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THE CENTER AND DISTRIBUTIVE ELEMENTS OF THE NEARRING OF COSET-PRESERVING FUNCTIONS

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Abstract. Let G be a finite group with $|G| \geq 4$, and let N be a nontrivial, proper, normal subgroup of G. Then $D(N) = \{f : G \to G \mid f(0) = 0 \text{ and } f(x+N) \subseteq f(x)+N \text{ for all } x \in G\}$ is a right nearring under the operations of function addition and composition. We characterize when $D(N)$ is abelian and find the distributive elements of $D(N)$. We also determine the center of $D(N)$ and show that $D(N)$ is never a ring.

1 Introduction

Definition 1.1. Let M be a nonempty set on which two binary operations called addition, + , and multiplication, $\cdot$, are defined. Then $(M, +, \cdot)$ is a (right) nearring provided $(M, +)$ is a group, $(M, \cdot)$ is a semigroup, and $(a + b)c = ac + bc$ for all a, b, and c in M.

Note that a right nearring does not require commutative addition or the left distributive law. If a nearring does have these two properties, then it is a ring. So every ring is a nearring. For more information on nearrings, see [9], [11], and [13].

Nearrings occur naturally as sets of functions. For any group $(G, +)$, not necessarily abelian, with identity 0, let $M(G) = \{f : G \to G\}$ denote the set of all functions mapping G to G. Under function addition and composition, $(M(G), +, \circ)$ is a right nearring. In addition, the set of all identity-preserving functions mapping G to G, $M_0(G) = \{f : G \to G \mid f(0) = 0\}$, is a subnearring of $M(G)$. In this paper, we explore a subnearring, $D(N)$, of $M_0(G)$. Recent papers concerning nearrings of functions include [2], [3], [7], and [8]. Papers investigating functions acting on nearring elements are [10] and [14].

Definition 1.2. Let G be a finite group with $|G| \geq 4$, and let N be a nontrivial, proper, normal subgroup of G. Define the set $D(N) = \{f : G \to G \mid f(0) = 0 \text{ and } f(x + N) \subseteq f(x) + N \text{ for all } x \in G\}$.

The set $D(N)$ is a subnearring of $M_0(G)$. The normality of N is required for $D(N)$ to be closed under addition and additive inverses. Since N is a nontrivial subgroup of G, it follows that $|N| \geq 2$. Furthermore, N being proper implies that $|G| \geq 4$.

We first state two results regarding $D(N)$ that are used throughout the paper.

Lemma 1.3. *Let $f \in D(N)$ and $g_1, g_2 \in G$ such that $f(g_1) \in g_2 + N$. Then $f(g_1 + N) \subseteq g_2 + N$. In particular, $f(N) \subseteq N$.*

Proof. Let $f \in D(N)$ and $g_1, g_2 \in G$ such that $f(g_1) \in g_2 + N$. Then using the definition of $D(N)$, for any $n \in N$ we get $f(g_1 + n) \in f(g_1) + N = g_2 + N$. Thus $f(g_1 + N) \subseteq g_2 + N$. Since $f(0) = 0 \in N = 0 + N$, we conclude that $f(N) \subseteq N$. □

Likewise, we note that if $f : G \to G$ is a function such that $f(0) = 0$, $f(N) \subseteq N$, and f maps cosets to cosets, then $f \in D(N)$. Therefore, we call the nearring $D(N)$ the nearring of coset-preserving functions. This nearring was previously studied in [5] and [6], where the complete lattice of ideals was determined. Here, we characterize when $D(N)$ is abelian and find the distributive elements of $D(N)$. We also find the center of $D(N)$ and show that $D(N)$ is never a ring.

We let id denote the identity function on G. Though we use 0 to denote the identity element of the group G and the zero-function on G, its use will be clear from context. Most of the results of this paper appear in [12].

2 Abelianness and Distributive Elements

In this section, we explore when $D(N)$ is abelian and find the distributive elements of $D(N)$. First we determine when $D(N)$ is abelian.

Theorem 2.1. *The nearring $D(N)$ is abelian if and only if G is an abelian group.*

Proof. We know that if G is an abelian group, then $M_0(G)$ is an abelian nearring. It follows that $D(N)$ is abelian since $D(N)$ is a subnearring of $M_0(G)$.

