i, q^{\delta-i}).
\end{aligned}$$

□

4 Congruences for $\overline{\mathrm{cpond}}_{\delta,\mathrm{i}}(\mathrm{n})$

Theorem 4.1. *We have*

$$\sum_{n=0}^{\infty} \overline{cpond}_{k,k}(n)q^n = 2\frac{f_4 f_6^2 f_{2k}^2}{f_2^2 f_3 f_{12} f_k}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4k,k}(n)q^n = \frac{f_4 f_6^2 f_{2k}^2}{f_2^2 f_3 f_{12} f_k}, \tag{4.2}$$

$$\overline{cpond}_{k,k}(n) = 2\ \overline{cpond}_{4k,k}(n). \tag{4.3}$$

Proof. Putting $\delta = k$ and $i = k$ in (3.1), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{cpond}_{k,k}(n)q^n &= \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}} f(q^k, 1) = 2\frac{f_4 f_6^2 f(q^k, q^{3k})}{f_2^2 f_3 f_{12}} \\
&= 2\frac{f_4 f_6^2 \psi(q^k)}{f_2^2 f_3 f_{12}},
\end{aligned} \tag{4.4}$$

which implies (4.1).
Putting $\delta = 4k$ and $i = k$ in (3.1), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{cpond}_{4k,k}(n)q^n &= \frac{f_4 f_6^2 f(q^k, q^{3k})}{f_2^2 f_3 f_{12}} = \frac{f_4 f_6^2 \psi(q^k)}{f_2^2 f_3 f_{12}} \\
&= \frac{f_4 f_6^2 f_{2k}^2}{f_2^2 f_3 f_{12} f_k},
\end{aligned} \tag{4.5}$$

which proves (4.2).
From (4.1) and (4.2), we obtain (4.3). □

5 Congruences for $\overline{\text{cpond}}_{4,1}(n)$

Theorem 5.1. *For $\alpha, n \geq 0$, then we have*

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(24 \cdot 7^{2\alpha} n + 5 \cdot 7^{2\alpha}) q^n \equiv 8 f_1 f_4 \pmod{16}, \tag{5.1}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(24 \cdot 7^{2\alpha+1} n + 11 \cdot 7^{2\alpha+1}) q^n \equiv 8q f_7 f_{28} \pmod{16}, \tag{5.2}$$

$$\overline{cpond}_{4,1}(24 \cdot 7^{2\alpha+2} n + c_1 \cdot 7^{2\alpha+1}) \equiv 0 \pmod{16}, \tag{5.3}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(24 \cdot 7^{2\alpha} n + 17 \cdot 7^{2\alpha}) q^n \equiv 16 f_1 f_{16} \pmod{32}, \tag{5.4}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(24 \cdot 7^{2\alpha+1} n + 23 \cdot 7^{2\alpha+1}) q^n \equiv 16q^4 f_7 f_{112} \pmod{32}, \tag{5.5}$$

$$\overline{cpond}_{4,1}(24 \cdot 7^{2\alpha+2} n + c_2 \cdot 7^{2\alpha+1}) \equiv 0 \pmod{32}, \tag{5.6}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(24 \cdot 5^{2\alpha} n + 11 \cdot 5^{2\alpha}) q^n \equiv 8 f_2 f_3^3 \pmod{16}, \tag{5.7}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(24 \cdot 5^{2\alpha+1} n + 7 \cdot 5^{2\alpha+1}) q^n \equiv 8q f_{10} f_{15}^3 \pmod{16}, \tag{5.8}$$

$$\overline{cpond}_{4,1}(24 \cdot 5^{2\alpha+2} n + c_3 \cdot 5^{2\alpha+1}) \equiv 0 \pmod{16}, \tag{5.9}$$

where $c_1 \in \{11, 59, 83, 107, 131, 155\}$, $c_2 \in \{23, 47, 71, 95, 143, 167\}$ and $c_3 \in \{7, 55, 77, 103\}$.

Proof. Putting $k = 1$ in (4.2), we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(n) q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}. \tag{5.10}$$

Using (2.13) in (5.10) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we find that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(3n) q^n = \frac{f_2^2 f_6^4}{f_1^4 f_{12}^2}, \tag{5.11}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(3n+1) q^n = \frac{f_2^4 f_3^3 f_{12}}{f_1^5 f_4 f_6^2} \tag{5.12}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(3n+2) q^n = 2\frac{f_2^3 f_6 f_{12}}{f_1^4 f_4}. \tag{5.13}$$

Using (2.3) in (5.13) and then extracting the terms involving q^{2n} and q^{2n+1}, we deduce that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(6n+2) q^n = 2\frac{f_2^{13} f_3 f_6}{f_1^{11} f_4^4} \tag{5.14}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(6n+5) q^n = 8\frac{f_2^7 f_3 f_6}{f_1^3} \pmod{32}. \tag{5.15}$$

Employing (2.3) in (5.15) and then extracting the terms involving q^{2n} and q^{2n+1}, we see that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(12n+5) q^n = 8\frac{f_2^2 f_4^2}{f_1^2} \pmod{32} \tag{5.16}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(12n+11)q^n = 24f_2^2f_3^2f_6^2 \pmod{32}. \tag{5.17}$$