To prove the converse, assume that $D(N)$ is abelian. Then $f_1 + f_2 = f_2 + f_1$ for all $f_1, f_2 \in D(N)$. Let $a, b \in G$ and $g \in G \setminus N$. Define $f_1(x) = \begin{cases} a & \text{if } x \in g + N \\ 0 & \text{otherwise} \end{cases}$ and $f_2(x) = \begin{cases} b & \text{if } x \in g + N \\ 0 & \text{otherwise} \end{cases}$. Since f_1 and f_2 both map cosets into cosets, it follows that $f_1, f_2 \in D(N)$.

Thus $a + b = f_1(g) + f_2(g) = (f_1 + f_2)(g) = (f_2 + f_1)(g) = f_2(g) + f_1(g) = b + a$. Hence, G is abelian. □

Definition 2.2. An element $f \in D(N)$ is a distributive element if $f \circ (f_1 + f_2) = f \circ f_1 + f \circ f_2$ for all $f_1, f_2 \in D(N)$.

We now characterize the distributive elements of $D(N)$.

Theorem 2.3. *Let $f \in M(G)$. Then f is a distributive element of $D(N)$ if and only if f is an endomorphism of G with $f(N) \subseteq N$.*

Proof. Assume f is a distributive element of $D(N)$. Then $f(N) \subseteq N$. Let $a, b \in G$ and $g \in G \backslash N$. Using $f_1, f_2 \in D(N)$ defined in Theorem 2.1, we get $f(a + b) = f(f_1(g) + f_2(g)) = f(f_1(g)) + f(f_2(g)) = f(a) + f(b)$, and f is an endomorphism of G.

Now assume f is a endomorphism of G with $f(N) \subseteq N$. First we show that $f \in D(N)$. Let $x \in G$ and $n \in N$. Then $f(x+n) = f(x)+f(n) \in f(x)+N$. So $f(x+N) \subseteq f(x)+N$. Since f is a homomorphism, $f(0) = 0$. Therefore $f \in D(N)$. To show f is distributive, let $f_3, f_4 \in D(N)$ and $x \in G$. Since f is a homomorphism, we get $[f \circ (f_3 + f_4)](x) = f(f_3(x) + f_4(x)) = f(f_3(x)) + f(f_4(x)) = (f \circ f_3)(x) + (f \circ f_4)(x)$, and f is a distributive element of $D(N)$. □

It is well-known that the distributive elements of $M_0(G)$ are the endomorphisms of G (Lemma 9.6, [11]). Thus Theorem 2.3 shows that the distributive elements of $D(N)$ are the distributive elements of $M_0(G)$ that are also in $D(N)$.

The next result identifies a class of distributive elements in $D(N)$.

Corollary 2.4. *Let $\varphi_g : G \to G$ be the inner automorphism of G determined by $g \in G$, namely, $\varphi_g(x) = g + x - g$ for all $x \in G$. Then φ_g is a distributive element of $D(N)$.*

Proof. Clearly φ_g is a homomorphism of G. Since N is a normal subgroup of G, we have $x + N = N + x$ for all $x \in G$. Thus, $x + N - x = N$ as well. Therefore, for all $n \in N$, $\varphi_g(n) = g + n - g \in N$. We conclude that $\varphi_g(N) \subseteq N$, and φ_g is a distributive element of $D(N)$ by Theorem 2.3. □

Definition 2.5. A subgroup H of a group G is fully invariant if $\alpha(H) \subseteq H$ for all endomorphisms α of G.

A fully invariant subgroup is necessarily a normal subgroup. The next result is clear.

Corollary 2.6. *If N is a fully invariant subgroup of G, then the distributive elements of $D(N)$ are precisely the endomorphisms of G.*

3 The Center of $D(N)$

Definition 3.1. The (multiplicative) center of a nearring M is $C(M) = \{c \in M \mid cm = mc \text{ for all } m \in M\}$.

In this section, we determine the center of $D(N)$ and show that $D(N)$ is never a ring. We find restrictions on a function $c \in C(D(N))$ by defining specific functions $f \in D(N)$ and considering $c \circ f = f \circ c$. First, we examine the action of c on the subgroup N. As a reminder, we use 0 to denote the identity element of G and the additive identity of $D(N)$. Every proof in this section uses both notations, except for the last proof.

Lemma 3.2. *Let $c \in C(D(N))$. Then $c|_N = 0$ or $c|_N = id$.*

Proof. We use a technique given in the proof of Proposition 1.1 of [1]. Let $0 \neq n \in N$ and $c \in C(D(N))$. Define $\lambda_n : G \to G$ by $\lambda_n(x) = \begin{cases} n & \text{if } 0 \neq x \in N \\ 0 & \text{otherwise} \end{cases}$. Then $\lambda_n \in D(N)$.

Let $0 \neq x \in N$. Then $c(x) \in N$ and either $c(x) = 0$ or $c(x) \neq 0$. If $c(x) = 0$, then $c(n) = c(\lambda_n(x)) = (c \circ \lambda_n)(x) = (\lambda_n \circ c)(x) = \lambda_n(c(x)) = \lambda_n(0) = 0$. Since $0 \neq n \in N$ is arbitrary, we get $c|_N = 0$. If $c(x) \neq 0$, then $c(n) = c(\lambda_n(x)) = (c \circ \lambda_n)(x) = (\lambda_n \circ c)(x) = \lambda_n(c(x)) = n$. Since $0 \neq n \in N$ is arbitrary, we get $c|_N = id$. □

Next, we examine the action of c on a coset $a + N$ where $a \notin N$.

Lemma 3.3. *Let $a \in G$ with $a \notin N$ and $c \in C(D(N))$. Then $c|_{a+N} = 0$ or $c|_{a+N} = id$.*

Proof. Since $c \in C(D(N))$ maps cosets into cosets by hypothesis, it suffices to determine the case for a single $x \in a + N$. We consider two cases. In the first case, we assume that $c(x) \in N$. Since $x \in a + N$, $c(x) \in N$, and c maps cosets into cosets, it follows that $c(a + N) \subseteq N$. By means of contradiction, assume that $c(x) \neq 0$. Using the function λ_n in the previous lemma, we get $0 = c(0) = c(\lambda_n(x)) = (c \circ \lambda_n)(x) = (\lambda_n \circ c)(x) = \lambda_n(c(x)) = n$, a contradiction. Hence $c(x) = 0$ and $c|_{a+N} = 0$.

For the second case, assume $c(x) \notin N$. We first show that $c(x) \in a + N$. On the contrary, assume $c(x) \notin a + N$. We define $\sigma : G \to G$ by $\sigma(x) = \begin{cases} x & \text{if } x \in a + N \\ 0 & \text{otherwise} \end{cases}$. So $\sigma \in D(N)$. Thus $c(x) = c(\sigma(x)) = (c \circ \sigma)(x) = (\sigma \circ c)(x) = \sigma(c(x)) = 0 \in N$, a contradiction since $c(x) \notin N$. Therefore $c(x) \in a + N$ and $c(a + N) \subseteq a + N$.

Fix $y \in a + N$ and define $\phi : G \to G$ by $\phi(x) = \begin{cases} y & \text{if } x \in a + N \\ 0 & \text{otherwise} \end{cases}$. So $\phi \in D(N)$. Since $c(x) \in a + N$, we conclude that $c(y) = c(\phi(x)) = (c \circ \phi)(x) = (\phi \circ c)(x) = \phi(c(x)) = y$. Since $y \in a + N$ is arbitrary, it follows that $c|_{a+N} = id$. □

Note that if there are exactly two cosets determined by N, say N and $a + N$, Lemmas 3.2 and 3.3 imply that for any $c \in C(D(N))$ we have $c|_N = 0$ or $c|_N = id$, and also $c|_{a+N} = 0$ or $c|_{a+N} = id$. Thus in this situation, there are at most four functions in $C(D(N))$.

Lemma 3.4. *Let $c \in C(D(N))$. Then $c|_{G\setminus N} = 0$ or $c|_{G\setminus N} = id$.*

Proof. If there are two cosets determined by N, the result follows from the comments above. So assume there are at least three distinct cosets determined by N, say N, $a + N$, and $b + N$. So $a, b \notin N$ and $b \notin a + N$.

We prove by contradiction. Without a loss of generality, assume there exists $c \in C(D(N))$ such that $c|_{a+N} = 0$ and $c|_{b+N} = id$. Define $\epsilon : G \to G$ by $\epsilon(x) = \begin{cases} b & \text{if } x \in a + N \\ 0 & \text{otherwise} \end{cases}$. So $\epsilon \in D(N)$. Thus for $x \in a + N$, $c(b) = c(\epsilon(x)) = (c \circ \epsilon)(x) = (\epsilon \circ c)(x) = \epsilon(c(x)) = \epsilon(0) = 0$, a contradiction since $c(b) = b$. It follows that $c|_{G\setminus N} = 0$ or $c|_{G\setminus N} = id$. □

Lemmas 3.2 and 3.4 imply that there are at most four functions in $C(D(N))$. The next theorem eliminates two of these functions.