Using (2.2) in (5.16) and then extracting the terms involving q^{2n} and q^{2n+1} , we find that

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24n+5)q^n = 8f_1f_4 \pmod{16} \tag{5.18}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24n+17)q^n = 16f_1f_{16} \pmod{32}. \tag{5.19}$$

The equation (5.19) is $\alpha = 0$ case of (5.4).
Suppose that the equation (5.4) is true for all $\alpha \geq 0$.
Using (2.15) in (5.4) and then extracting the term involving q^{7n+6}, we obtain (5.5).
From the equation (5.5), we obtain (5.6) and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24\cdot 7^{2\alpha+2}n + 17\cdot 7^{2\alpha+2})q^n \equiv 8f_1f_{16} \pmod{32}, \tag{5.20}$$

which is $\alpha+1$ case of (5.4). By mathematical induction, the equation (5.4) hold for all $\alpha \geq 0$.
Using (2.1) in (5.17), we find that

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24n+11)q^n = 8f_2f_3^3 \pmod{16} \tag{5.21}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24n+23)q^n = 16qf_2f_3^3f_{12}^3 \pmod{32}. \tag{5.22}$$

The equation (5.21) is $\alpha = 0$ case of (5.7).
Suppose that the equation (5.7) is true for all $\alpha \geq 0$.
Using (2.14) in (5.7) and then extracting the term involving q^{5n+1}, we obtain (5.8).
From the equation (5.8), we obtain (5.9) and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24\cdot 5^{2\alpha+2}n + 11\cdot 5^{2\alpha+2})q^n \equiv 8f_2f_3^3 \pmod{32}, \tag{5.23}$$

which is $\alpha+1$ case of (5.7). By mathematical induction, the equation (5.7) hold for all $\alpha \geq 0$. □

Theorem 5.2. *For α and $n \geq 0$, then we have*

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}\left(3^{2\alpha+1}(8n+5)\right)q^n = 8f_1^3f_4^3 + 16qf_1^3f_{12}^3 \pmod{32}, \tag{5.24}$$

$$\overline{cpond}_{4,1}\left(3^{2\alpha+2}(24n+23)\right) \equiv 0 \pmod{32}, \tag{5.25}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}\left(3^{2\alpha+1}(8n+1)\right)q^n = 4\frac{f_4}{f_1} \pmod{16}, \tag{5.26}$$

$$\overline{cpond}_{4,1}\left(3^{2\alpha+1}(24n+1)\right) = \begin{cases} 4 \pmod 8 & \text{if } n = \dfrac{k(3k-1)}{2} \\ 0 \pmod 8 & \text{otherwise} \end{cases}, \tag{5.27}$$

$$\overline{cpond}_{4,1}\left(3^{2\alpha+2}(24n+11)\right) = 0 \pmod{16}, \tag{5.28}$$

$$\overline{cpond}_{4,1}\left(3^{2\alpha+2}(24n+19)\right) = 0 \pmod{16}, \tag{5.29}$$

$$\overline{cpond}_{4,1}(24n+9) = 0 \pmod{16}, \tag{5.30}$$

$$\overline{cpond}_{4,1}(24n+17) = 0 \pmod{16}, \tag{5.31}$$

$$\overline{cpond}_{4,1}(24n+21) = 0 \pmod{16}, \tag{5.32}$$

$$\overline{cpond}_{4,1}(48n+14) = 0 \pmod{16}, \tag{5.33}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+1}(8n+1)\right) q^n = 12\frac{f_4}{f_1} \pmod{16}, \tag{5.34}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+1}(4n+3)\right) q^n = 12f_3^6 \pmod{16}, \tag{5.35}$$

$$\overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+1}(12n+7)\right) \equiv 0 \pmod{16}, \tag{5.36}$$

$$\overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+1}(12n+11)\right) \equiv 0 \pmod{16}, \tag{5.37}$$

Proof. Using (2.6) in (5.10), we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(2n)q^n = \frac{f_4^2 f_6^4}{f_1^2 f_3^2 f_{12}^2} \tag{5.38}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(2n+1)q^n = \frac{f_2^6 f_{12}^2}{f_1^4 f_4^2 f_6^2}. \tag{5.39}$$

Employing (2.3) in (5.39), we find that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(4n+1)q^n = \frac{f_2^{12} f_6^2}{f_1^8 f_3^2 f_4^4} \tag{5.40}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(4n+3)q^n = 4\frac{f_4^4 f_6^2}{f_1^4 f_3^2}. \tag{5.41}$$

Using (2.2) and (2.3) in (5.41), we see that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(8n+3)q^n = 4\frac{f_1^6 f_{12}}{f_3^3} \pmod{16} \tag{5.42}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(8n+7)q^n = 8qf_1^6 f_3 f_{12}^3 + 16f_2^9 f_3 \pmod{32}. \tag{5.43}$$

Using (2.8) in (5.43) and then extracting the term involving q^{3n+1}, we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(24n+15)q^n = 8f_1^3 f_4^3 + 16qf_1^3 f_{12}^3 \pmod{32}, \tag{5.44}$$

which is $\alpha = 0$ case of (5.24).
Suppose that the equation (5.24) is hold for $\alpha \geq 0$.
Using (2.8) in (5.44) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(3^{2\alpha+1}(24n+5)\right) q^n \equiv 8f_1 f_4 \pmod{16}, \tag{5.45}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(3^{2\alpha+1}(24n+13)\right) q^n \equiv 8qf_1 f_{12}^3 + 8f_4 f_3^3 \pmod{16} \tag{5.46}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(3^{2\alpha+2}(8n+7)\right) q^n \equiv 8qf_3^3 f_{12}^3 + 16f_3^3 f_4^3 \pmod{32}. \tag{5.47}$$

Using (2.8) in (5.47) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we get (5.25)

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(3^{2\alpha+2}(24n+7)\right) q^n \equiv 8f_1 f_2^3 \pmod{32} \tag{5.48}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(3^{2\alpha+3}(8n+5)\right) q^n \equiv 8f_1^3 f_4^3 + 16qf_1^3 f_{12}^3 \pmod{32}, \tag{5.49}$$

which is $\alpha+1$ case of (5.24). By induction, the equation (5.24) is true for all $\alpha \geq 0$.
Using (2.8) in (5.42) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(24n+3\right) q^n \equiv 4\frac{f_4}{f_1} \pmod{16}, \tag{5.50}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(24n+11\right) q^n \equiv 8f_2 f_3^3 \pmod{16} \tag{5.51}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(24n+19\right) q^n \equiv 4f_1 f_6^3 \pmod{8}. \tag{5.52}$$

The equation (5.50) is $\alpha = 0$ case of (5.26).
Suppose that the equation (5.26) is true for all $\alpha \geq 0$.
Using (2.13) in (5.26), we obtain (5.27) and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(3^{2\alpha+2}(8n+3)\right) q^n = 4\frac{f_{12}}{f_3} \pmod{16}, \tag{5.53}$$

which implies (5.28), (5.29) and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(3^{2\alpha+3}(8n+1)\right) q^n = 4\frac{f_4}{f_1} \pmod{16}, \tag{5.54}$$

which is $\alpha+1$ case of (5.26), By mathematical induction, the equation (5.26) is true for all $\alpha \geq 0$.
Using (2.2) and (2.4) in (5.40), we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(8n+1\right) q^n = \frac{f_{12}^5}{f_3^3 f_{24}^2} \pmod{16} \tag{5.55}$$

and

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(8n+5\right) q^n = 8f_2^6 f_3 + 2q\frac{f_6^2 f_{24}^2}{f_3^3 f_{12}} \pmod{16}. \tag{5.56}$$

From the equation (5.55), we obtain (5.30) and (5.31).
Using (2.8) in (5.56), we deduce that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(24n+5\right) q^n = 8f_1 f_4 \pmod{16}, \tag{5.57}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}\left(24n+13\right) q^n = 2f_1 f_4^3 \pmod{8} \tag{5.58}$$

and (5.32).
Using (2.6) in (5.38), we get

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(4n+2)\,q^n = 2\frac{f_1^2f_2^2f_6^2}{f_3^2} \pmod{16}. \tag{5.59}$$

Using (2.5) in (5.59), we find that

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(8n+2)\,q^n = 2\frac{f_1^3f_2^2f_6^4}{f_3^3f_4f_{12}} \pmod{16} \tag{5.60}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(8n+6)\,q^n = 12\frac{f_2f_4f_6f_{12}}{f_3^2} \pmod{16}. \tag{5.61}$$

Employing (2.12) in (5.61), we deduce that

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24n+6)\,q^n = 12\frac{f_4^2}{f_1^2} \pmod{16}, \tag{5.62}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24n+14)\,q^n = 8qf_2f_{12}^3 \pmod{16}, \tag{5.63}$$

which gives (5.33),

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(48n+38)\,q^n = 8f_1f_6^3 \pmod{16} \tag{5.64}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24n+22)\,q^n = 4\frac{f_2f_4f_6f_{12}}{f_1^2} \pmod{16}. \tag{5.65}$$

From the equation (5.65) after using (2.2), we obtain

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(48n+22)\,q^n = 4f_2f_3^3 \pmod{8} \tag{5.66}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(48n+46)\,q^n = 4f_2f_3^3f_4^3 \pmod{16}. \tag{5.67}$$

Using (2.2) in (5.62), we see that

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(48n+6)\,q^n = 12\frac{f_4}{f_1} \pmod{16} \tag{5.68}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(48n+30)\,q^n = 8f_1^3f_8 \pmod{16}. \tag{5.69}$$

The equation (5.68) is $\alpha = 0$ case of (5.34).
The rest of the proof of (5.68) is similar to the proof of (5.26). So, we omit the details.
Using (2.8) and (2.9) in (5.60) and then extracting the terms involving q^{3n+1} and q^{3n+2}, we get

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}(24n+10)\,q^n = 2f_1f_3^3 \pmod{8} \tag{5.70}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}\left(24n+18\right)q^n \equiv 12f_3^6 \pmod{16}, \tag{5.71}$$

which is $\alpha = 0$ case of (5.35).
Suppose that the equation (5.35) is true for all $\alpha \geq 0$.
From the equation (5.35), we obtain (5.36), (5.37) and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+2}(4n+1)\right)q^n \equiv 12f_1^6 \pmod{16}. \tag{5.72}$$