Theorem 3.5. *The center of $D(N)$ consists of exactly two functions, namely $C(D(N)) = \{0, id\}$.*

Proof. Let $0 \neq n \in N$. Define $\beta : G \to G$ by $\beta(x) = \begin{cases} n & \text{if } x \notin N \\ 0 & \text{otherwise} \end{cases}$. So $\beta \in D(N)$.

Assume there exists $c \in C(D(N))$ such that $c|_N = id$ and $c|_{G\setminus N} = 0$. Thus for $x \notin N$, we get $n = c(n) = c(\beta(x)) = (c \circ \beta)(x) = (\beta \circ c)(x) = \beta(c(x)) = \beta(0) = 0$, a contradiction since $n \neq 0$.

Now assume there exists $c \in C(D(N))$ such that $c|_N = 0$ and $c|_{G\setminus N} = id$. Thus for $x \notin N$, we get $n = \beta(x) = \beta(c(x)) = (\beta \circ c)(x) = (c \circ \beta)(x) = c(\beta(x)) = c(n) = 0$, a contradiction since $n \neq 0$.

Hence, for $c \in C(D(N))$, either $c|_N = id$ and $c|_{G\setminus N} = id$, or $c|_N = 0$ and $c|_{G\setminus N} = 0$. We conclude that $C(D(N)) = \{0, id\}$. □

It may not be surprising that $C(D(N))$ has only two elements. If a nearring of functions has many elements, one may expect very few elements in the nearring to commute with all other elements. We see this in other nearrings of functions. If G is a finite group and H is a nontrivial, proper subgroup of G, define $M_1 = \{f : G \to G \mid f(0) = 0 \text{ and } f(H) \subseteq H\}$ and $M_2 = \{f : G \to G \mid f(0) = 0 \text{ and } f(G) \subseteq H\}$. In [7], it was shown that $C(M_1) = \{0, id\}$. In [3], it was shown that $C(M_2) = \{0\}$.

On the other hand, the centralizer nearring $M_E(S_n) = \{f : S_n \to S_n \mid f((1)) = (1) \text{ and } f \circ s = s \circ f \text{ for all } s \in \text{End } S_n\}$ was investigated in [2]. There it was shown that $|M_E(S_5)| = 180$, $|C(M_E(S_5))| = 40$, $|M_E(S_6)| = 720$, and $|C(M_E(S_6))| = 70$. Thus, nearrings of functions may have large or small centers.

Unlike the situation in rings, the center of a nearring is not necessarily a subnearring (see [1] and [4]).

Corollary 3.6. *The center of $D(N)$, $C(D(N))$, is a subnearring of $D(N)$ if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.*

Proof. If $C(D(N))$ is a subnearring of $D(N)$, then $id + id = 0$ by Theorem 3.5. So for any $g \in G$, $g + g = id(g) + id(g) = (id + id)(g) = 0(g) = 0$. Therefore, $g + g = 0$ and the exponent of G is 2. It follows that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. The converse is clear by Theorem 3.5. □

The final result shows that $D(N)$ is never a ring.

Theorem 3.7. *The nearring $D(N)$ is not a ring.*

Proof. Assume $D(N)$ is a ring. Then the left distributive law holds. Let $0 \neq a \in N$ and $b \in G\setminus N$. Then $a + b \notin N$. Since $D(N)$ is a ring, it follows that $C(D(N))$ is a subring of $D(N)$. By Corollary 3.6, $b + b = 0$. Define $f_1, f_2, f_3 : G \to G$ by $f_1(x) = \begin{cases} 0 & \text{if } x \in N \\ x & \text{if } x \notin N \end{cases}$, $f_2(x) = \begin{cases} 0 & \text{if } x \in N \\ a + b & \text{if } x \notin N \end{cases}$, and $f_3(x) = \begin{cases} 0 & \text{if } x \in N \\ b & \text{if } x \notin N \end{cases}$. Then $f_1, f_2, f_3 \in D(N)$.

Let $g \in G\setminus N$. Then $0 = f_1(a) = f_1(a + b + b) = f_1(f_2(g) + f_3(g)) = [f_1 \circ (f_2 + f_3)](g) = [f_1 \circ f_2](g) + [f_1 \circ f_3](g) = f_1(f_2(g)) + f_1(f_3(g)) = f_1(a + b) + f_1(b) = a + b + b = a$, a contradiction since $a \neq 0$. We conclude that $D(N)$ is not a ring. □

Note that if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, then $C(D(N))$ is a subnearring of $D(N)$ even though $D(N)$ is not a ring by Theorem 3.7.

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