Using (2.8) in (5.72), we get

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+2}(12n+1)\right)q^n \equiv 12\frac{f_2^2}{f_1^2} \pmod{16}, \tag{5.73}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+2}(12n+5)\right)q^n \equiv 8f_1f_3^3 \pmod{16} \tag{5.74}$$

and

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+3}(4n+3)\right)q^n \equiv 12f_3^6 \pmod{16}, \tag{5.75}$$

which is $\alpha+1$ case of (5.35), By mathematical induction, the equation (5.35) is true for all $\alpha \geq 0$.
Using (2.2) in (5.73), we get

$$\sum_{n=0}^{\infty}\overline{cpond}_{4,1}\left(2\cdot 3^{2\alpha+2}(24n+13)\right)q^n \equiv 8f_1f_4^3 \pmod{16}. \tag{5.76}$$

□

6 Congruences for $\overline{\mathrm{cpond}}_{12,3}(\mathrm{n})$

Theorem 6.1. *For $\alpha \geq 0$, then we have for* (mod 16)

$$\overline{cpond}_{12,3}\left(9n+8\right) \equiv 0, \tag{6.1}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(3^{4\alpha+2}n+\frac{3^{4\alpha+2}-1}{4}\right)q^n \equiv 2\frac{f_2^3f_3^6f_6}{f_{12}^2}, \tag{6.2}$$

$$\overline{cpond}_{12,3}\left(3^{4\alpha+3}n+\frac{5\cdot 3^{4\alpha+2}-1}{4}\right) \equiv 0, \tag{6.3}$$

$$\overline{cpond}_{12,3}\left(3^{4\alpha+4}n+\frac{11\cdot 3^{4\alpha+3}-1}{4}\right) \equiv 0, \tag{6.4}$$

$$\overline{cpond}_{12,3}\left(3^{4\alpha+5}n+\frac{5\cdot 3^{4\alpha+4}-1}{4}\right) \equiv 0, \tag{6.5}$$

$$\overline{cpond}_{12,3}\left(3^{4\alpha+6}n+\frac{11\cdot 3^{4\alpha+5}-1}{4}\right) \equiv 0. \tag{6.6}$$

Proof. Putting $k=3$ in (4.2), we get

$$\overline{cpond}_{12,3}(n)q^n = \frac{f_4f_6^4}{f_2^2f_3^2f_{12}}. \tag{6.7}$$

Using (2.10) in (6.7), we deduce that

$$\overline{cpond}_{12,3}(3n+2)q^n = 2\frac{f_4^2 f_6^3}{f_1^2 f_2^3}. \tag{6.8}$$

Employing (2.10) in (6.8), we find that

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}(9n+2)q^n \equiv 2\frac{f_2^3 f_3^6 f_6}{f_{12}^2} \pmod{16}, \tag{6.9}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}(9n+5)q^n \equiv 4\frac{f_1 f_2^2 f_6^2}{f_3} \pmod{8} \tag{6.10}$$

and (6.1).
Using (2.7) in (6.10), we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}(18n+5)q^n = 4f_1 f_2^3 \pmod{8}, \tag{6.11}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}(18n+14)q^n = 4f_1 f_6^3 \pmod{8}. \tag{6.12}$$

The equation (6.9) is $\alpha = 0$ case of (6.2).
Suppose that the equation (6.2) hold for all $\alpha \geq 0$.
Using (2.8) in (6.2), we obtain (6.3),

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(3^{4\alpha+3}n + \frac{3^{4\alpha+2}-1}{4}\right) q^n \equiv 2\frac{f_2^4 f_{12}}{f_1^2 f_4 f_6^2} \pmod{8}, \tag{6.13}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(3^{4\alpha+3}n + \frac{3^{4\alpha+4}-1}{4}\right) q^n \equiv 10\frac{f_2^5 f_6^3}{f_1^2 f_4^2} \pmod{16}. \tag{6.14}$$

Using (2.7) in (6.13), we obtain

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+3}n + \frac{13\cdot 3^{4\alpha+2}-1}{4}\right) q^n \equiv 4f_1 f_4^3 \pmod{8}. \tag{6.15}$$

Using (2.9) and (2.10) in (6.14), we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(3^{4\alpha+4}n + \frac{3^{4\alpha+4}-1}{4}\right) q^n \equiv 10\frac{f_2^3 f_3^6 f_6}{f_{12}^2} \pmod{16}, \tag{6.16}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(3^{4\alpha+4}n + \frac{7\cdot 3^{4\alpha+3}-1}{4}\right) q^n \equiv 4\frac{f_1 f_2^2 f_6^2}{f_3} \pmod{8} \tag{6.17}$$

and (6.4).
Using (2.7) in (6.17), we get

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+4}n + \frac{7\cdot 3^{4\alpha+3}-1}{4}\right) q^n \equiv 4f_1 f_2^3 \pmod{8}, \tag{6.18}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+4}n + \frac{19\cdot 3^{4\alpha+3}-1}{4}\right) q^n \equiv 4f_1 f_6^3 \pmod{8}. \tag{6.19}$$

Using (2.8) in (6.16), we obtain (6.5),

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(3^{4\alpha+5}n + \frac{3^{4\alpha+4}-1}{4}\right) q^n \equiv 2\frac{f_4 f_{12}}{f_1^2 f_6^2} \pmod{8}, \tag{6.20}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(3^{4\alpha+5}n+\frac{3^{4\alpha+6}-1}{4}\right)q^n\equiv 2\frac{f_2^5f_6^3}{f_1^2f_4^2}\pmod{16}. \tag{6.21}$$

Using (2.2) in (6.20), we get

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+5}n+\frac{13\cdot 3^{4\alpha+4}-1}{4}\right)q^n\equiv 4f_1f_4^3\pmod{8}. \tag{6.22}$$

Using (2.9) and (2.10) in (6.21), we see that

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(3^{4\alpha+6}n+\frac{3^{4\alpha+6}-1}{4}\right)q^n\equiv 2\frac{f_2^3f_3^6f_6}{f_{12}^2}\pmod{16}, \tag{6.23}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(3^{4\alpha+6}n+\frac{7\cdot 3^{4\alpha+5}-1}{4}\right)q^n\equiv 4\frac{f_1f_2^2f_6^2}{f_3}\pmod{8} \tag{6.24}$$

and (6.6).
The equation (6.23) is $\alpha+1$ case of (6.2). By mathematical induction, the equation (6.2) hold for all $\alpha\geq 0$.
Using (2.7) in (6.24), we get

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+6}n+\frac{7\cdot 3^{4\alpha+5}-1}{4}\right)q^n\equiv 4f_1f_2^3\pmod{8}, \tag{6.25}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+6}n+\frac{19\cdot 3^{4\alpha+5}-1}{4}\right)q^n\equiv 4f_1f_6^3\pmod{8}. \tag{6.26}$$

□

Theorem 6.2. *For $\alpha, n\geq 0$, then we have for* $\pmod{32}$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+1}n+\frac{7\cdot 3^{4\alpha+1}-1}{4}\right)q^n\equiv 4\frac{f_3^3f_8^2}{f_4}, \tag{6.27}$$

$$\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+2}n+\frac{23\cdot 3^{4\alpha+1}-1}{4}\right)\equiv 0, \tag{6.28}$$

$$\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+3}n+\frac{5\cdot 3^{4\alpha+2}-1}{4}\right)\equiv 0, \tag{6.29}$$

$$\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+4}n+\frac{23\cdot 3^{4\alpha+3}-1}{4}\right)\equiv 0, \tag{6.30}$$

$$\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+5}n+\frac{5\cdot 3^{4\alpha+4}-1}{4}\right)\equiv 0. \tag{6.31}$$

Proof. Using (2.2) in (6.8), we find that

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}(6n+5)q^n\equiv 4\frac{f_3^3f_8^2}{f_4}\pmod{32}, \tag{6.32}$$

which is $\alpha=0$ case of (6.27).
Suppose that the equation (6.27) is true for all $\alpha\geq 0$.
Using (2.11) in (6.27) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we see that

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+2}n+\frac{7\cdot 3^{4\alpha+1}-1}{4}\right)q^n\equiv 4f_1f_2^3\pmod{8}, \tag{6.33}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+2}n+\frac{5\cdot 3^{4\alpha+2}-1}{4}\right)q^n\equiv 4q\frac{f_1^3f_{24}^2}{f_{12}}\pmod{32} \tag{6.34}$$

and (6.28).
Using (2.8) in (6.34) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (6.29),

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+3}n+\frac{13\cdot 3^{4\alpha+2}-1}{4}\right)q^n\equiv 4f_1f_4^3\pmod{8}, \tag{6.35}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+3}-1}{4}\right)q^n\equiv 20\frac{f_3^3f_8^2}{f_4}\pmod{32}. \tag{6.36}$$

Using (2.11) in (6.36) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we find that

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+4}n+\frac{13\cdot 3^{4\alpha+3}-1}{4}\right)q^n\equiv 4f_1f_2^3\pmod{8}, \tag{6.37}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+4}n+\frac{5\cdot 3^{4\alpha+4}-1}{4}\right)q^n\equiv 20q\frac{f_1^3f_{24}^2}{f_{12}}\pmod{32} \tag{6.38}$$

and (6.30).
Using (2.8) in (6.38) and then extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (6.31),

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+5}n+\frac{13\cdot 3^{4\alpha+4}-1}{4}\right)q^n\equiv 4f_1f_4^3\pmod{8}, \tag{6.39}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+5}n+\frac{5\cdot 3^{4\alpha+5}-1}{4}\right)q^n\equiv 20q\frac{f_3^3f_8^2}{f_4}\pmod{32}, \tag{6.40}$$

which is $\alpha+1$ case of (6.27). By mathematical induction, the equation (6.27) hold for all $\alpha\geq 0$. □

Theorem 6.3. *For $\alpha, \beta, n \geq 0$, then we have for* $\pmod 8$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(18\cdot 5^{2\beta}n+\frac{21\cdot 5^{2\beta}-1}{4}\right)q^n\equiv 4f_1f_2^3, \tag{6.41}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+4}5^{2\beta}n+\frac{7\cdot 3^{4\alpha+3}5^{2\beta}-1}{4}\right)q^n\equiv 4f_1f_2^3, \tag{6.42}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+6}5^{2\beta}n+\frac{7\cdot 3^{4\alpha+5}5^{2\beta}-1}{4}\right)q^n\equiv 4f_1f_2^3, \tag{6.43}$$

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+2}5^{2\beta}n+\frac{7\cdot 3^{4\alpha+1}5^{2\beta}-1}{4}\right)q^n\equiv 4f_1f_2^3. \tag{6.44}$$

Proof. The equation (6.11) is $\beta=0$ case of (6.41).
Suppose that the equation (6.41) is true for $\beta\geq 0$.
Using (2.14) in (6.41) and then extracting the term involving q^{5n+2}, we find that

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(18\cdot 5^{2\beta+1}n+\frac{33\cdot 5^{2\beta+1}-1}{4}\right)q^n\equiv 4qf_5f_{10}^3, \tag{6.45}$$

implies

$$\sum_{n=0}^{\infty}\overline{cpond}_{12,3}\left(18\cdot 5^{2\beta+2}n+\frac{21\cdot 5^{2\beta+2}-1}{4}\right)q^n\equiv 4f_1f_2^3, \tag{6.46}$$

which is $\beta+1$ case of (6.41). By mathematical induction, the equation (6.41) hold for all $\beta\geq 0$.
Since the proofs of (6.42)-(6.44) are similar to the proof of (6.41). So, we omit the details. □

Theorem 6.4. *For $\alpha, \beta, n \geq 0$, then we have for* $\pmod 8$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+3}5^{2\beta}n + \frac{13\cdot 3^{4\alpha+2}5^{2\beta}-1}{4}\right)q^n \equiv 4f_1f_4^3, \tag{6.47}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+5}5^{2\beta}n + \frac{13\cdot 3^{4\alpha+4}5^{2\beta}-1}{4}\right)q^n \equiv 4f_1f_4^3. \tag{6.48}$$

Since the proofs of (6.47)-(6.48) are similar to the proof of (6.41). So, we omit the details.

Theorem 6.5. *For $\alpha, \beta, n \geq 0$, then we have for* $\pmod 8$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(18\cdot 5^{2\beta}n + \frac{57\cdot 5^{2\beta}-1}{4}\right)q^n \equiv 4f_1f_6^3, \tag{6.49}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+4}5^{2\beta}n + \frac{19\cdot 3^{4\alpha+3}5^{2\beta}-1}{4}\right)q^n \equiv 4f_1f_6^3, \tag{6.50}$$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+6}5^{2\beta}n + \frac{19\cdot 3^{4\alpha+5}5^{2\beta}-1}{4}\right)q^n \equiv 4f_1f_6^3. \tag{6.51}$$

Since the proofs of (6.49)-(6.51) are similar to the proof of (6.41). So, we omit the details.

7 Ramanujan-type congruences for $\overline{\mathrm{cpond}}_{\delta,\mathrm{i}}(\mathrm{n})$

Theorem 7.1. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then for all $k, m, \alpha \geq 0$ with $p \nmid m$, we have*

$$\overline{cpond}_{4,1}(7^{2\alpha}\cdot p^{2k+1}(24m+5p)) \equiv 0 \pmod{16}. \tag{7.1}$$

Proof. Using (1.1) in the right hand side of (5.1), we see that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(7^{2\alpha}(24n+5))q^n = 8\sum_{k,l=-\infty}^{\infty} q^{\frac{k(3k-1)}{2}+2l(3l-1)} \pmod{16}, \tag{7.2}$$

which implies $n = \frac{k(3k-1)}{2} + 2l(3l-1)$. This is equivalent to

$$24n+5 = (6k-1)^2 + (12l-2)^2 = x^2+y^2.$$

Thus if $24n+5 \neq (6k-1)^2+(12l-2)^2 = x^2+y^2$, then

$$\overline{cpond}_{4,1}(7^{2\alpha}(24n+5)) \equiv 0 \pmod{16}. \tag{7.3}$$

Since $24n+5 = p^{2k+1}(24m+5p) \neq x^2+y^2$, we have

$$24\cdot 7^{2\alpha}n + 5\cdot 7^{2\alpha} = 24\cdot 7^{2\alpha}\cdot p^{2k+1}m + 5\cdot 7^{2\alpha}\cdot p^{2k+2},$$

which proves the theorem. □

Theorem 7.2. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then for all $k, m, \alpha \geq 0$ with $p \nmid m$, we have*

$$\overline{cpond}_{4,1}(7^{2\alpha+1}\cdot p^{2k+1}(24m+11p)) \equiv 0 \pmod{16}. \tag{7.4}$$

Proof. Using (1.1) in the right hand side of (5.2), we see that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(7^{2\alpha+1}(24n+5))q^n = 8\sum_{k,l=-\infty}^{\infty} q^{1+\frac{7k(3k-1)}{2}+14l(3l-1)} \pmod{16}. \tag{7.5}$$

The rest of the proof is similar to the proof of (7.1). So, we omit the details. □

Theorem 7.3. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then for all $k, m, \alpha \geq 0$ with $p \nmid m$, we have*

$$\overline{cpond}_{4,1}(7^{2\alpha} \cdot p^{2k+1}(24m + 17p)) \equiv 0 \pmod{32}. \tag{7.6}$$

Proof. Using (1.1) in the right hand side of (5.4), we see that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(7^{2\alpha}(24n + 17)q^n = 16 \sum_{k,l=-\infty}^{\infty} q^{\frac{k(3k-1)}{2}+8l(3l-1)} \pmod{32}.$$

which gives $n = \frac{k(3k-1)}{2} + 8l(3l - 1)$. This is equivalent to

$$24n + 17 = (6k - 1)^2 + [4(12l - 2)]^2 = x^2 + y^2.$$

Thus if $24n + 17 \neq (6k - 1)^2 + [4(12l - 2)]^2 = x^2 + y^2$, then

$$\overline{cpond}_{4,1}(7^{2\alpha}(24n + 17)) \equiv 0 \pmod{32}. \tag{7.7}$$

Since $24n + 17 = p^{2k+1}(24m + 17p)$, we have

$$24 \cdot 7^{2\alpha}n + 17 \cdot 7^{2\alpha} = 24 \cdot 7^{2\alpha} \cdot p^{2k+1}m + 17 \cdot 7^{2\alpha} \cdot p^{2k+2},$$

which proves the theorem. □

Theorem 7.4. *Let p be a prime such that $p \equiv 3 \pmod 4$. Then for all $k, m, \alpha \geq 0$ with $p \nmid m$, we have*

$$\overline{cpond}_{4,1}(7^{2\alpha+1} \cdot p^{2k+1}(24m + 23p)) \equiv 0 \pmod{32}. \tag{7.8}$$

Since the proof is similar to the proof of (7.6). So, we omit the details.

Theorem 7.5. *Let p be a prime such that $p \equiv 5$ or $7 \pmod 8$. Then for all $k, m, \alpha \geq 0$ with $p \nmid m$, we have*

$$\overline{cpond}_{4,1}(5^{2\alpha} \cdot p^{2k+1}(24m + 11p)) \equiv 0 \pmod{16} \tag{7.9}$$

and

$$\overline{cpond}_{4,1}(5^{2\alpha+1} \cdot p^{2k+1}(24m + 7p)) \equiv 0 \pmod{16}. \tag{7.10}$$

Proof. Using (1.1) in the right hand side of (5.7), we see that

$$\sum_{n=0}^{\infty} \overline{cpond}_{4,1}(5^{2\alpha}(24n + 11))q^n = 8 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{k(3k-1)+\frac{3l(l+1)}{2}} \pmod{16},$$

which implies $n = k(3k - 1) + \frac{3l(l+1)}{2}$. This is equivalent to

$$24n + 11 = 2(6k - 1)^2 + [3(2l + 1)]^2 = 2x^2 + y^2.$$

Thus if $24n + 11 \neq 2(6k - 1)^2 + [3(2l + 1)]^2 = 2x^2 + y^2$, then

$$\overline{cpond}_{4,1}(5^{2\alpha}(24n + 11)) \equiv 0 \pmod{16}. \tag{7.11}$$

Since $24n + 11 = p^{2k+1}(24m + 11p) \neq 2x^2 + y^2$, we have

$$\overline{cpond}_{4,1}(5^{2\alpha} \cdot p^{2k+1}(24m + 11p)) \equiv 0 \pmod{16},$$

which proves (7.9).
Since the proof (7.10) is similar to the proof of (7.9). So, we omit the details. □

Theorem 7.6. *Let p be a prime such that $p \equiv 13, 17, 19$ or $23 \pmod{24}$. Then for all $k, m, \alpha, \beta \geq 0$ such that $p \nmid m$, we have for* $\pmod 8$

$$\overline{cpond}_{12,3}\left(18 \cdot 5^{2\beta}p^{2k+1}n + \frac{21 \cdot 5^{2\beta}p^{2k+2} - 1}{4}\right) \equiv 0, \tag{7.12}$$

$$\overline{cpond}_{12,3}\left(2 \cdot 3^{4\alpha+4}5^{2\beta}p^{2k+1}n + \frac{7 \cdot 3^{4\alpha+3}5^{2\beta}p^{2k+2} - 1}{4}\right) \equiv 0, \tag{7.13}$$

$$\overline{cpond}_{12,3}\left(2 \cdot 3^{4\alpha+6}5^{2\beta}p^{2k+1}n + \frac{7 \cdot 3^{4\alpha+5}5^{2\beta}p^{2k+2} - 1}{4}\right) \equiv 0. \tag{7.14}$$

Proof. Using (1.1) and (1.2) in the right hand side of (6.41), we see that for $\pmod 8$

$$\sum_{n=0}^{\infty} \overline{cpond}_{12,3}\left(18\cdot 5^{2\beta}n+\frac{21\cdot 5^{2\beta}-1}{4}\right)q^n \equiv 4\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} q^{k(3k-1)/2+l(l+1)},$$

which implies $n=k(3k-1)/2+l(l+1)$. This is equivalent to

$$24n+7=(6k-1)^2+6(2l+1)^2=x^2+6y^2.$$

Thus if $24n+7\neq (6k-1)^2+6(2l+1)^2=x^2+6y^2$, then

$$\overline{cpond}_{12,3}\left(18\cdot 5^{2\beta}n+\frac{21\cdot 5^{2\beta}-1}{4}\right)\equiv 0. \tag{7.15}$$

Since $24n+7=p^{2k+1}(24m+7p)\neq x^2+6y^2$, we have

$$\overline{cpond}_{12,3}\left(18\cdot 5^{2\beta}p^{2k+1}n+\frac{21\cdot 5^{2\beta}p^{2k+2}-1}{4}\right)\equiv 0,$$

which proves (7.12).
Since the proofs of (7.13)-(7.14) are similar to the proof of (7.12). So, we omit the details. □

Theorem 7.7. *Let p be a prime such that $p\equiv 5$ or $11 \pmod{12}$. Then for all $k,m,\alpha,\beta\geq 0$ such that $p\nmid m$, we have for $\pmod 8$*

$$\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+3}5^{2\beta}p^{2k+1}n+\frac{13\cdot 3^{4\alpha+2}5^{2\beta}p^{2k+2}-1}{4}\right)\equiv 0, \tag{7.16}$$

$$\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+5}5^{2\beta}p^{2k+1}n+\frac{13\cdot 3^{4\alpha+4}5^{2\beta}p^{2k+2}-1}{4}\right)\equiv 0. \tag{7.17}$$

Since the proofs of (7.16)-(7.17) are similar to the proof of (7.12). So, we omit the details.

Theorem 7.8. *Let p be a prime such that $p\equiv 5$ or $7 \pmod 8$. Then for all $k,m,\alpha,\beta\geq 0$ such that $p\nmid m$, we have for $\pmod 8$*

$$\overline{cpond}_{12,3}\left(18\cdot 5^{2\beta}p^{2k+1}n+\frac{57\cdot 5^{2\beta}p^{2k+2}-1}{4}\right)\equiv 0, \tag{7.18}$$

$$\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+4}5^{2\beta}p^{2k+1}n+\frac{19\cdot 3^{4\alpha+3}5^{2\beta}p^{2k+2}-1}{4}\right)\equiv 0, \tag{7.19}$$

$$\overline{cpond}_{12,3}\left(2\cdot 3^{4\alpha+6}5^{2\beta}p^{2k+1}n+\frac{19\cdot 3^{4\alpha+5}5^{2\beta}p^{2k+2}-1}{4}\right)\equiv 0. \tag{7.20}$$

Since the proofs of (7.18)-(7.20) are similar to the proof of (7.12). So, we omit the details.

References

[1] G. E. Andrews, Singular overpartitions, Int. J. Number Theory 5 (11) (2015) 1523-1533.

[2] C. Ballantine and A. Welch, Generalizations of POD and PED partitions, (preprint)

[3] N. D. Baruah and K. K. Ojah, Partitions with designated summands in which all parts are odd, Integers, 15, # A9, (2015).

[4] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.

[5] S. Corteel and J. Lovejoy, Overpartions, Trans. Amer. Math. Soc., 356 (2004), 1623-1634.

[6] M. D. Hirschhorn, The power of q, Springer International Publishing, Switzerland, 2017.

[7] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of partition with odd distinct, Ramanujan J., 22, (2010), 273-284.

[8] M. D. Hirschhorn and J. A. Sellers, A congruence modulo 3 for partition into distinct non multiples of four, J. integer seq., 17, Article 14.9.6, (2014).

[9] M. S. Mahadeva Naika and D. S. Gireesh, Congruences for Andrews' singular overpartitions, J. Number theory, 165, (2016), 109-130.

[10] M. S. Mahadeva Naika and S. Shivaprasada Nayaka, Some new congruences for Andrews' singular overpartition pairs, Vietnam J.math., 46(3), (2018), 609-628.

[11] M. S. Mahadeva Naika and C. Shivashankar, Congruences for Andrews' singular overpartition pairs, International J. Number theory, 14(4), (2018), 989-1008.

[12] M. S. Mahadeva Naika, T. Harishkumar and T. N. Veeranayaka , On some congruence for $(j, k)-$ regular overpartitions, Gulf J. Math., 10 (2021), 43-68.

[13] M. S. Mahadeva Naika, T. Harishkumar and T. N. Veeranayaka, Congruences for $[j, k]-$ overpartitions with even parts distinct, Bol. Soc. Mat. Mex., 28, Article 44, (2022).

[14] M. Prasad, Kaneez Raheela and M. S. Mahadeva Naika, Arithmetic properties of OPOND partitions, Proceedings of the Jangjeon Mathematical Society, 28(2), (2025), 229-251.

[15] M. Prasad, Kaneez Raheela and M. S. Mahadeva Naika, Singular PEND Overpartitions, Montes Taurus J. Pure Appl. Math., 8(1),(2026), 77-104 ; Article ID:MTJPAM-D-25-00135.

Author information

M. Prasad, Department of Mathematics, PES College of Engineering, Mandya, Karnataka, India.
E-mail: prasadmj1987@gmail.com

Kaneez Raheela, Department of Mathematics, Farooqia PU College, Mysuru, Karnataka, India.
E-mail: kaneezraheela92@gmail.com

M. S. Mahadeva Naika, Department of Mathematics, Bengaluru City University, Bengaluru, Karnataka, India.
E-mail: msmnaika@rediffmail.com

Received: 2025-05-29
Accepted: 2026-04-28